A higher-order oscillator, including a nonlinear unit and an 8th-order low-pass active Bessel filter is described. The Bessel unit plays the role of “three-in-one”: a delay line, an amplifier and a filter. Results of hardware experiments and numerical simulation are presented. Depending on the parameters of the nonlinear unit the oscillator operates either in a one-scroll or two-scroll mode. Two positive Lyapunov exponents, found at larger values of the negative slopes of the nonlinear function, characterize the oscillations as hyperchaotic.

Keywords: Delay systems; chaotic oscillators; higher-order systems.
1. Introduction

A large variety of chaotic and hyperchaotic oscillators based on delayed feedback have been described in literature.\textsuperscript{1–23} The common units of an electronic delay oscillator are: (1) a nonlinear element, (2) a delay line, (3) an amplifier, and (4) a filter. Various types of the delay line have been used to build the oscillators: (i) an ultrasonic delay device,\textsuperscript{1} (ii) a so-called “bucket-brigade” tunable delay device,\textsuperscript{2,10–13} (iii) a transmission cable\textsuperscript{3,4,14–20} (most suitable for very high and ultrahigh frequency range), (iv) high-order low-pass filter, composed of T-type LCL passive circuits.\textsuperscript{5,7,21} The filter used in the oscillators is either low-pass or band-pass unit. Regarding possible application in communication systems chaotic oscillators with band-pass filters can better match transmission channels, which are mostly band-pass by nature.

Another approach for building higher-order chaotic and hyperchaotic oscillators is based on the coupling of either simple chaotic or linear oscillators in arrays. Various combinations of linearly coupled nonlinear (chaotic) and nonlinearly coupled linear oscillators have been proposed.\textsuperscript{25–36}

In this paper, we suggest exploiting in the oscillator a low-pass active Bessel filter. The main motivation behind such a choice is that the transfer function of the Bessel filter is characterized by linear phase shift versus frequency, i.e., by a constant delay time over a wide frequency band.\textsuperscript{37} In addition, the Bessel filter when implemented as an RC active unit has inherent gain necessary for the oscillator. Finally, the Bessel unit plays its designated role of a filter. We show that complicated chaotic behaviors including one-scroll and two-scroll modes can be achieved with a single 8th-order Bessel filter. Presumably higher-order Bessel filters would work even better, providing higher dimensionality and higher complexity of chaos. We note, however, the 8th-order filter is the highest order filter which circuit parameters are easily available in handbooks.\textsuperscript{37} Therefore it makes sense to design the 8th-order filter as a main building block, and then to couple several of them in series, if necessary. In addition, we propose a very simple scalar delay differential equation to describe chaotic dynamical behavior of the oscillator and characterize the oscillations by the Lyapunov exponents.

2. Circuit Description

The oscillator (Fig. 1) has a ring structure and comprises only two units, namely a nonlinear unit\textsuperscript{21} (Fig. 2) and an 8th-order low-pass active Bessel filter\textsuperscript{37} (Fig. 3). An auxiliary 1st-order low-pass RC filter is not involved in the feedback loop of the oscillator. It produces an auxiliary variable $V_{\text{out2}}(t)$ for plotting the phase portraits, since the oscillator itself has only one easily accessible independent variable $V_{\text{out1}}(t)$.
The amplifiers in the nonlinear unit and the active Bessel filter are the LM741 type operational amplifiers; the diodes in the nonlinear unit are the 1N4148 or similar general-purpose devices. The absolute values of the positive slope $\alpha$ and the negative slopes $\beta$ of the transfer function of the nonlinear unit are related to the resistor values in Fig. 2, left, and can be estimated as follows:

$$\alpha = r_2/(r_1 + r_2), \quad \beta = (r_4/r_3)(1 - \alpha) - \alpha. \quad (1)$$

In order to demonstrate the advantages of the Bessel filter over a simple RC filter and to explain some differences between the measured and the simulated power spectra the corresponding transfer functions have been calculated (Fig. 4). The components of the 8th-order active Bessel filter were set to values, which provide the cut-off frequency $f_c(-3\text{dB}) = 3$ kHz (Fig. 4, left). The low-frequency gain of the 8th-order Bessel filter is 12.7 dB, the delay time of the Bessel filter is $T_{\text{del}} = 170 \mu$s (Fig. 4, right) and is rather constant over wide frequency range. The same cut-off frequency $f_c$ exhibits the 1st-order passive RC filter with time constant of 53 $\mu$s. However, its

Fig. 1. Block diagram of the Bessel filter based chaotic oscillator.

Fig. 2. Nonlinear unit (left) and its transfer function (right). $r_1 = 430 \Omega$, $r_2 = 1 \text{k}\Omega$, $r_3 = 1 \text{k}\Omega$, $r_4 = 6.2 \text{k}\Omega$.

Fig. 3. Circuit diagram of the 8th-order active Bessel filter. Circuit values $C = 10 \text{nF}$, $R = 5.31 \text{k}\Omega$, $R_1 = 10 \text{k}\Omega$.

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3. Experimental Results

The transfer function of the nonlinear unit is shown in Fig. 5. The dynamic behavior of the overall system is illustrated with the phase portraits in Fig. 6.

The corresponding frequency spectra taken by means of an analogue spectrum analyzer are shown in Fig. 7. The results presented in Figs. 6 and 7 including the existence of the one-scroll and the two-scroll attractors, resemble very much the results of the recent paper\textsuperscript{21} for delayed feedback chaotic oscillator with a separate delay line, a separate amplifier and a separate 1st-order RC filter.

4. Mathematical Model and Numerical Results

The transfer function of the nonlinear unit shown in Figs. 2 and 5 can be presented by a three-segment piecewise-linear approximation (see also Fig. 8)

\[
N(x) = \begin{cases} 
-\beta(x + 1) - \alpha, & x \leq -1, \\
\alpha x, & -1 < x \leq 1, \\
-\beta(x - 1) + \alpha, & x > 1 
\end{cases}
\]

Fig. 4. Transfer characteristics of the 8th-order Bessel filter (solid lines) and of the 1st-order RC filter with time constant 53\,\mu s (dashed lines). Gain (left), delay (right). Gain of the RC filter is intentionally shifted up from 0 dB to 12.7 dB for better comparison with the Bessel filter. Circuit values $C = 10$ nF, $R = 5.31$ k\,\Omega, $R_1 = 10$ k\,\Omega.

Fig. 5. Measured transfer function of the nonlinear unit. Slopes estimated from (1): $\alpha = 0.7$, $\beta = 1.2$. Average experimental slopes slightly differ from the estimated values: slope $\alpha_{\text{exp}} \approx 0.65$, slope $\beta_{\text{exp}} \approx 0.85$.

low-frequency gain is only 0 dB, as expected, the delay time is essentially smaller and has noticeable dispersion.
where $x$ is the normalized input voltage $V_{in}$ of the nonlinear unit (when in the closed loop the $V_{in}$ is just the output voltage of the Bessel filter $V_{out}$).

Since the circuit includes the 8th-order filter, the overall system should be described by a set of 8 ordinary differential equations. However, we replace in the model the 8th-order filter with the 1st-order RC filter of the same cut-off frequency ($f_c = 1/2\pi RC$). Then we make use of the fact, that the Bessel filter has a constant
delay $T_{\text{del}}$, and describe this value as a sum of an intrinsic delay $T_{\text{del0}}$ and the $RC$ time constant ($T_{\text{del}} = T_{\text{del0}} + RC$). This gives the following one-variable delay differential equation:

$$\dot{x} = kN[x(t - \tau)] - x,$$

(3)

where $k$ is the low frequency gain of the active Bessel filter, the dimensionless delay $\tau = T_{\text{del0}}/RC = (T_{\text{del}} - RC)/RC = T_{\text{del}}/RC - 1$. The auxiliary integrating circuit is described by the 1st-order ordinary differential equation:

$$\dot{y} = N(x) - y.$$

(4)

Numerical integration of Eqs. (3) and (4) gives the results (Figs. 9 and 10) very similar to the experimental phase portraits in Fig. 6 and frequency spectra in Fig. 7. Nevertheless, there are some discrepancies between the measured and the calculated spectra. For both, the one-scroll and the two-scroll oscillations the spectra obtained from the model (Fig. 10) have noticeably lower falloff rate of the spectral density at higher frequencies compared to the measured spectra (Fig. 7). This difference can be explained by the fact that the Bessel filter used in the experiment is

![Fig. 9. Phase portraits from Eqs. (3) and (4) with $\alpha = 0.65$. One-scroll, $\beta = 0.60$ (left). Two-scroll, $\beta = 0.85$ (right).](image)

![Fig. 10. Frequency spectra of variable $x$ from Eq. (3) with $\alpha = 0.65$. One-scroll, $\beta = 0.60$ (left). Two-scroll, $\beta = 0.85$ (right).](image)
the 8th-order filter, while the mathematical model includes the 1st-order RC filter. The difference in the fall-off rate is evident from Fig. 4. Though the both filters have the same cut-off frequency \( f_c = 3 \text{ kHz at } -3 \text{ dB} \) attenuation in the stop-band with the 8th-order Bessel filter is by 30 dB (at 10 kHz) more effective than with the 1st-order RC filter. Consequently, the spectral density of the output signal from the Bessel filter is by 30 dB lower than from the 1st-order RC filter, practically by the same value as observed experimentally (Fig. 7).

The difference between the calculated and the measured spectra observed for the two-scroll oscillations (Figs. 7 and 10) at low frequencies (< 1 kHz), i.e., a nearly 20 dB excess rise in the calculated spectrum could have several reasons. This frequency band reflects the rate of switching of two-scroll attractor between the first and the third quadrants of the phase plane (Figs. 6 and 9). This process is very sensitive to three factors: (1) the slope values \( \alpha \) and \( \beta \) of the nonlinear unit, (2) the unavoidable offsets in the operational amplifiers of the oscillator, (3) the manufacturing mismatch of the characteristics of the two diodes in the nonlinear unit. The second factor has been clearly demonstrated, both numerically and experimentally, for a related delayed feedback chaotic oscillator\(^{21}\): the peak emerging at frequencies close to zero can be removed by adding constant DC bias. In a real chaotic oscillator the above-mentioned factors could all together play their roles.

Obviously, the auxiliary signal \( V_{\text{out}2} \) in the oscillator (Fig. 1), correspondingly the dynamical variable \( y \) in Eq. (4), can also be taken for the output signal. There is a third possible output node in the oscillator, namely the output of the nonlinear unit \( V_{\text{out}3} \) (Fig. 1). In this case the spectra (Fig. 11) are more flat than those obtained from the \( V_{\text{out}1} \) or \( V_{\text{out}2} \). The reason is in the fact, that the \( V_{\text{out}3} \) is taken directly from the nonlinear element, in contrast to \( V_{\text{out}1} \) or \( V_{\text{out}2} \), thus is not filtered by the low-pass filters and consequently has higher power content at higher frequencies.

The bifurcation diagram (Fig. 12) plotted against the control parameter \( \beta \) demonstrates relatively dense chaotic states with only narrow periodic windows in

![Fig. 11. Frequency spectra of the output signal \( V_{\text{out}3} \) of the nonlinear unit in the two-scroll mode. Simulated spectrum from Eq. (3) with \( \alpha = 0.65, \beta = 0.85 \) (left). Experimental spectrum at \( \alpha_{\text{exp}} \approx 0.65, \beta_{\text{exp}} \approx 0.85 \) (right); frequency range is from 0 to 10 kHz, horizontal scale 1 kHz/div, resolution 100 Hz, vertical scale 10 dB/div.](image)
the one-scroll mode \(0.4 < \beta < 0.6\) also in the developed two-scroll chaotic mode \(0.7 < \beta < 1.0\).

We analyzed the system (3) theoretically making use of the calculations of the Lyapunov exponents (LEs). The hyperchaotic attractor possesses a chaotic behavior with at least two positive LEs. Thus we did not need to estimate the whole spectrum of LEs, it sufficed to determine only positive LEs. As the model of the system was known the calculations were based on the algorithm presented in Refs. 38 and 39. The variational equation

\[
\delta \dot{x} = -\delta x + k\delta N[x(t - \tau)]
\]  

was derived from the Eq. (3). Here \(\delta x\) defines the separation of infinitesimally close trajectories and

\[
\delta N(x) = \begin{cases} 
-\beta \delta x, & x < -1, \quad x > 1; \\
\alpha \delta x, & -1 \leq x \leq 1.
\end{cases}
\]

Two positive LEs \(\lambda_1\) and \(\lambda_2\) presented in Fig. 13 indicate, that hyperchaotic behavior of the oscillator can be expected.

5. Conclusions

Higher-order (e.g., the 8th-order) low-pass active Bessel filter is a convenient building block for chaotic and hyperchaotic oscillators. The mathematical model
suggested in the form of an extremely simple scalar delay differential equation including three-segment piecewise-linear function describes qualitatively well the experimental results. The type of oscillations, i.e., either one-scroll or two-scroll, either chaotic with a single positive Lyapunov exponent or hyperchaotic with two positive Lyapunov exponents, can be controlled by the slope parameters of the nonlinear unit.

Since the agreement between many experimental and numerical results is rather good, for example between the phase portraits (Figs. 6 and 9), between the power spectra of the signals taken from the main output $V_{\text{out}1}$ (Figs. 7 and 10) also the power spectra of the signals taken from the nonlinear unit $V_{\text{out}3}$ (Fig. 11), we hope that the Lyapunov exponents calculated from Eq. (3) apply to the experimental circuit as well.

References


