Improved statistical inference for the two-parameter Birnbaum–Saunders distribution
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Abstract
We develop nearly unbiased estimators for the two-parameter Birnbaum–Saunders distribution [Birnbaum, Z.W., Saunders, S.C., 1969a. A new family of life distributions. J. Appl. Probab. 6, 319–327], which is commonly used in reliability studies. We derive modified maximum likelihood estimators that are bias-free to second order. We also consider bootstrap-based bias correction. The numerical evidence we present favors three bias-adjusted estimators. Different interval estimation strategies are evaluated. Additionally, we derive a Bartlett correction that improves the finite-sample performance of the likelihood ratio test in finite samples. © 2006 Elsevier B.V. All rights reserved.

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1. Introduction

Birnbaum and Saunders (1969a) proposed a family of two-parameter distributions to model failure time due to fatigue under cyclic loading and the assumption that failure follows from the development and growth of a dominant crack. Desmond (1985) derived the distribution under a more general setting, using a biological model and relaxing several of the assumptions made by Birnbaum and Saunders (1969a), and Desmond (1986) explored the relationship between the Birnbaum–Saunders and inverse Gaussian distributions.

The random variable $T$ is Birnbaum–Saunders distributed with parameters $\alpha, \beta > 0$, denoted $\mathcal{B}-\mathcal{S}(\alpha, \beta)$, if its distribution function is given by

$$F_T(t) = P(T \leq t) = \Phi \left( \frac{1}{\alpha} \left( \frac{t}{\beta} - \sqrt{\frac{t}{\beta}} \right) \right), \quad t > 0,$$

(1)

where $\Phi(\cdot)$ denotes the standard normal distribution function. $\alpha$ is a shape parameter, and as $\alpha$ decreases towards zero the Birnbaum–Saunders distribution approaches the normal distribution with mean $\beta$ and variance $\tau$, where $\tau \to 0$ when $\alpha \to 0$. Also, $\beta$ is a scale parameter, i.e., $T/\beta \sim \mathcal{B}-\mathcal{S}(\alpha, 1)$. Additionally, $\beta$ is the median of the distribution: $F_T(\beta) = \Phi(0) = 0.5$. It is noteworthy that the reciprocal property holds for the Birnbaum–Saunders distribution: $T^{-1} \sim \mathcal{B}-\mathcal{S}(\alpha, \beta^{-1})$; see Saunders (1974).

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Mann et al. (1974, p. 155) noted that, although the hazard rate implied by the Birnbaum–Saunders distribution is not increasing, the average hazard rate is nearly nondecreasing. Engelhardt et al. (1981) developed interval estimation for $x$ considering $\beta$ as an unknown nuisance parameter, and likewise for $\beta$. They also considered hypothesis testing for performing inference on the two parameters that index the distribution. Rieck and Nedelman (1991) developed a log-linear model for the $\beta$-$\alpha$ distribution and showed that it can be used for accelerated life testing or to compare median lives of several populations. Achcar (1993) developed Bayesian estimation approaches for the parameters in (1) using approximations to the posterior marginal distributions of $x$ and $\beta$. Rieck (1999) obtained the moment generating function of the sinh-normal (normal hyperbolic sine) distribution, which can be used to obtain integer and non-integer moments of $\beta$-$\alpha$ distribution. Lu and Chang (1997) used bootstrap methods to construct prediction intervals for future realizations of the distribution $\beta$-$\alpha$, and concluded that such intervals have good coverage when the sample contains more than 30 observations. Dupuis and Mills (1998) used robust methods to estimate the two parameters that index the Birnbaum–Saunders distribution when the sample contains outlying data. Other references related to the Birnbaum–Saunders distribution are Chang and Tang (1993, 1994), Díaz–García and Leiva–Sánchez (2005), Galea et al. (2004), Jin and Kawczak (2003), Ng et al. (2003), Owen and Padgett (1999, 2000), Rieck (1995), Wang et al. (2006) and Wu and Wong (2004). For further details on the Birnbaum–Saunders distribution, see Johnson et al. (1995).

A bias correction to the maximum likelihood estimators (MLEs) of the parameters that index the Birnbaum–Saunders distribution was proposed by Ng et al. (2003). However, their correction is ad hoc in the sense that it was obtained by inspecting the pattern of the bias of the MLEs in a large Monte Carlo experiment. It is not possible, thus, to guarantee that their bias-adjusted estimators are unbiased to second order.

The chief goal of our paper is to obtain modified MLEs that are nearly free of bias, in particular, modified MLEs that are unbiased to second order. That is, we remove the bias of the MLEs of $x$ and $\beta$ to order $O(n^{-1})$. This is done analytically and closed-form expressions for the second order biases of the MLEs are provided. Bootstrap-based (Efron, 1979) bias-adjusted estimators are also considered. Additionally, we numerically evaluate the finite-sample behavior of different interval estimation strategies and derive a Bartlett correction to the likelihood ratio test, thus obtaining a modified test with superior finite-sample performance.

The paper unfolds as follows. Section 2 introduces the Birnbaum–Saunders distribution, point estimation and asymptotic confidence intervals. In Section 3, we derive the second order biases of the MLEs and consider bias-correction estimators using both analytical and numerical correction schemes. We also introduce four alternative confidence intervals, three of them are bootstrap-based. In Section 4, we derive a Bartlett correction to the likelihood ratio test used to perform inference on the shape parameter of the Birnbaum–Saunders distribution. A bootstrap-based test is also considered. Numerical results from Monte Carlo simulation experiments are presented and discussed in Section 5. Two empirical examples are considered in Section 6. Finally, Section 7 concludes the paper.

2. The Birnbaum–Saunders distribution

The distribution function of the random variable $T$ is given in (1), the corresponding density function being

$$f_T(t; \alpha, \beta) = \frac{1}{2\alpha\beta\sqrt{2\pi}} \left[ \left( \frac{\beta}{t} \right)^{1/2} + \left( \frac{\beta}{t} \right)^{3/2} \right] \exp \left\{ -\frac{1}{2\alpha^2} \left( \frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right\},$$

$t > 0$, $\alpha, \beta > 0$. The expected value, variance, skewness and kurtosis are, respectively,

$$E(T) = \beta(1 + \frac{1}{2} \alpha^2), \quad \text{Var}(T) = (\alpha\beta)^2(1 + \frac{5}{3}\alpha^2),$$

$$\mu_3 = \frac{4\alpha(11\alpha^2 + 6)}{(5\alpha^2 + 4)^{3/2}} \quad \text{and} \quad \mu_4 = 3 + \frac{6\alpha^2(93\alpha^2 + 40)}{(5\alpha^2 + 4)^2},$$

the expressions we give for the skewness and kurtosis correct those given by Johnson et al. (1995, p. 653). As noted earlier, if $T \sim \beta$-$\alpha$, then $T^{-1} \sim \beta$-$\beta$. It then follows that

$$E(T^{-1}) = \beta^{-1}(1 + \frac{1}{2}\alpha^2) \quad \text{and} \quad \text{Var}(T^{-1}) = \alpha^2 \beta^{-2}(1 + \frac{5}{4}\alpha^2).$$
The density (2) is skewed to the right. However, the asymmetry of the distribution decreases with $\alpha$. Fig. 1 plots the Birnbaum–Saunders density for some values of $\alpha$ with $\beta = 1$. Note that, as $\alpha$ decreases, the distribution becomes more symmetric around $\beta$, the median; note also that the variance decreases with $\alpha$.

Let $t = (t_1, \ldots, t_n)^T$ denote a random sample of size $n$ from the Birnbaum–Saunders distribution. The log-likelihood function, apart from an unimportant constant, is

$$
\ell(\alpha, \beta) = -n \log(\alpha \beta) + \sum_{i=1}^{n} \log \left[ \left( \frac{\beta}{t_i} \right)^{1/2} + \left( \frac{\beta}{t_i} \right)^{3/2} \right] - \frac{1}{2\alpha^2} \sum_{i=1}^{n} \left( \frac{t_i}{\beta} + \frac{\beta}{t_i} - 2 \right). 
$$

The MLEs $\hat{\alpha}$ and $\hat{\beta}$ of $\alpha$ and $\beta$, respectively, are obtained from the maximization of (3), as the solution to the following system of equations:

$$
\begin{align*}
\frac{\partial \ell(\alpha, \beta)}{\partial \alpha} &= -n \left( 1 + \frac{2}{\alpha^2} \right) + \frac{1}{\alpha^2} \beta \sum_{i=1}^{n} t_i + \frac{\beta}{\alpha^2} \sum_{i=1}^{n} \frac{1}{t_i} = 0, \\
\frac{\partial \ell(\alpha, \beta)}{\partial \beta} &= -n \frac{2}{2\beta} + \sum_{i=1}^{n} \frac{1}{t_i + \beta} + \frac{1}{2\beta^2} \sum_{i=1}^{n} t_i - \frac{1}{2\alpha^2} \sum_{i=1}^{n} \frac{1}{t_i} = 0.
\end{align*}
$$

From (4), Birnbaum and Saunders (1969b) showed that $\hat{\alpha}$ can be written as

$$
\hat{\alpha} = \left( \frac{s}{\bar{\beta}} + \frac{\bar{\beta}}{r} - 2 \right)^{1/2},
$$

where

$$
s = \frac{1}{n} \sum_{i=1}^{n} t_i \quad \text{and} \quad r = \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{t_i} \right)^{-1}.
$$
In order to find \( \hat{\beta} \), it is necessary to solve a nonlinear equation in \( \beta \), that is, \( \hat{\beta} \) is obtained as the positive root of
\[
\beta^2 - \beta [2r + \mathcal{K}(\beta)] + r[s + \mathcal{K}(\beta)] = 0,
\]
(5)
where
\[
\mathcal{K}(\delta) = \left[ \frac{1}{n} \sum_{i=1}^{n} (\delta + \tau_i)^{-1} \right]^{-1} \quad \text{for} \quad \delta > 0.
\]

Birnbaum and Saunders (1969b) proposed two iterative schemes to find \( \hat{\beta} \) from (5), which they called ‘method I’ and ‘method II’. They noted that the former works well when \( x < 0.5 \), but not when \( x > 2 \); the latter also does not work well for some values of \( x \). In what follows, we shall find the MLEs of \( x \) and \( \beta \) by maximizing the log-likelihood function in (3) using the BFGS quasi-Newton nonlinear optimization method with analytical first derivatives, which is generally regarded as the most reliable nonlinear optimization algorithm (Mittelhammer et al., 2000, p. 199).

Let \( \hat{\theta} = (\hat{x}, \hat{\beta})^T \) be the parameter vector and let \( \hat{\theta} = (\tilde{x}, \tilde{\beta})^T \) denote its MLE. From the asymptotic normality of the MLE, it follows that
\[
\hat{\theta} \sim \mathcal{N}_2(\theta, K(\theta)^{-1}),
\]
when \( n \) is large, \( \sim \) denoting approximately distributed. Here, \( K(\theta) \) is Fisher’s information matrix, \( K(\theta)^{-1} \) being its inverse, where
\[
K(\theta) = \begin{pmatrix}
\frac{2n}{\tilde{x}^2} & 0 \\
0 & \frac{n[\pi(\pi^{-1/2}h(x) + 1)]}{\tilde{x}^2 \beta^2}
\end{pmatrix}
\]
and
\[
K(\theta)^{-1} = \begin{pmatrix}
\frac{\tilde{x}^2}{2n} & 0 \\
0 & \frac{\tilde{x}^2 \beta^2}{n[\pi(\pi^{-1/2}h(x) + 1)]}
\end{pmatrix}
\]
with \( h(x) = \pi \sqrt{(\pi/2)} - \pi e^{2/\beta^2} [1 - \Phi(2/x)] \). Note that the expression we give for Fisher’s information matrix only involves numerical integration through the evaluation of the standard normal distribution function \( \Phi(.) \), unlike the expression for \( K(\theta) \) given by Ng et al. (2003, p. 286), which involves an integral that requires numerical solution. We then have that
\[
\hat{\theta} = \begin{pmatrix}
\tilde{x} \\
\tilde{\beta}
\end{pmatrix}
\sim \mathcal{N}_2 \left[ \begin{pmatrix}
x \\
\beta
\end{pmatrix}, \begin{pmatrix}
\frac{\tilde{x}^2}{2n} & 0 \\
0 & \frac{\tilde{x}^2 \beta^2}{n[\pi(\pi^{-1/2}h(x) + 1)]}
\end{pmatrix} \right].
\]
Note from (6) that \( x \) and \( \beta \) are orthogonal, i.e., \( \tilde{x} \) and \( \tilde{\beta} \) are asymptotically independent. Therefore, from (6), one can construct asymptotic confidence intervals (ACIs) for \( x \) and \( \beta \), which are given by (Ng et al., 2003) \( \tilde{x} \pm z_{1-\gamma/2}(K(\tilde{\theta})^{xx})^{1/2} \) and \( \tilde{\beta} \pm z_{1-\gamma/2}(K(\tilde{\theta})^{\beta \beta})^{1/2} \), where \( K(\tilde{\theta})^{xx} \) and \( K(\tilde{\theta})^{\beta \beta} \) are the (1, 1) and (2, 2) elements of the inverse of Fisher’s information matrix evaluated at \( \tilde{\theta} \). These are 100(1 - \( \gamma \))% level confidence intervals, \( 0 < \gamma < 1/2 \). The quantity \( z_{\gamma/2} \), \( 0 < \gamma < 1 \), is such that \( \Phi(Z) = \gamma \), where \( Z \) is standard normally distributed. The asymptotic variances of \( \tilde{x} \) and \( \tilde{\beta} \) are \( K(\tilde{\theta})^{xx} \) and \( K(\tilde{\theta})^{\beta \beta} \), respectively, \( K(\tilde{\theta})^{xx} \) denoting the (1,1) element of \( K(\tilde{\theta})^{-1} \) evaluated at \( \tilde{\theta} \) and \( K(\tilde{\theta})^{\beta \beta} \), the (2,2) element of \( K(\tilde{\theta})^{-1} \) evaluated at \( \tilde{\theta} \). It should be noted that these intervals can display poor coverage in small samples and that they can include values that do not belong to the parameter space with positive probability; see Efron and Tibshirani (1993, Chapter 12).

Ng et al. (2003) considered the estimation of \( x \) and \( \beta \) by maximum likelihood. They proposed modified MLEs for these parameters by visual inspection of the estimated biases from a Monte Carlo experiment. Their modified estimators are
\[
\tilde{x} = \left( \frac{n}{n-1} \right) \hat{x} \quad \text{and} \quad \tilde{\beta} = \left( 1 + \frac{\hat{x}^2}{4n} \right)^{-1} \hat{\beta}.
\]
We note that it is possible to construct asymptotic confidence intervals for the parameters that index the Birnbaum–Saunders distribution using \( \hat{\theta} = (\tilde{x}, \tilde{\beta})^T \). From the consistency and asymptotic normality of the MLEs, we have that
\( \hat{\theta} \sim \mathcal{N}_2(\theta, K(\theta)^{-1}) \). Thus, \( \tilde{x} \pm z_{1-\gamma/2}(K(\tilde{\theta})^{xx})^{1/2} \) and \( \tilde{\beta} \pm z_{1-\gamma/2}(K(\tilde{\theta})^{\beta \beta})^{1/2} \) are asymptotic confidence intervals.
of level 100(1 − γ)%<\textsubscript{c}, 0 < γ < 1/2, for x and β, respectively. We shall call such intervals ACINKB. The asymptotic variances of \( \hat{x} \) and \( \hat{\beta} \) are \( K(\hat{\theta})^{xx} \) and \( K(\hat{\theta})^{\beta\beta} \), respectively, where \( K(\hat{\theta})^{xx} \) is the (1,1) element of \( K(\hat{\theta})^{-1} \) evaluated at \( \hat{\theta} \) and \( K(\hat{\theta})^{\beta\beta} \), the (2,2) element of \( K(\hat{\theta})^{-1} \) evaluated at \( \hat{\theta} \).

Ng et al. (2003) have also proposed jackknife estimators for \( \hat{x} \) and \( \hat{\beta} \). The underlying idea is to remove observation \( t_j \) from the random sample \( t = (t_1, t_2, \ldots, t_n)^T \), and to estimate the parameters based on the remaining \( n - 1 \) observations; this is done for \( j = 1, \ldots, n \).

Suppose observation \( j \) is deleted from the sample. Ng et al. (2003, p. 288) give the following expressions for \( r, s \in \mathcal{H}(\hat{\delta}) \):

\[
\begin{align*}
s(j) &= \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} t_i = \frac{ns - t_j}{n - 1}, \\
r(j) &= \left[ \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} t_i^{-1} \right]^{-1} = \frac{nr - t_j^{-1}}{n - 1}, \\
\mathcal{H}(j)(\hat{\delta}) &= \left[ \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} (\hat{\delta} + t_i)^{-1} \right]^{-1} = \frac{n\mathcal{H}(\hat{\delta}) - (\hat{\delta} + t_j)^{-1}}{n - 1}.
\end{align*}
\]

However, the expressions for \( r(j) \) and \( \mathcal{H}(j)(\hat{\delta}) \) given above are in error. The correct expressions are

\[
\begin{align*}
r(j) &= \left[ \frac{nr^{-1} - t_j^{-1}}{n - 1} \right]^{-1} \quad \text{and} \quad \mathcal{H}(j)(\hat{\delta}) = \left[ \frac{n\mathcal{H}(\hat{\delta}) - (\hat{\delta} + t_j)^{-1}}{n - 1} \right]^{-1}.
\end{align*}
\]

We obtain \( \hat{\beta}_j(j) \) as the positive root of

\[
\beta^2 - \beta[2r(j) + \mathcal{H}(j)(\hat{\beta})] + r(j)[s(j) + \mathcal{H}(j)(\hat{\beta})] = 0.
\]

Also,

\[
\hat{x}_j(j) = \left( \frac{s(j)}{\hat{\beta}_j(j)} + \frac{\hat{\beta}_j(j)}{r(j)} - 2 \right)^{1/2}.
\]

Let

\[
\hat{x}_j = \frac{1}{n} \sum_{j=1}^{n} \hat{x}_j(j) \quad \text{and} \quad \hat{\beta}_j = \frac{1}{n} \sum_{j=1}^{n} \hat{\beta}_j(j).
\]

The bias corrected jackknife estimators are given by (see, e.g., Efron, 1982)

\[
\bar{x} = n\hat{x} - (n - 1)\hat{x}_j \quad \text{and} \quad \bar{\beta} = n\hat{\beta} - (n - 1)\hat{\beta}_j.
\]

3. Bias-corrected MLEs and interval estimation

In what follows, we shall derive closed-form expressions for the second order biases of the MLEs of the parameters that index the Birnbaum–Saunders distribution. To that end, we shall use the general expression given by Cox and Snell

\footnote{The jackknife estimate of bias was first proposed by Quenouille (1949).}
At the outset, we shall introduce some notation. The derivatives of the log-likelihood function are \( U_x = \partial \ell / \partial x, \) \( U_\beta = \partial \ell / \partial \beta, \) \( U_{x\beta} = \partial^2 \ell / \partial x \partial \beta, \) \( U_{x\beta\beta} = \partial^3 \ell / \partial x^2 \partial \beta, \) and so on. The notation used for the moments of such derivatives is that of Lawley (1968): \( \kappa_{xx} = E(U_{xx}), \) \( \kappa_{x\beta} = E(U_{x\beta}), \) \( \kappa_{\beta\beta} = E(U_{\beta\beta}), \) \( \kappa_{x\beta\beta} = E(U_{x\beta\beta}), \) etc., where all \( \kappa \)'s are moments under the sample and are typically of order \( O(n) \). Finally, their derivatives are denoted as \( \kappa_{x\beta} = \partial \kappa_{x\beta}/\partial x, \) \( \kappa_{\beta\beta} = \partial \kappa_{\beta\beta}/\partial \beta, \) etc. Additionally, \( \kappa_{x\beta}^2 \) denotes the \((1,1)\) element of the inverse of Fisher’s information matrix, \( K(\theta)^{-1} \); likewise, \( \kappa_{\beta\beta}^2 \) denotes the \((1,2)\) element of this matrix and \( \kappa_{x\beta}^2 \), the \((2,2)\) element.

Cox and Snell (1968) obtained a general formula for the bias, to order \( O(n^{-1}) \), of the MLE of the parameter vector \( \theta = (\theta_1, \ldots, \theta_p)^T \). The expression is

\[
B(\tilde{\theta}) = \sum_{r,t,u} \kappa_r^{s,r} \kappa_t^{s,t,u} \left\{ \frac{1}{2} K_{r tu} + K_{r t u} \right\},
\]

where \( r, s, t, u \) index the parameter space. We refer to \( B(\tilde{\theta}) \) as the second order bias of the MLE \( \tilde{\theta}_s, s = 1, \ldots, p \).

Using (7) and the orthogonality between the parameters that index the Birnbaum–Saunders distribution, it is possible to show that the biases, to order \( O(n^{-1}) \), of the MLEs \( \tilde{\alpha} \) and \( \tilde{\beta} \) are, respectively,

\[
B(\tilde{\alpha}) = \kappa_{\alpha\alpha}^2 \kappa_{\alpha\alpha}^{-1} \left( \kappa_{\alpha\alpha} - \frac{1}{2} \kappa_{\alpha\alpha}^2 \right) + \kappa_{\alpha\alpha}^2 \kappa_{\alpha\beta} \left( \kappa_{\beta\beta} - \frac{1}{2} \kappa_{\beta\beta}^2 \right)
\]

and

\[
B(\tilde{\beta}) = \kappa_{\beta\beta}^2 \kappa_{\beta\beta}^{-1} \left( \kappa_{\beta\beta} - \frac{1}{2} \kappa_{\beta\beta}^2 \right) + \kappa_{\beta\beta}^2 \kappa_{\beta\alpha} \left( \kappa_{\alpha\alpha} - \frac{1}{2} \kappa_{\alpha\alpha}^2 \right).
\]

The quantities needed to write \( B(\tilde{\alpha}) \) and \( B(\tilde{\beta}) \) in closed form are given in the Appendix. After extensive algebra, we obtained the following expressions for the second order biases of \( \tilde{\alpha} \) and \( \tilde{\beta} \):

\[
B(\tilde{\alpha}) = -\frac{\alpha}{4n} \left( 1 + \frac{2 + x^2}{\alpha(2\pi)^{-1/2} h(x) + 1} \right)
\]

and

\[
B(\tilde{\beta}) = \frac{\beta x^2}{2n [\alpha (2\pi)^{-1/2} h(x) + 1]},
\]

respectively.

Using (8) and (9), we define the bias-adjusted MLEs \( \tilde{\alpha} \) and \( \tilde{\beta} \) as

\[
\tilde{\alpha} = \tilde{\alpha} - \tilde{B}(\tilde{\alpha}) \quad \text{and} \quad \tilde{\beta} = \tilde{\beta} - \tilde{B}(\tilde{\beta}),
\]

where \( \tilde{B}(\tilde{\alpha}) \) and \( \tilde{B}(\tilde{\beta}) \) are the MLEs of \( B(\tilde{\alpha}) \) and \( B(\tilde{\beta}) \), respectively, i.e., the unknown parameters in the second order biases are replaced by their corresponding MLEs. We say that \( \tilde{\alpha} \) and \( \tilde{\beta} \) are bias-adjusted MLEs to order \( O(n^{-1}) \).

Their biases are of order \( O(n^{-2}) \), since \( E(\tilde{\alpha}) = z + O(n^{-1}) \) and \( E(\tilde{\beta}) = \beta + O(n^{-2}) \). It is expected that \( \tilde{\alpha} \) and \( \tilde{\beta} \) have superior finite-sample behavior relative to \( \tilde{\alpha} \) and \( \tilde{\beta} \), respectively, whose biases are of order \( O(n^{-1}) \). It can be shown that \( \tilde{\theta} \sim A^2(\theta, K(\theta)^{-1}) \), where \( \theta = (\tilde{\alpha}, \tilde{\beta})^T \). Thus, \( \tilde{\alpha} \pm z_{1-\gamma/2} K(\tilde{\alpha})^{1/2} \) and \( \tilde{\beta} \pm z_{1-\gamma/2} K(\tilde{\beta})^{1/2} \) are asymptotic confidence intervals of level 100(1-\gamma)%, \( 0 \leq \gamma < 1/2 \), for \( \alpha \) and \( \beta \), respectively. We shall call such intervals ACICS.

The asymptotic variances of \( \tilde{\alpha} \) and \( \tilde{\beta} \) are \( K(\tilde{\alpha}) \) and \( K(\tilde{\beta}) \), respectively, where \( K(\tilde{\theta}) \) is the \((1,1)\) element of \( K(\tilde{\theta})^{-1} \) evaluated at \( \tilde{\theta} \) and \( K(\tilde{\alpha})^{-1} \) evaluated at \( \tilde{\alpha} \).

An alternative strategy for bias-correcting MLEs is through Efron’s (1979) bootstrap resampling. Let \( y = (y_1, y_2, \ldots, y_n)^T \) denote a random sample from the random variable \( Y \) with distribution function \( F \). Let \( \theta = t(F) \) be a function of \( F \) known as parameter and let \( \bar{\theta} = s(y) \) be an estimator of \( \theta \). In the parametric bootstrap, we obtain, from the sample \( y \), a large number of pseudo-samples \( y^* = (y_1^*, y_2^*, \ldots, y_n^*)^T \), compute the corresponding bootstrap replicates of \( \tilde{\theta}, \tilde{\theta}^* = s(y^*) \), and, using the empirical distribution of \( \tilde{\theta}^* \), estimate the distribution function of \( \tilde{\theta} \). We assume that \( F \) belongs to a parametric family which is known and has finite dimension, \( F_\xi \). We can obtain a parametric estimate for \( F \) by using a consistent estimator for \( \hat{\xi} : F_\hat{\xi} \). The bias of the estimator \( \bar{\theta} = s(y) \) can be written as \( B_F(\bar{\theta}, \theta) = E_F[s(y)] - t(F), \)
the subscript $F$ denoting that expectation is taken with respect to $F$. The bootstrap bias estimate is obtained by replacing $F$, from which the original sample was obtained, by $F_\hat{\gamma}$. Hence, the bias can be written as

$$B_{F_\hat{\gamma}}(\hat{\gamma}, \theta) = E_{F_\hat{\gamma}}[s(y)] - t(F_\hat{\gamma}).$$

Suppose now that $B$ bootstrap samples $(y^{s_1}, y^{s_2}, \ldots, y^{s_B})$ are generated independently from the original sample $y$ and the corresponding bootstrap replicates $(\hat{\gamma}^{s_1}, \hat{\gamma}^{s_2}, \ldots, \hat{\gamma}^{s_B})$ are computed, where $\hat{\gamma}^{s_b} = s(y^{s_b}), b = 1, 2, \ldots, B$. It is then possible to approximate the expected value $E_{F_\hat{\gamma}}[s(y)]$ as $(1/B) \sum_{b=1}^{B} \hat{\gamma}^{s_b}$. Hence, the bootstrap bias estimate, obtained from the $B$ replicates of $\hat{\gamma}$, is $B_{F_\hat{\gamma}}(\hat{\gamma}, \theta) = \hat{\gamma} - s(y)$. We can now define a second order bias-corrected estimator:

$$\tilde{\gamma} = s(y) - B_{F_\hat{\gamma}}(\hat{\gamma}, \theta) = 2\hat{\gamma} - \hat{\gamma}^{s_b}.$$

Following MacKinnon and Smith (1998), the estimator $\tilde{\gamma}$ shall be called CBC (constant bias-corrected).

After obtaining the empirical distribution $\hat{F}$ of $\tilde{\gamma}$ by bootstrap, one can construct percentile confidence intervals (CIP), with approximate coverage $1 - \gamma$, $0 < \gamma < 1/2$, by computing the percentiles $\gamma/2$ and $1 - \gamma/2$ of $\hat{F}$. The interval is

$$(\hat{F}^{-1}(\gamma/2), \hat{F}^{-1}(1 - \gamma/2)).$$

After arranging the $B$ bootstrap replicates of $\tilde{\gamma}$, $\hat{\gamma}^{s_b} = s(y^{s_b})$, in increasing order, we obtain the lower and upper limits of the percentile interval as the integer parts of $B \times (\gamma/2)$ and $B \times (1 - \gamma/2)$, respectively. Note that, unlike the usual asymptotic confidence interval, the percentile interval can be asymmetric and will not include values that do not belong to the parameter space.

In what follows, in addition to computing the percentile confidence interval for $\theta = (\alpha, \beta)^T$, the parameter vector that index the Birnbaum–Saunders distribution, based on the empirical distribution of $\tilde{\gamma} = (\hat{\alpha}, \hat{\beta})^T$, we shall also construct confidence intervals for $\theta = (\alpha, \beta)^T$ using the empirical distributions of the bias-corrected estimator derived in the previous section, $\tilde{\gamma} = (\hat{\alpha}, \hat{\beta})^T$, and the adjusted estimator proposed by Ng et al. (2003), $\tilde{\gamma} = (\hat{\alpha}, \hat{\beta})^T$; we shall refer to such intervals as CIPCS and CIPNKB, respectively.

Efron (1981, 1987) introduced a bootstrap method for interval estimation that follows from letting the standard error vary with $\tilde{\gamma}$, the bias-corrected and accelerated confidence interval (BCa). The BCa interval for $\tilde{\gamma}$ with coverage $1 - \gamma$, $0 < \gamma < 1/2$, is (DiCiccio and Tibshirani, 1987)

$$(\hat{F}^{-1}(\Phi(z_{\gamma/2}^*)), \hat{F}^{-1}(\Phi(z_{1-\gamma/2}^*))),$$

where

$$z_{\gamma/2}^* = v_0 + \frac{v_0 + z_{\gamma/2}}{1 - a(v_0 + z_{\gamma/2})}, \quad z_{1-\gamma/2}^* = v_0 + \frac{v_0 + z_{1-\gamma/2}}{1 - a(v_0 + z_{1-\gamma/2})},$$

and $a$ is a constant that measures the rate of change of the standard error of $\tilde{\gamma}$ with respect to the true parameter value; this constant is known as the acceleration constant and can be estimated as

$$\hat{a} = \frac{1}{6}\text{Skew}(\hat{e}_\theta(\tilde{\gamma}))|_{\tilde{\gamma} = \tilde{\gamma}},$$

where $\text{Skew}(\cdot)$ denotes the skewness coefficient and $\hat{e}_\theta(\tilde{\gamma})$ is the derivative of the log-likelihood function evaluated at $\tilde{\gamma}$. The constant $v_0$ is a measure of the bias of the median of the bootstrap distribution with respect to $\tilde{\gamma}$ and can be estimated as

$$\hat{v}_0 = \Phi^{-1}\left(\frac{\#(\hat{\gamma}^{s_b} < \tilde{\gamma})}{B}\right).$$
The lower and upper limits of the BCa interval are
\[ \delta_1 = \Phi \left( \overline{t}_0 + \frac{\overline{t}_0 + z_{\gamma/2}}{1 - \overline{a}(\overline{t}_0 + z_{\gamma/2})} \right) \quad \text{and} \quad \delta_2 = \Phi \left( \overline{t}_0 + \frac{\overline{t}_0 + z_{1-\gamma/2}}{1 - \overline{a}(\overline{t}_0 + z_{1-\gamma/2})} \right), \]
respectively.

The main disadvantage of the BCa method is the large number of bootstrap replications that are needed; typically between 1000 and 2000 replications are used, thus increasing the computational burden relative to other bootstrap-based approaches.

A different bootstrap confidence interval for the parameter of interest \( \theta \), known as percentile \( t \) confidence interval, is obtained from the estimation of the distribution of the statistic \( \mathcal{F} \) based on the observed sample \( y = (y_1, \ldots, y_n)^T \), where \( \mathcal{F} \) is given by
\[ \mathcal{F} = \frac{\hat{\theta} - \theta}{\hat{\text{se}}(\hat{\theta})}, \]
\( \hat{\text{se}}(\hat{\theta}) \) being the standard error of \( \hat{\theta} \). Under normality, \( \mathcal{F} \) is known as the \( t \) statistic. According to Efron and Tibshirani (1993, Section 12.5), one should proceed as follows. First, generate \( B \) bootstrap samples \( (y_1^*, \ldots, y_B^*) \) from the original sample \( y \). Then, for each pseudo-sample, compute
\[ \mathcal{F}^{*B} = \frac{\hat{\theta}^{*B} - \hat{\theta}}{\hat{\text{se}}^{*B}}, \]
\( b = 1, 2, \ldots, B \), where \( \hat{\theta} = s(y) \) is the estimate of \( \theta \) obtained from the original sample \( y \), \( \hat{\theta}^{*B} = s(y^{*B}) \) is the estimate of \( \theta \) obtained from the bootstrap sample \( y^{*B} \) and \( \hat{\text{se}}^{*B} \) is the standard error of \( \hat{\theta}^{*B} \) computed from the bootstrap sample \( y^{*B} \). Finally, the \( \gamma/2 \) and \( 1 - \gamma/2 \) percentiles of \( \mathcal{F}^{*B} \) are estimated by \( \mathcal{T}^{(\gamma/2)} \) and \( \mathcal{T}^{(1-\gamma/2)} \), respectively, such that
\[ \frac{\# \{ \mathcal{F}^{*B} \leq \mathcal{T}^{(\gamma/2)} \}}{B} = \frac{\gamma}{2} \quad \text{and} \quad \frac{\# \{ \mathcal{F}^{*B} \leq \mathcal{T}^{(1-\gamma/2)} \}}{B} = 1 - \frac{\gamma}{2}, \]
respectively. Therefore, the bootstrap-\( t \) confidence interval is
\[ (\hat{\theta} - \mathcal{T}^{(1-\gamma/2)} \hat{\text{se}}, \hat{\theta} - \mathcal{T}^{(\gamma/2)} \hat{\text{se}}), \]
where \( \hat{\text{se}} \equiv \hat{\text{se}}(\hat{\theta}) \). The quantities \( \mathcal{T}^{(\gamma/2)} \) and \( \mathcal{T}^{(1-\gamma/2)} \) are obtained as follows: we arrange the \( B \) bootstrap replicates \( \mathcal{F}^{*B} \) in increasing order, and \( \mathcal{T}^{(\gamma/2)} \) and \( \mathcal{T}^{(1-\gamma/2)} \) are, respectively, the replicates corresponding to the integer parts of \( B \times (\gamma/2) \) and \( B \times (1 - \gamma/2) \). Efron and Tibshirani (1993, p. 178) note that “the bootstrap-\( t \) intervals have good theoretical coverage probabilities, but tend to be erratic in actual practice. The percentile intervals are less erratic, but have less satisfactory coverage properties.” In what follows, we shall refer to percentile \( t \) bootstrap confidence intervals for the parameters \( (\alpha, \beta) \) that index the Birnbaum–Saunders distribution as CIBt. For a good discussion of bootstrap-based confidence intervals, the reader is referred to Hall (1988).

4. Hypothesis tests

Consider the hypotheses
\[ \mathcal{H}_0 : \alpha = \alpha^{(0)} \quad \text{and} \quad \mathcal{H}_1 : \alpha \neq \alpha^{(0)}. \] (10)
The interest lies in testing the null hypothesis \( \mathcal{H}_0 \) against the alternative hypothesis \( \mathcal{H}_1 \).

4.1. Likelihood ratio test

Let \( t = (t_1, t_2, \ldots, t_n)^T \) be a random sample of size \( n \) from the two-parameter Birnbaum–Saunders distribution indexed by \( \theta = (\alpha, \beta)^T \), each \( t_i, i = 1, \ldots, n \), having density (2); the associated log-likelihood function is given in (3). The likelihood ratio statistic for the test of the null hypothesis in (10) is
\[ LR = 2(\ell(\hat{\theta}; t) - \ell(\hat{\theta}' ; t)), \] (11)
where \( \hat{\theta}^* = (\hat{\alpha}(0), \hat{\beta}^*)^T \) and \( \hat{\theta} = (\hat{\alpha}, \hat{\beta})^T \) are the restricted and unrestricted maximum likelihood estimators of \( \theta = (\alpha, \beta)^T \), respectively. Under the null hypothesis,
\[
LR \xrightarrow{D} \chi^2_1,
\]
(12)
where \( \xrightarrow{D} \) denotes convergence in distribution. A proof of (12) can be found in Rao (1973); see also Buse (1982) for a geometric interpretation of the likelihood ratio test.

4.2. Bootstrap likelihood ratio test

One can use bootstrap resampling to estimate the null distribution of the statistic \( LR \) directly from the observed sample \( t = (t_1, \ldots, t_n)^T \). To that end, one generates, under \( \mathcal{H}_0 \) (i.e., imposing the restrictions stated in the null hypothesis), \( B \) bootstrap samples \( (t^1, \ldots, t^B) \) from the original sample \( t \), and, for each pseudo-sample, one computes
\[
LR^{sb} = 2\ell(\hat{\theta}^{sb}; t^{sb}) - \ell(\hat{\theta}^{*sb}; t^{sb}),
\]
(13)
where \( \hat{\theta}^{*sb} = (\hat{\alpha}(0), \hat{\beta}^{*sb})^T \) and \( \hat{\theta}^{sb} = (\hat{\alpha}^{sb}, \hat{\beta}^{sb})^T \) are the maximum likelihood estimators of \( \theta = (\alpha, \beta)^T \) obtained from the maximization of \( \ell(\theta; t^{sb}) \) under \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \), respectively. The \( 1 - \gamma \) percentile of \( LR^{sb} \) is estimated by \( q_{1-\gamma} \), such that
\[
\frac{\#\{LR^{sb} \leq q_{1-\gamma}\}}{B} = 1 - \gamma.
\]
When performing the bootstrap likelihood ratio test, one rejects the null hypothesis if
\[
LR > q_{1-\gamma}.
\]
The quantity \( q_{1-\gamma} \) is obtained as follows. We order the \( B \) bootstrap replicates \( LR^{sb} \) in increasing order, and the estimated percentile \( q_{1-\gamma} \) is taken to be the replicate corresponding to the integer part of \( B \times (1 - \gamma) \). For a good discussion of bootstrap tests, see Efron and Tibshirani (1993, Chapter 16).

4.3. Bartlett correction

Large sample tests are commonly used in statistics and econometrics, since exact tests are not always available. Such tests are usually based on first order asymptotics, i.e., they employ critical values obtained from the limiting distribution of the test statistic under the null hypothesis. A natural shortcoming is that the asymptotic approximation used in the test may not be reliable when the sample size is not large. The most commonly used large sample tests are the likelihood ratio, score and Wald tests, whose test statistics converge in distribution, under the null hypothesis, to \( \chi^2_q \), where \( q \) is the number of restrictions under test.

Bartlett (1937) proposed an approach for improving the quality of the large sample approximation used in the likelihood ratio test. He suggested computing the expected value of the statistic under \( \mathcal{H}_0 \) and up to order \( n^{-1} \), and then using it to define a modified test statistic. A similar correction for Rao’s score test statistic was obtained by Cordeiro and Ferrari (1991), and a correction to Wald tests of nonlinear restrictions was derived by Ferrari and Cribari-Neto (1993). For a review of Bartlett corrections in statistics and econometrics, see Cribari-Neto and Cordeiro (1996).

Under regularity conditions, Lawley (1956) has shown that \( E(LR) = 1 + b(\theta) + O(n^{-2}) \), where \( b(\theta) = e_2 - e_1 \), with
\[
e_1 = \sum_{\beta} (\lambda_{rstuv} - \lambda_{rstuvw}) \quad \text{and} \quad e_2 = \sum_{\alpha, \beta} (\lambda_{rstu} - \lambda_{rstuvw}),
\]
where the summations \( \sum_{\beta} \) and \( \sum_{\alpha, \beta} \) run through the parameters \( \beta \) and \( (\alpha, \beta) \), respectively. Here, \( \lambda_{rstu} = K^{r,s,t,u} K^{s,t,u} \left[ K_{rstu} / 4 - K_{rst}^{(u)} + K_{rst}^{(u)} \right] \) and \( \lambda_{rstuvw} = K^{r,s,t,u,w} \left[ K_{rstuw} / 6 - K_{rst}^{(u)} + K_{rst}^{(u)} \right] + K_{rstuw} / 4 - K_{rst}^{(u)} + K_{rst}^{(u)} + \left[ K_{rst}^{(u)} + K_{rst}^{(u)} \right] \). Using the orthogonality between \( \alpha \) and \( \beta \), we obtain \( e_1 = \lambda_{\beta \beta \beta \beta} - \lambda_{\beta \beta \beta \beta} \) and \( e_2 = \lambda_{\alpha \alpha \alpha \alpha} + \lambda_{\alpha \alpha \alpha \alpha} + \lambda_{\beta \beta \beta \beta} - \lambda_{\alpha \alpha \alpha \alpha} + \lambda_{\beta \beta \beta \beta} - \lambda_{\alpha \alpha \alpha \alpha} + \lambda_{\beta \beta \beta \beta} \). Since \( b(\theta) = e_2 - e_1 \), it follows that
\[
b(\theta) = \lambda_{\beta \beta \beta \beta} + \lambda_{\beta \beta \beta \beta} - \lambda_{\beta \beta \beta \beta} - \lambda_{\beta \beta \beta \beta} - \lambda_{\beta \beta \beta \beta} - \lambda_{\beta \beta \beta \beta}.\]

We note that there is a typographical error in the formula presented by Lawley (1956). This has been noted by several authors; see, e.g., Stafford and Andrews (1993, p. 729). The correct formula is given by Cribari-Neto and Cordeiro (1996, p. 367). The quantities needed for obtaining \( b(\theta) \) in closed form are given in the Appendix. After obtaining the \( \lambda \)'s and after some algebra, we can write

\[
b(\theta) = \frac{1}{n} \left\{ \frac{23}{12} + \frac{1 - 4x^2 - 5x^4/4 + 8(1 + x^2)h(x)/\sqrt{2\pi}}{2[x(2\pi)^{-1/2}h(x) + 1]^2} \right\}.
\]  

(14)

Therefore, the Bartlett-corrected likelihood ratio test statistic for the test of the null hypothesis in (10) is given by

\[
LR_c = \frac{LR}{1 + b(\theta)}.
\]

(15)

It is possible to show that, under the null hypothesis, \( E(LR_c) = 1 + O(n^{-2}) \). Also, whereas \( P(LR \leq x) = P(\chi_1^2 \leq x) + O(n^{-1}) \), the correction yields \( P(LR_c \leq x) = P(\chi_1^2 \leq x) + O(n^{-2}) \), a clear improvement.

Note that the correction factor \( b(\theta) \) in (14) does not depend on the nuisance parameter \( \beta \), depending only on the value of \( x \) specified in the null hypothesis. Fig. 2 shows the plot of \( b(\theta) \) against \( x \) for \( n = 1 \). Note that \( b(\theta) \) is greater than 3 for values of \( x \) close to zero, decreasing when \( x \) gets larger. Note also that \( b(\theta) \) is negative when \( x > 2 \).

Since the Bartlett correction factor does not involve the nuisance parameter \( \beta \), it is possible to obtain \( LR_c \) given in (15) explicitly for different null hypotheses. Suppose that, based on a given sample size \( n \), we wish to test the following hypotheses: (i) \( H_0: x = 0.1 \) versus \( H_1: x \neq 0.1 \); (ii) \( H_0: x = 0.25 \) versus \( H_1: x \neq 0.25 \); (iii) \( H_0: x = 0.5 \) versus \( H_1: x \neq 0.5 \); (iv) \( H_0: x = 0.75 \) versus \( H_1: x \neq 0.75 \); (v) \( H_0: x = 1.0 \) versus \( H_1: x \neq 1.0 \). From (15), we obtain the corresponding corrected test statistics:

\[
LR_c = \frac{LR}{1 + 3.4017/n}, \quad LR_c = \frac{LR}{1 + 3.3227/n}, \quad LR_c = \frac{LR}{1 + 3.0414/n}, \quad LR_c = \frac{LR}{1 + 2.5924/n} \quad \text{and} \quad LR_c = \frac{LR}{1 + 2.0307/n}.
\]
5. Numerical evaluation

Birnbaum–Saunders random number generation was performed using the transformation

\[ X = \frac{1}{2} \left( \sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}} \right) , \]

where, from (1), \( X \sim \mathcal{N}(0, \frac{1}{4} x^2) \). Thus, the Birnbaum–Saunders random variable \( T \) can be written as

\[ T = \beta \left[ 1 + 2X^2 + 2X \left( 1 + X^2 \right)^{1/2} \right] . \quad (16) \]

We then generated realizations of \( T \) through normal random numbers, i.e., through realizations of \( X \), by exploring the above relationship. The Monte Carlo simulation experiments were performed using the \( \text{Ox} \) matrix programming language (Cribari-Neto and Zarkos, 2003; Doornik, 2001). The number of Monte Carlo replications was \( R = 5000 \) and the number of bootstrap replications was \( B = 600 \).

5.1. Numerical results: estimation

We shall now evaluate the finite-sample behavior of the MLEs of \( x \) and \( \beta \) and their bias-adjusted counterparts. The sample sizes considered were \( n = 10, 20, 40, 60 \), and the values of the shape parameter were \( x = 0.1, 0.25, 0.5, 0.75, 1.0 \). Without loss of generality we set the value of \( \beta \) at 1.0, i.e., \( \beta = 1.0 \) in all of the experiments.

After obtaining the MLEs, \( \hat{x} \) and \( \tilde{\beta} \), we computed the modified estimates \( \tilde{x} \) and \( \tilde{\beta} \) (proposed in this paper) and \( \hat{x} \), \( \bar{\beta} \), \( \tilde{x} \) and \( \tilde{\beta} \) (Ng et al., 2003). We then performed \( B \) replications of the parametric bootstrap in order to compute \( \tilde{x} \) and \( \tilde{\beta} \), the bootstrap bias-corrected estimates. The bootstrap scheme was also used to produce bootstrap-based confidence intervals. The following interval estimates were computed: asymptotic, percentile, BCa and bootstrap-\( t \) (percentile \( t \)).

The evaluation of point estimation was performed based on the following quantities for each sample size: (i) relative bias (the relative bias of an estimator \( \hat{\theta} \) of a parameter \( \theta \) is defined as \( \{ \mathbb{E}(\hat{\theta}) - \theta \} / \theta \), its estimate being obtained by estimating \( \mathbb{E}(\hat{\theta}) \) by Monte Carlo), and (ii) root mean squared error, i.e., \( \sqrt{\text{MSE}} \), where MSE is the mean squared error estimated from the \( R \) Monte Carlo replications. For interval estimation evaluation we present graphics that display the average values of the lower and upper limits of all intervals together with the empirical coverages. We also inform with what frequency the true parameter value was smaller (larger) than the lower (upper) limit.

Table 1 presents the relative biases of the different point estimators. It is noteworthy that the corrected estimators of the shape parameter display relative biases that are smaller, in absolute values, than that of the corresponding MLE for all sample sizes considered in the experiment. Amongst the corrected estimators, the one with poorest performance was \( \hat{x} \), the estimator proposed by Ng et al. (2003), under all settings. For instance, when \( n = 10 \) and \( x = 0.50 \), the absolute relative bias of \( \hat{x} \) was 0.02287, whereas the corresponding measures for \( \tilde{x} \), \( \bar{x} \) and \( \tilde{x} \) were 0.00788, 0.00600 and 0.00242, respectively. It is noteworthy that the jackknife estimator \( \hat{x} \) outperforms the competition when the sample size is very small. However, when for samples of 40 observations or more, the estimator \( \tilde{x} \) (the analytically corrected estimator) is the least biased.

Note also that the estimators \( \bar{\beta} \), \( \tilde{\beta} \) and \( \bar{\beta} \), were the best performing ones for all sample sizes. For example, when \( n = 20 \) and \( x = 0.25 \), their absolute relative biases were nearly five times smaller than that of \( \tilde{\beta} \), the MLE of \( \beta \).

In Table 2 we present the root mean squared errors of the different point estimators. Note that the figures for the corrected estimators are similar to those obtained for the MLEs.

It is interesting to note that, as the value of shape parameter \( x \) increases, the finite-sample performances of the point estimators of \( \beta \), the scale parameter, deteriorate (see Tables 1 and 2). For instance, when \( n = 10 \), the relative biases of \( \tilde{\beta} \) (MLE) were 0.00046 when \( x = 0.1 \) and 0.04116 when \( x = 1.00 \), which amounts to an increase in relative bias of nearly 90 times.

It is also noteworthy that the bias-correction schemes we propose outperform the ad hoc correction of Ng et al. (2003) by delivering associated adjusted estimators with smaller biases. Figs. 3 and 4 plot the second order biases of \( \tilde{x} \) and \( \bar{x} \), respectively, against \( x \) along with the approximations obtained by Ng et al. (2003). (We set \( n = 5 \) and \( \beta = 1.0 \).)

We notice that their approximations deteriorate when \( x \) increases, especially the approximation to the bias of \( \tilde{\beta} \). As a consequence, the second order bias corrected estimator derived in this paper delivers more accurate point estimation.
### Table 1
Relative biases ($\beta = 1.0$)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x$</th>
<th>$\hat{\beta}$</th>
<th>$\bar{\beta}$</th>
<th>$\tilde{\beta}$</th>
<th>$\check{\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.10</td>
<td>0.07702</td>
<td>0.00768</td>
<td>0.00563</td>
<td>0.02553</td>
</tr>
<tr>
<td>0.25</td>
<td>0.07760</td>
<td>0.00773</td>
<td>0.00572</td>
<td>0.02488</td>
<td>0.00254</td>
</tr>
<tr>
<td>0.50</td>
<td>0.07942</td>
<td>0.00788</td>
<td>0.00600</td>
<td>0.02287</td>
<td>0.00242</td>
</tr>
<tr>
<td>0.75</td>
<td>0.08193</td>
<td>0.00828</td>
<td>0.00650</td>
<td>0.02098</td>
<td>0.00227</td>
</tr>
<tr>
<td>1.00</td>
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<td>0.00884</td>
<td>0.00707</td>
<td>0.01716</td>
<td>0.00222</td>
</tr>
</tbody>
</table>

### Table 2
Root mean squared errors ($\beta = 1.0$)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x$</th>
<th>$\hat{\beta}$</th>
<th>$\bar{\beta}$</th>
<th>$\tilde{\beta}$</th>
<th>$\check{\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.10</td>
<td>0.02333</td>
<td>0.02369</td>
<td>0.02376</td>
<td>0.02460</td>
</tr>
<tr>
<td>0.25</td>
<td>0.05834</td>
<td>0.05929</td>
<td>0.05947</td>
<td>0.06144</td>
<td>0.06007</td>
</tr>
<tr>
<td>0.50</td>
<td>0.11686</td>
<td>0.11900</td>
<td>0.11933</td>
<td>0.12266</td>
<td>0.12070</td>
</tr>
<tr>
<td>0.75</td>
<td>0.17586</td>
<td>0.17932</td>
<td>0.17981</td>
<td>0.18371</td>
<td>0.18233</td>
</tr>
<tr>
<td>1.00</td>
<td>0.23561</td>
<td>0.24025</td>
<td>0.24102</td>
<td>0.24496</td>
<td>0.24520</td>
</tr>
</tbody>
</table>

| 20  | 0.10| 0.01594       | 0.01605        | 0.01608        | 0.01633      |
| 0.25| 0.03985| 0.04016       | 0.04023        | 0.04081        | 0.04025      |
| 0.50| 0.07975| 0.08045       | 0.08059        | 0.08151        | 0.08066      |
| 0.75| 0.11982| 0.12095       | 0.12116        | 0.12214        | 0.12135      |
| 1.00| 0.16017| 0.16167       | 0.16197        | 0.16284        | 0.16236      |

| 40  | 0.10| 0.01116       | 0.01121        | 0.01126        | 0.01131      |
| 0.25| 0.02790| 0.02804       | 0.02816        | 0.02826        | 0.02805      |
| 0.50| 0.05583| 0.05613       | 0.05637        | 0.05649        | 0.05617      |
| 0.75| 0.08381| 0.08431       | 0.08467        | 0.08470        | 0.08438      |
| 1.00| 0.11191| 0.11257       | 0.11305        | 0.11293        | 0.11269      |

| 60  | 0.10| 0.09002       | 0.09095        | 0.09097        | 0.09100      |
| 0.25| 0.02255| 0.02262       | 0.02267        | 0.02273        | 0.02262      |
| 0.50| 0.04510| 0.04526       | 0.04535        | 0.04544        | 0.04526      |
| 0.75| 0.06764| 0.06790       | 0.06805        | 0.06810        | 0.06791      |
| 1.00| 0.09025| 0.09059       | 0.09078        | 0.09077        | 0.09061      |
that the adjusted estimator proposed by Ng et al. (2003). The jackknife estimator proposed by these authors, however, is competitive with our analytically corrected estimator.

Which of the two resampling schemes is to be preferred, the jackknife or the bootstrap? The numerical results show that the former works very well when the sample size is small. Additionally, the jackknife resampling is computationally less costly than the bootstrap resampling when the sample size is not large. However, the bootstrap is a more general resampling scheme, and can also be used for interval estimation and hypothesis testing; see the remaining numerical
results in this section. It should also be noted that the jackknife estimate of bias is, in fact, an approximation to the bootstrap estimate of bias (Efron and Tibshirani, 1993, pp. 147–148).

The MLEs of $\alpha$ and $\beta$ are considerably biased in small samples (see Table 1). When bias is a concern, the corrected estimators proposed in this paper should be used instead of the MLEs of $\alpha$ and $\beta$. The analytically adjusted estimators have the advantage of not requiring data resampling, being available in closed form.

We shall now turn to interval estimation. Our goal is to evaluate the relative merits of the interval estimators discussed in Sections 2 and 3. Altogether, we evaluated 480 confidence intervals with nominal coverages $1 - \gamma = 0.90, 0.95, 0.99$, the sample sizes being $n = 10, 20, 40, 60$ and the parameter values varying as $\alpha = 0.1, 0.25, 0.5, 0.75, 1.0$; again, $\beta = 1.0$. All intervals were constructed so that the expected coverage was $1 - \gamma$, the noncoverage probability being equally divided ($\gamma/2$) into both tails. Figs. 5–8 contain histograms of the MLEs constructed from the 5000 Monte Carlo replications. (For brevity, we only present results for $n = 10$, $\alpha = 0.1, 0.75$ and $\beta = 1.0$.) In the figures, the intervals are represented by straight lines, whose lengths indicate the average lengths of the intervals in the Monte Carlo experiment. The upper (lower) figures indicate the empirical frequency with which the lower (upper) limit was greater (smaller) than the true parameter value.

For the three nominal coverage levels, the four sample sizes and all values of the parameters considered, the intervals ACICS, ACINKB and CIBt (bootstrap-$t$) had the best empirical coverages, followed by the BCa bootstrap interval. However, the average length of the CIBt interval exceeded those of ACICS, ACINKB and BCa, especially when the sample size was small.

Note from Figs. 5 and 7 that the bootstrap interval CIBt for the shape parameter $\alpha$ was the most asymmetric. From Figs. 6 and 8 we note that all confidence intervals for the scale parameter $\beta$ were approximately symmetric and well balanced.

When the value of the shape parameter $\alpha$ is small (less than 0.50), the empirical and limiting distributions of the MLE of $\beta$ are similar, which translates into well balanced and symmetric confidence intervals.

In order to evaluate the sensitiveness of BCa intervals, which typically requires a larger number of bootstrap replications, to the number of pseudo-samples we have performed a set of Monte Carlo simulations using $R = 5000$ (number
of Monte Carlo replications), \( z = 0.25 \), \( \beta = 1.0 \), \( n = 40 \) and \( B = 600, 1200, 2000 \). The results are given in Table 3. ‘Coverage’ indicates the percentage of Monte Carlo replications in which the interval contained the true parameter value (\( z = 0.25 \)). Similarly, ‘% Left’ (‘% Right’) is the percentage of the 5000 intervals in which the true parameter value was smaller (greater) than the lower (upper) limit of the interval. The figures in Table 3 show that the finite-sample performance of the BCa confidence interval improves with \( B \), the number of bootstrap replications; the empirical coverages approach their nominal counterparts and the interval becomes more well balanced. This occurs, nevertheless, at the expense of a higher computational burden. For instance, the simulation that used \( B = 2000 \) entailed a total of ten million replications. BCa intervals are, however, very competitive when the number of bootstrap replications is not an issue.

5.2. Numerical results: hypothesis testing

We shall now turn to the evaluation of the finite-sample behavior of the likelihood ratio test \( LR \) (test statistic given in (11)), the Bartlett-corrected likelihood ratio test \( LR_c \) (test statistic given in (15)), and the bootstrap likelihood ratio test (see (13)) for the shape parameter \( \beta \) of the Birnbaum–Saunders distribution. The Monte Carlo simulation was performed with \( \beta(0) = 0.75 \), i.e., we test

\[
\mathcal{H}_0 : \beta = 0.75 \quad \text{versus} \quad \mathcal{H}_1 : \beta \neq 0.75,
\]

for the sample sizes \( n = 5, 10, 20, 40, 60 \) and nominal levels \( \gamma = 10\%, 5\%, 1\%, 0.5\% \).

Table 4 presents the null rejection rates, expressed as percentages, of the null hypothesis. The \( LR \) test is considerably liberal in small samples, i.e., it overrejects the null hypothesis, the size distortions decreasing as the sample size increases. Note that the bootstrap test \( LR_b \) and the Bartlett-corrected test \( LR_c \) outperform the likelihood ratio test \( LR \). For instance, when \( n = 10 \) and \( \gamma = 5\% \), the null rejection rates of the \( LR, LR_c \) and \( LR_b \) tests were 7.72\%, 4.80\% and 5.44\%, respectively. We also note that the \( LR_b \) test (bootstrap) did not perform well at \( \gamma = 0.5\% \) and 1\% relative to Bartlett-corrected test \( LR_c \), which performed well for all sample sizes and at nominal levels.
Fig. 7. Interval estimation for $x = 0.75$ when $\beta = 1.0$.

Table 3
BCa interval estimation

<table>
<thead>
<tr>
<th>$B$</th>
<th>Parameter</th>
<th>$y$ (%)</th>
<th>Coverage</th>
<th>% Left</th>
<th>% Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>600</td>
<td>$\alpha$</td>
<td>10</td>
<td>86.48</td>
<td>4.68</td>
<td>8.84</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>91.74</td>
<td>2.34</td>
<td>5.92</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>95.82</td>
<td>0.56</td>
<td>3.62</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>10</td>
<td>88.64</td>
<td>5.64</td>
<td>5.72</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>93.76</td>
<td>3.00</td>
<td>3.24</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>98.20</td>
<td>0.72</td>
<td>1.08</td>
</tr>
<tr>
<td>1200</td>
<td>$\alpha$</td>
<td>10</td>
<td>87.88</td>
<td>4.92</td>
<td>7.20</td>
</tr>
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<td></td>
<td>5</td>
<td>93.40</td>
<td>2.36</td>
<td>4.24</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>97.28</td>
<td>0.62</td>
<td>2.10</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>10</td>
<td>88.86</td>
<td>5.44</td>
<td>5.70</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>94.02</td>
<td>2.95</td>
<td>3.03</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>98.50</td>
<td>0.70</td>
<td>0.80</td>
</tr>
<tr>
<td>2000</td>
<td>$\alpha$</td>
<td>10</td>
<td>89.62</td>
<td>4.78</td>
<td>5.60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>94.80</td>
<td>2.34</td>
<td>2.86</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>98.90</td>
<td>0.52</td>
<td>0.58</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>10</td>
<td>89.80</td>
<td>5.02</td>
<td>5.18</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>94.85</td>
<td>2.53</td>
<td>2.62</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>98.95</td>
<td>0.51</td>
<td>0.54</td>
</tr>
</tbody>
</table>
Fig. 8. Interval estimation for $\beta = 1.0$ when $\alpha = 0.75$.

Table 4
Null rejection rates of the tests $LR$, $LR_c$ and $LR_b$ ($H_0 : \beta = 0.75$)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$LR$</th>
<th>$LR_c$</th>
<th>$LR_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10%</td>
<td>5%</td>
<td>1%</td>
</tr>
<tr>
<td>5</td>
<td>18.74</td>
<td>11.08</td>
<td>3.28</td>
</tr>
<tr>
<td>10</td>
<td>13.36</td>
<td>7.72</td>
<td>1.74</td>
</tr>
<tr>
<td>20</td>
<td>12.02</td>
<td>6.34</td>
<td>1.30</td>
</tr>
<tr>
<td>40</td>
<td>11.02</td>
<td>5.72</td>
<td>1.12</td>
</tr>
<tr>
<td>60</td>
<td>10.72</td>
<td>5.14</td>
<td>0.96</td>
</tr>
</tbody>
</table>

Fig. 9 presents quantile-quantile plots, where the exact quantiles (estimated by Monte Carlo) of the test statistics $LR$ (dash) and $LR_c$ (dash dot) are plotted against their asymptotic quantiles for different sample sizes. Note that quantiles of $LR$ exceed their asymptotic counterparts, especially when the number of observations is small. As the sample size increases, so do the discrepancies between exact and asymptotic quantiles. It is also noteworthy that the exact quantiles of the Bartlett-corrected test statistic $LR_c$ are in agreement with the corresponding asymptotic quantiles, thus revealing that its null distribution is well approximated by the limiting null distribution used in the test.

6. Empirical illustrations

6.1. First illustration

We shall now analyze a data set on the fatigue lives (in hours) of 10 bearings of a certain kind. The data were used by Cohen et al. (1984) as a illustration of the three-parameter Weibull distribution. The data are given in Table 5, and their source is McCool (1974).
The point and interval estimates are presented in Tables 6 and 7, respectively. Table 6 contains the point estimates and their asymptotic standard errors (in parentheses). Note that the corrected estimates \( \tilde{\alpha}, \tilde{\alpha}, \tilde{\alpha}, \tilde{z} \) exceeded the MLE \( \hat{\alpha} \), and that the corrected estimates \( \tilde{\beta}, \tilde{\beta}, \tilde{\beta}, \tilde{\beta} \) are smaller than the MLE \( \hat{\beta} \). That is, all of the corrections point to the same direction: they suggest that estimation by maximum likelihood is underestimating \( \alpha \) and overestimating \( \beta \).

Table 7 contains the interval estimates. Note that the confidence intervals for \( \alpha \) are somewhat different. For instance, at the 90% nominal level, the CIP and CIBt confidence intervals for the shape parameter \( \alpha \) are (0.129; 0.381) and (0.209; 0.616), respectively.
Table 7
Interval estimates; first illustration

<table>
<thead>
<tr>
<th>Interval</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>90%</td>
<td>95%</td>
</tr>
<tr>
<td>ACI</td>
<td>(0.179; 0.386)</td>
<td>(0.158; 0.406)</td>
</tr>
<tr>
<td>ACICS</td>
<td>(0.192; 0.415)</td>
<td>(0.170; 0.437)</td>
</tr>
<tr>
<td>ACINKB</td>
<td>(0.199; 0.429)</td>
<td>(0.176; 0.451)</td>
</tr>
<tr>
<td>CIP</td>
<td>(0.129; 0.381)</td>
<td>(0.122; 0.398)</td>
</tr>
<tr>
<td>CIPCS</td>
<td>(0.139; 0.410)</td>
<td>(0.131; 0.429)</td>
</tr>
<tr>
<td>CIPNKB</td>
<td>(0.143; 0.423)</td>
<td>(0.136; 0.443)</td>
</tr>
<tr>
<td>BCa</td>
<td>(0.142; 0.428)</td>
<td>(0.136; 0.444)</td>
</tr>
<tr>
<td>CIBt</td>
<td>(0.209; 0.616)</td>
<td>(0.200; 0.650)</td>
</tr>
</tbody>
</table>

Fig. 10 contains the Birnbaum–Saunders density (2) evaluated at the point estimates of $\alpha$ and $\beta$ given in Table 6. Note that the estimated densities obtained from evaluating the density function (2) at the bias-corrected estimates $(\tilde{\alpha}, \tilde{\beta})$, $(\bar{\alpha}, \bar{\beta})$, $(\hat{\alpha}, \hat{\beta})$ and $(\check{\alpha}, \check{\beta})$ are in agreement in the sense that they all suggest that the ML density estimate is too peaked. It is noteworthy that the estimated densities obtained from the bootstrap and analytically corrected parameter estimates are very similar, and that the jackknife density estimate is the least peaked among the corrected estimates.

6.2. Second illustration

The data were used by Birnbaum and Saunders (1969b) and correspond to the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps). The data are given in Table 8.
The point and interval estimates are presented in Tables 9 and 10, respectively. (The figures in parentheses in Table 9 are asymptotic standard errors.) Unlike the previous illustration, all point estimates are in agreement; the same occurs with the confidence intervals. This follows from the large sample size (101 observations) relative to the previous example.
Table 11 contains parameter estimates of $\alpha$ and $\beta$ based on 20 observations randomly selected from the sample of 101 observations, i.e., from the observations listed in Table 8. This was done in order to evaluate the impact of the finite-sample corrections when the sample size is small. Note that all corrections suggest that maximum likelihood point estimation underestimates $\alpha$ and overestimates $\beta$, just as in the previous illustration. The corrections thus contain useful information when the sample size is not large.

7. Concluding remarks

We have derived closed-form expressions for the second order biases of the MLEs of the parameters that index the Birnbaum–Saunders distribution, and used such results to construct bias-corrected estimators. The biases of the proposed estimators are of order $O(n^{-2})$ whereas for the MLEs they are $O(n^{-1})$. That is, the biases of the newly proposed estimators converge to zero considerably faster than those of the MLEs. We note that it is not possible to guarantee such improvement for the modified estimators proposed by Ng et al. (2003); the improvement holds, however, for their jackknife estimators. We have also described a numerical approach to bias-correcting MLEs via the bootstrap. The Monte Carlo simulation results showed that the estimators we proposed have good finite-sample behavior, even outperforming the modified MLEs of Ng et al. (2003). Several different strategies for interval estimation were considered and numerically evaluated. We have also derived a Bartlett correction factor for likelihood ratio inference on the shape parameter. The numerical results showed that our Bartlett-corrected test outperforms the likelihood ratio test and also a bootstrap-based likelihood ratio test.

Acknowledgments

We gratefully acknowledge partial financial support from CAPES and CNPq. We also thank the two referees for comments and suggestions.

Appendix

We differentiate the log-likelihood function (3) up to fourth order with respect to the unknown parameters and obtain the relevant moments of such derivatives. The second, third and fourth order derivatives are

$$U_{2x} = \frac{n}{\alpha^2} + \frac{6n}{\alpha^4} - \frac{3n}{\alpha^3 \beta} \sum_{i=1}^{n} t_i - \frac{3n}{\alpha^3 \beta^2} \sum_{i=1}^{n} \frac{1}{t_i} + \frac{1}{\alpha^3} \sum_{i=1}^{n} \frac{1}{t_i},$$

$$U_{x\beta} = -\frac{n}{2 \beta^2} - \frac{1}{(t_i + \beta)^2} - \frac{1}{\alpha^2 \beta^3} \sum_{i=1}^{n} t_i,$$

$$U_{2\beta} = \frac{3}{2 \alpha \beta^2} \sum_{i=1}^{n} t_i - \frac{3}{\alpha^4} \sum_{i=1}^{n} \frac{1}{t_i} + \frac{2}{\alpha^3 \beta^3} \sum_{i=1}^{n} t_i,$$

$$U_{x\beta \beta} = -\frac{n}{\beta^3} + \frac{2}{(t_i + \beta)^3} + \frac{3}{(t_i + \beta)^4} \sum_{i=1}^{n} t_i,$$

$$U_{2\beta \beta} = \frac{12}{\alpha^2 \beta^2} \sum_{i=1}^{n} t_i + \frac{12}{\alpha^2} \sum_{i=1}^{n} \frac{1}{t_i} + \frac{12}{\alpha^2 \beta^2} \sum_{i=1}^{n} t_i,$$

$$U_{2x \beta \beta} = -\frac{6}{\alpha^3 \beta^3} \sum_{i=1}^{n} t_i,$$

$$U_{x \beta \beta \beta} = \frac{3n}{\beta^4} - \frac{6}{(t_i + \beta)^4} - \frac{12}{\alpha^2 \beta^2} \sum_{i=1}^{n} t_i.$$
Taking expected values of such derivatives, we obtain the following cumulants:

\[ \kappa_{2\alpha} = \frac{2n}{\alpha^2}, \quad \kappa_{2\beta} = 0, \quad \kappa_{\beta\beta} = -\frac{n[1 + \alpha^2\beta^2\xi_{\beta\beta}]}{\alpha^2\beta^2}, \]

where

\[ \xi_{\beta\beta} = \int_0^\infty \frac{1}{(t + \beta)^2} \frac{1}{2\alpha^2\sqrt{2\pi}} \left[ \left( \frac{\beta}{t} \right)^{1/2} + \left( \frac{\beta}{t} \right)^{3/2} \right] \exp \left\{ -\frac{1}{2\alpha^2} \left( \frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right\} \, dt. \]

Making a change of variable in the previous integral and after some algebra we obtain

\[ \xi_{\beta\beta} = \frac{1}{\alpha\beta^2\sqrt{2\pi}} h(\alpha), \]

where

\[ h(\alpha) = \int_0^\infty \frac{1}{v^2(1 + v^2)} \exp \left\{ -\frac{1}{2\alpha^2} (v - v^{-1})^2 \right\} \, dv. \]

After some algebra, we were able to write \( h(\alpha) \) as

\[ h(\alpha) = \sqrt{\frac{\pi}{2}} - \pi e^{2/\alpha^2} \left[ 1 - \Phi \left( \frac{2}{\alpha} \right) \right]. \]

Thus,

\[ \kappa_{\beta\beta} = -\frac{n[1 + (2\pi)^{-1/2}h(\alpha)]}{\alpha^2\beta^2}, \quad \kappa_{2\alpha} = \frac{10n}{\alpha^3}, \quad \kappa_{2\alpha^2} = 0, \quad \kappa_{\alpha\beta^2} = \frac{2n}{\alpha^3\beta^2} \left( 1 + \frac{\alpha^2}{2} \right), \]

\[ \kappa_{\beta\beta\beta} = \frac{n}{2\beta^3} + \frac{3n}{\alpha^2\beta^3} + 2n \xi_{\beta\beta\beta}, \]

with

\[ \xi_{\beta\beta\beta} = \int_0^\infty \frac{1}{(t + \beta)^3} \frac{1}{2\alpha^2\sqrt{2\pi}} \left[ \left( \frac{\beta}{t} \right)^{1/2} + \left( \frac{\beta}{t} \right)^{3/2} \right] \exp \left\{ -\frac{1}{2\alpha^2} \left( \frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right\} \, dt, \]

\[ \kappa_{3\alpha} = -\frac{54n}{\alpha^4}, \quad \kappa_{3\alpha^2} = 0, \quad \kappa_{3\alpha\beta} = -\frac{6n}{\alpha^4\beta^2} \left( 1 + \frac{\alpha^2}{2} \right), \]

\[ \kappa_{3\beta\beta} = -\frac{6n}{\alpha^3\beta^3} \left( 1 + \frac{\alpha^2}{2} \right), \quad \kappa_{\beta\beta\beta\beta} = -\frac{3n}{\beta^4} - \frac{12n}{\alpha^2\beta^4} - 6n \xi_{\beta\beta\beta\beta}, \]

where

\[ \xi_{\beta\beta\beta\beta} = \int_0^\infty \frac{1}{(t + \beta)^4} \frac{1}{2\alpha^2\sqrt{2\pi}} \left[ \left( \frac{\beta}{t} \right)^{1/2} + \left( \frac{\beta}{t} \right)^{3/2} \right] \exp \left\{ -\frac{1}{2\alpha^2} \left( \frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right\} \, dt. \]

Differentiating the cumulants with respect to the unknown parameters, we obtain

\[ \kappa^{(2)}_{2\alpha} = \frac{4n}{\alpha^3}, \quad \kappa^{(2)}_{2\beta} = 0, \quad \kappa^{(2)}_{\alpha\beta} = 0, \quad \kappa^{(2)}_{\beta\beta} = \frac{2n}{\alpha^3\beta^2} - n \xi^{(2)}_{\beta\beta}, \]
where
\[ \mathcal{L}_{\beta\beta}^{(\alpha)} = \frac{\partial \mathcal{L}_{\beta\beta}}{\partial \beta} = \frac{\partial}{\partial \beta} \int_0^\infty \frac{1}{(t + \beta)^2} f(t) \, dt, \]

and hence
\[ \kappa_{\beta\beta}^{(\alpha)} = n \left( \frac{1}{\alpha} + \frac{4}{3 \alpha^3} \right) \mathcal{L}_{\beta\beta} + \frac{n}{\alpha^3 \beta^2} - \frac{n}{2 \alpha \beta^2}. \]

Also,
\[ \kappa_{\beta\beta}^{(\beta)} = \frac{2n}{2^2 \beta^3} - n \mathcal{L}_{\beta\beta}, \]

where
\[ \mathcal{L}_{\beta\beta}^{(\beta)} = \frac{\partial \mathcal{L}_{\beta\beta}}{\partial \beta} = \frac{\partial}{\partial \beta} \int_0^\infty \frac{1}{(t + \beta)^2} f(t) \, dt. \]

We can write
\[ \mathcal{L}_{\beta\beta}^{(\beta)} = \int_0^\infty \frac{\partial}{\partial \beta} \frac{1}{(t + \beta)^2} f(t) \, dt = \int_0^\infty \frac{1}{(t + \beta)^2} \frac{\partial}{\partial \beta} f(t) \, dt - \int_0^\infty \frac{2}{(t + \beta)^3} f(t) \, dt = -\frac{1}{2\beta^2} \mathcal{L}_{\beta\beta} - \mathcal{L}_{\beta\beta}^{(\beta)} - \frac{1}{4\beta^3}. \]

Therefore,
\[ \kappa_{\beta\beta}^{(\beta)} = \frac{2n}{2^2 \beta^3} + \frac{n}{4\beta^3} - \frac{n}{2\beta} \mathcal{L}_{\beta\beta} + n \mathcal{L}_{\beta\beta}^{(\beta)}. \]

Also,
\[ \kappa_{2,\beta}^{(\alpha)} = -\frac{30n}{\alpha^4}, \quad \kappa_{2,\beta}^{(2\alpha)} = -\frac{12n}{\alpha^4}, \quad \kappa_{2,\beta}^{(\alpha\beta)} = 0, \quad \kappa_{2,\beta}^{(\beta)} = 0, \]

\[ \kappa_{2,\beta\beta}^{(\alpha)} = -\frac{6n}{\alpha^4 \beta^2} - \frac{n}{\alpha^2 \beta^2}, \quad \kappa_{2,\beta\beta}^{(\beta)} = 0, \]

\[ \kappa_{2,\beta\beta}^{(\beta)} = \frac{n}{2\beta^2} (\beta \mathcal{L}_{\beta\beta}^{(\beta)} - \mathcal{L}_{\beta\beta}^{(\beta)}) + n \mathcal{L}_{\beta\beta}^{(\beta)} - \frac{3n}{4\beta^4} - \frac{4n}{2^2 \beta^3}, \]

where
\[ \mathcal{L}_{\beta\beta}^{(\beta)} = \frac{\partial \mathcal{L}_{\beta\beta}^{(\beta)}}{\partial \beta} = \frac{\partial}{\partial \beta} \int_0^\infty \frac{1}{(t + \beta)^3} f(t) \, dt. \]

Additionally, we obtain
\[ \kappa_{\beta\beta}^{(\beta)} = -\frac{3n}{2\beta^4} - \frac{9n}{\alpha^2 \beta^4} + 2n \mathcal{L}_{\beta\beta}^{(\beta)}. \]
Notice that
\[
\xi_{\beta \beta \beta}^{(\beta)} = \int_0^\infty \frac{1}{(t + \beta)^3} f(t) \, dt = \int_0^\infty \left[ \frac{1}{(t + \beta)^3} \frac{\partial f(t)}{\partial \beta} - \frac{3}{(t + \beta)^4} f(t) \right] \, dt
\]
\[
= \int_0^\infty \frac{1}{(t + \beta)^3} \frac{\partial f(t)}{\partial \beta} \, dt - 3\xi_{\beta \beta \beta}^{(\beta)}
\]
\[
= - \frac{1}{4x^2 \beta^4} \frac{1}{4\beta^4} + \frac{1}{x^2 \beta^2} \xi_{\beta \beta} - \frac{1}{2\beta} \xi_{\beta \beta} - 2\xi_{\beta \beta \beta}^{(\beta)}.
\]

Thus,
\[
\kappa_{\beta \beta}^{(\beta)} = - \frac{n}{\beta^3} \left( \frac{4}{x^2} + \frac{9}{8\beta} + \frac{1}{4x^2 \beta} \right) - \frac{n}{\beta^2} \left( \frac{3}{4} - \frac{1}{x^2} \right) \xi_{\beta \beta} - \frac{n}{\beta} \xi_{\beta \beta \beta}^{(\beta)} - 2n\xi_{\beta \beta \beta}^{(\beta)},
\]
\[
\kappa_{\beta \beta \beta}^{(\beta)} = - \frac{n}{\beta^4} \left( 2 + \frac{19}{2x^2} + \frac{2n}{x^2 \beta^2} \xi_{\beta \beta} - \frac{n}{\beta} \xi_{\beta \beta \beta}^{(\beta)} - 4n\xi_{\beta \beta \beta}^{(\beta)}.
\]

In what follows, we shall obtain the quantities needed for obtaining the Bartlett correction. We have that
\[
\kappa_{\beta \beta} = \frac{x^2}{2n} \quad \text{and} \quad \kappa_{\beta \beta \beta} = \frac{x^2 \beta^2}{n[\pi(2\pi)^{-1/2}h(x) + 1]}.
\]

Hence,
\[
\hat{\lambda}_{\beta \beta \beta} = \kappa_{\beta \beta \beta} \kappa_{\beta \beta} \left[ \kappa_{\beta \beta \beta} \kappa_{\beta \beta \beta} / 4 - \kappa_{\beta \beta \beta}^{(\beta)} + \kappa_{\beta \beta \beta}^{(\beta)} \right] = \frac{9}{8n},
\]
\[
\hat{\lambda}_{\beta \beta \beta} = \kappa_{\beta \beta \beta} \kappa_{\beta \beta} \left[ \kappa_{\beta \beta \beta} / 4 - \kappa_{\beta \beta \beta}^{(\beta)} + \kappa_{\beta \beta \beta}^{(\beta)} \right] = - \frac{3}{4n[\pi(2\pi)^{-1/2}h(x) + 1]} \left( 1 + \frac{x^2}{2} \right),
\]
\[
\hat{\lambda}_{\beta \beta \beta} = \kappa_{\beta \beta \beta} \kappa_{\beta \beta} \left[ \kappa_{\beta \beta \beta} / 4 - \kappa_{\beta \beta \beta}^{(\beta)} + \kappa_{\beta \beta \beta}^{(\beta)} \right] = \frac{3}{2n[\pi(2\pi)^{-1/2}h(x) + 1]} \left( \frac{9}{2x^2} + \frac{1}{4x^2} \right),
\]
\[
\hat{\lambda}_{\beta \beta \beta} = \kappa_{\beta \beta \beta} \kappa_{\beta \beta} \left[ \kappa_{\beta \beta \beta} / 4 - \kappa_{\beta \beta \beta}^{(\beta)} + \kappa_{\beta \beta \beta}^{(\beta)} \right] = - \frac{9}{24n},
\]
\[
\hat{\lambda}_{\beta \beta \beta} = \kappa_{\beta \beta \beta} \kappa_{\beta \beta} \left[ \kappa_{\beta \beta \beta} / 4 - \kappa_{\beta \beta \beta}^{(\beta)} + \kappa_{\beta \beta \beta}^{(\beta)} \right] = \frac{5}{4n[\pi(2\pi)^{-1/2}h(x) + 1]} \left( 1 + \frac{x^2}{2} \right),
\]
\[
\hat{\lambda}_{\beta \beta \beta} = \kappa_{\beta \beta \beta} \kappa_{\beta \beta} \left[ \kappa_{\beta \beta \beta} / 4 - \kappa_{\beta \beta \beta}^{(\beta)} + \kappa_{\beta \beta \beta}^{(\beta)} \right] = - \frac{3}{4n[\pi(2\pi)^{-1/2}h(x) + 1]} \left( 1 + \frac{x^2}{2} \right).
\]
\[
\hat{\lambda}_{\beta \beta \beta} = \kappa_{\beta \beta \beta} \kappa_{\beta \beta} \left[ \kappa_{\beta \beta \beta} / 4 - \kappa_{\beta \beta \beta}^{(\beta)} + \kappa_{\beta \beta \beta}^{(\beta)} \right] = 0.
\]
\[
\hat{\lambda}_{2x\beta\beta\beta\beta} = \kappa^{2\beta} \kappa^{2\beta} \kappa^{2\beta} \kappa^{2\beta} \left[ \kappa^{2\beta}_{2\beta} \left( \frac{\kappa \beta_2}{6} - \kappa^{2\beta}_{2\beta} \left( \kappa \beta_2 / 6 - \frac{\kappa^{2\beta}_{2\beta}}{4} \right) + \kappa^{2\beta}_{2\beta} \kappa^{2\beta}_{2\beta} + \kappa^{2\beta}_{2\beta} \kappa^{2\beta}_{2\beta} \right) \right]
\]
\[
= \frac{5}{6n \left[ \alpha(2\pi)^{1/2} h(z) + 1 \right]^2 \left( 1 + \frac{\alpha^2}{2} \right) ^2},
\]
\[
\hat{\lambda}_{\beta\beta\beta\beta\beta\beta} = \kappa^{2\beta} \kappa^{2\beta} \kappa^{2\beta} \kappa^{2\beta} \left[ \kappa^{2\beta}_{\beta\beta} \left( \kappa^{2\beta}_{\beta\beta} / 6 - \frac{\kappa^{2\beta}_{\beta\beta}}{4} \right) + \kappa^{2\beta}_{\beta\beta} \kappa^{2\beta}_{\beta\beta} + \kappa^{2\beta}_{\beta\beta} \kappa^{2\beta}_{\beta\beta} \right]
\]
\[
= \frac{1}{n \left[ \alpha(2\pi)^{1/2} h(z) + 1 \right]^2 \left( 1 + \frac{\alpha^2}{2} \right) ^2 \left[ \frac{1}{3\alpha^2 \beta^2} \left( 1 + \frac{\alpha^2}{2} \right) + \frac{1}{1 + \frac{\alpha^2}{2}} \right]},
\]
\[
\hat{\lambda}_{\beta\beta\beta\beta\beta\beta} = \kappa^{2\beta} \kappa^{2\beta} \kappa^{2\beta} \kappa^{2\beta} \left[ \kappa^{2\beta}_{\beta\beta} \left( \kappa^{2\beta}_{\beta\beta} / 6 - \frac{\kappa^{2\beta}_{\beta\beta}}{4} \right) + \kappa^{2\beta}_{\beta\beta} \kappa^{2\beta}_{\beta\beta} + \kappa^{2\beta}_{\beta\beta} \kappa^{2\beta}_{\beta\beta} \right]
\]
\[
= \frac{1}{3n \left[ \alpha(2\pi)^{1/2} h(z) + 1 \right]^2 \left( 1 + \frac{\alpha^2}{2} \right) ^2}.
\]

References


