Treble dodging minor methods: ringing the cosets, on six bells

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Abstract
A general theory is developed to analyze Treble Dodging methods, for change ringing extents, on \( n = 2 \mod 4 \) bells, and then specialized to six bells, by means of Schreier right coset graphs of comparatively low order. The theory is then applied to the eight Treble Bob Minor methods, the six Delight Minor methods, and the fifteen Surprise Minor methods that appear in the book *Diagrams*. In each case all possible extents of the form \([W(P, B)]^n\) are obtained and exhibited, where \( P \) and \( B \) denote the plain and bob leads, respectively, and \( W(P, B) \) is a word in these two letters. This application is then extended to all 152 Treble Dodging methods appearing in *Collection of Minor Methods*, Fifth Edition and beyond, to the classification and construction of additional Treble Dodging minor methods, as appearing in *Collection of Minor Methods*, Sixth Edition and in *Treble Dodging Minor Methods*.

1. Introduction

The aim of this paper is to present a relatively efficient means of analyzing, by the use of Schreier right coset graphs, Treble Dodging methods of change ringing extents. The analysis is particularly simple for Treble Dodging Minor methods, and the general theory is applied to all twenty-nine such appearing in *Diagrams* [12]. In each case, all possible compositions of the most natural form are obtained and exhibited. We begin with the necessary definitions. (For additional terminology and background in change ringing and/or algebraic and topological graph theory, see [3, 11, 16 and/or 4], respectively.) Denote the \( n \) bells in a tower by the natural numbers \( 1, 2, \ldots n \) — arranged in descending order of pitch. Bell 1 is called the *treble*; bell \( n \) is called the *tenor*. Any composition on \( n = 6 \) bells will have the designation *minor* as the last part of its nomenclature. (We illustrate the general theory developed in this paper for \( n = 6 \), as this is the first value which is both tractable and nontrivial.) A *change* is a ringing of all \( n \) bells, in one of the \( n! \) possible orderings. (Alternatively — see, for example, [3] or [16] — each ringing of the \( n \) bells is called a *row*, and a *change* is a transition between
two successive rows. Unfortunately, the literature reflects this ambiguity. Our choice for the meaning of ‘change’ here is consistent with the mathematical predecessors of this paper, such as [1, 2, 5–10]. We regard it as a permutation \( f: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \), where the domain numbers represent positions and the range numbers represent bells. Thus a change \( f \), recorded as \( f(1), f(2), \ldots, f(n) \), would ring bell \( f(1) \) first, bell \( f(2) \) second, \ldots, and bell \( f(n) \) last. The very special change given by the identity permutation is called rounds. The central problem in change ringing is to ring an extent on the \( n \) bells. This is a sequence of \( n! + 1 \) changes satisfying:

(i) The first and last change are both rounds.
(ii) No other change is repeated; thus every other change is rung exactly once.
(iii) From one change to the next, no bell changes its order of ringing by more than one position.

It is well known (see, for example, [5 or 6]) that an extent on \( n \) bells, using transition rules from \( \Delta \) (to pass from one change to another; thus each member of \( \Delta \), by (iii), is a disjoint product of transpositions of adjacent numbers from 1 to \( n \) in the symmetric group \( S_n \) — and, by (ii), \( \Delta \) must generate \( S_n \)) can be composed if and only if the Cayley graph \( G_\Delta(S_n) \) is hamiltonian. (That we return to the starting point is required by (i).)

Certain additional conditions that an extent might meet are often regarded as desirable (or perhaps even necessary); but it is only (i)–(iii) which are always required. Among these additional conditions, the three following are the most noteworthy:

(iv) In the plain course, no bell occupies the same position in its order of ringing for more than two successive changes. (This has been relaxed to “four successive changes” [16].)
(v) The working bells all do the same work in the plain course.
(vi) Each lead (or division) of the extent is palindromic in the transitions employed.

We need additional terminology to understand these last three conditions. There are two basic types of construction for change ringing composition: methods and principles. A method is treble-dominated. This may occur if the treble plain hunts — occupying successively positions

\[1, 2, \ldots, n; \quad n, \ldots, 2, 1; \quad 1, 2, \ldots, n; \quad n, \ldots, 2, 1;\]

and so on — or if it dodge hunts (i.e. dodges while hunting) — occupying successively positions

\[1, 2, 1, 2, 3, 4, 3, 4, \ldots, n - 1, n, n - 1, n; \quad n, n - 1, n, n - 1, \ldots, 4, 3, 4, 3, 2, 1, 2, 1;\]

and so on. (Other forms of treble domination — with the hunt bell making internal places — occur in methods such as Treble Place and Alliance, but we shall not be concerned with such methods here.) A hunting bell is not considered to be working. In a principle all the bells are working — that is, performing more intricate tasks, such as dodging around other bells, making internal places, and so forth. (A plain hunt bell makes only the external places 1 and \( n \).)

The basic unit of composition for a method is called a lead; for a plain hunt method, this consists of the \( 2n \) changes from one treble lead \( (f(1) = 1) \) to the next. For a Treble
Dodging method, a lead consists of $4n$ changes from one treble lead to the third following treble lead. The basic unit of composition for a principle is called a division. In both cases, the plain course is the aggregate of changes commencing with rounds and continuing without calls (special generating transitions, either bobs or singles) until rounds occurs again. (If this occurs before the extent is completed, then calls are required. If not, then the plain course is the extent, said to be a no-call extent.) The plain course consists of a succession of leads (or divisions), and condition (v) can be expressed by saying that the plain course must consist of the same number of leads (or divisions) as there are working bells. In particular, if there are $m$ working bells, and if $w = \delta_1 \delta_2 \cdots \delta_k$ describes the first lead (division) and the transition to the second, then $w^m = I$ (the identity in $S_n$) describes the plain course. Moreover, $w$ must be an $m$-cycle, so there is only one orbit in its action on the working bells. Condition (vi) is that $w' = \delta_1 \delta_2 \cdots \delta_{k-1}$ is palindromic in the letters $\delta_i$, $1 \leq i \leq k - 1$.

We remark that the twenty-nine methods we analyze in Section 4 all satisfy conditions (iv)–(vi) as well as (i)–(iii). We observe that (iv) fails in the fourth position each time a bob is called, for eight of the methods; however, as a bob is not part of the plain course, this is not considered to be a violation.

Recall that an extent on $n$ bells using transitions from $\Delta$ can be composed precisely when $G_{\Delta}(S_n)$ is hamiltonian. This point of view was used in [6] to compose what is now called ‘White’s No-call Doubles’ (a principle on 5 bells); this was first rung to quarterpeal length (eleven replications, 1,320 changes in all) in Oxford, on December 9, 1984. The corresponding hamiltonian cycle in $G_{\Delta}(S_5)$ was discovered by utilizing a 5-fold symmetry in the Cayley graph (imbedded in a nonorientable surface of genus ten). Modding out that symmetry, we obtain a Schreier right coset graph of $Z_5$ in $S_5$. Then a hamiltonian cycle (of length 24) in this quotient structure which corresponds to the nonidentity element in $Z_5$ lifts to a hamiltonian cycle (of length 120) in $G_{\Delta}(S_5)$, and hence produces a composition. (It is a no-call extent.) If the 120 distinct changes beginning with rounds are arranged in five columns of 24 changes each (the divisions of the principle), then row one corresponds to $Z_5$ and the other rows give the right cosets of $Z_5$ in $S_5$. This point of view is exploited in [10].

Left coset decompositions are also useful in change ringing. For example, in the most basic method — Plain Bob — the first lead corresponds to the dihedral group $D_n$ (called the hunting group), and every other lead is a left coset of $D_n$ in $S_n$. Under appropriate conditions, a hamiltonian cycle in a Cayley color graph for the alternating group $A_{n-1}$ will determine the connections among these cosets to give an extent of Plain Bob on $n$ bells. (Note that $((n - 1)!/2)(2n) = n!$) This point of view is developed in [8].

Here are some of the results that have been obtained by these considerations.

$n = 4$. There are exactly four no-call minimus principles. [10]

$n = 5$. All 102 no-call doubles principles on three generating transitions are catalogued in [10]. These include Western Michigan University Doubles, first rung to
quarter-peal length, in Oxford, on July 19, 1987. Also in [10], an analysis of the Schreier right coset graph for $Z_5$ in $A_5$ produces all four 'pure' doubles extents (all transitions move four bells, except for a single, which is called twice).

**General $n$.** (1) An extent on $n$ bells using transitions from $A$ exists if and only if $G_n(S_n)$ is hamiltonian [5–8].

(2) For each $n \in \mathbb{N}$, an extent exists on $n$ bells; moreover, for $n \geq 4$, we can always take $|A| = 3$ [6–7].

(3) A standard algorithm for generating all the permutations of $n$ objects also produces an extent on $n$ bells. (The extent fails conditions (iv) and (v) badly, however.) [7]

(4) For $n \equiv 0 \pmod{4}$, Plain Bob on $n$ bells can be rung on plain and single leads only [9].

(5) For $n$ even ($n \geq 6$), there is no extent of Plain Bob using plain and bob leads only [8].

(6) For $n$ even ($n \geq 6$), there is a Plain Bob $2n!$ on $n$ bells (each change rung exactly twice, followed by rounds for a third time) using plain and single leads only [8].

(7) For $n$ odd ($n \geq 5$), there is no extent of Grandsire (a method employing two plain hunt bells) using plain and bob leads only [8].

Now we turn our attention to Treble Dodging methods. We use right cosets, and begin with a specific example.

### 2. Oxford Treble Bob Minor

In the group $S_n$, let

$x = (12)(34)(56),
\quad y = (12)(56),
\quad z = (34)(56),
\quad b = (23)(56),
\quad c = (23)(45),

and take $A = \{x, y, z, b, c\}$. Put $a = xyxc(xzxc)^*xyx$

and note that this is a palindrome. We let $a$ determine the leads of the extent. The first lead, beginning with rounds, is given in Table 1. We see that the dodging pattern of the treble is clearly displayed by these twenty-four rows (changes). We also make the following two observations (at this stage, these can be checked empirically; in
Section 3 we will prove these two properties, in a general setting:

(1) Two corresponding rows (row \(i\) and row \(25-i\), for \(1 \leq i \leq 12\); the correspondence is by horizontal reflection, invoking the palindromic nature of \(a\)) are in the same right coset of \(A_5\) in \(S_6\). For example, if we take \(i = 3\), then the rows \(1 2 4 3 5 6\) and \(1 2 3 4 6 5\) are represented in \(S_6\), respectively, by \((3 4)\) and \((5 6)\); then \(A_5(3 4) = A_5(5 6)\), since \((3 4)(5 6) \in A_5\).

(2) The rows of a half-lead (rows \(1-12\) or \(13-24\)) form a right transversal to \(A_5\) in \(S_6\); that is, each of the twelve right cosets of \(A_5\) in \(S_6\) is represented exactly once in each half-lead.

We readily calculate that \(a = (3 4)(5 6)\) (so \(a = z\)). Thus the changes corresponding to rows 1 and 24 (the lead head and lead end, respectively) will be in the same coset of \(A_5\) in \(S_6\); that is, they both have the same parity. (Note that the treble is leading — is in position 1 — in both these rows. Thus we are identifying \(S_5\) with \((S_6)_1\), the stabilizer of 1 in \(S_6\).) Note also that the other two treble leads, occurring in rows 3 and 22, are both in the other coset of \(A_5\) in \(S_5\); that is, they both have parity opposite to that of rows 1 and 24. Thus the word \(a\) partitions \(A_5\) into 30 pairs of even permutations, each pair bounding a lead of 24 changes. As we well show in Section 3, these 30 leads are mutually disjoint, and thus if we can connect them in some reasonable fashion, we will have an extent \((30 \times 24 = 6!)\).

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The connection is achieved by means of transitions $b$ and $c=p$. Set $P=ap$ and $B=ab$; these are the Plain lead and Bob lead, respectively. (Note that we are now regarding each lead as 24 transitions, instead of 24 rows.) Since both $b$ and $p$ are in $A_5$, each joins two rows corresponding to two elements in $A_5$. Now consider the identity word

$$\left[ (ap)^4(ab)^2(ab)^3(ab) \right]^3 = [P^4B^2P^3B]^3$$

in $A_5$. This is the extent Oxford Treble Bob Minor. We see that there are nine calls, all bobs, arranged in a systematic manner. The non-identity word

$$(ap)^4(ab)^2(ab)^3(ab)$$

describes a hamiltonian cycle (starting at the designated vertex) in Fig. 1, which is a Schreier right coset graph for $Z_3$ in $A_5$. Thus this Treble Dodging Minor extent determines a hamiltonian cycle, with $a$ alternating and the hamiltonian word in $a, b, p$ of order three, in a Schreier right coset graph $S_3(A_5/Z_3)$, where $\mathcal{A} = \{a, b, p\}$. The main idea of this paper is that the converse is true also (we prove this, in a more general setting, in Section 3). Moreover, the diagram can be analyzed for alternate callings.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{schreier_graph.png}
\caption{The Schreier right coset graph for $Z_3$ in $A_5$.}
\end{figure}
For example, the labelled hamiltonian cycle in Fig. 1 is unique (with respect to the
alternation of a and the order three); thus there is just the one (up to cyclic rotations
and reversal) sequence of plain and bob leads for Oxford Treble Bob Minor.

What we prove in Section 3, therefore, is that, in general, Treble Dodging music can
be composed graphically. We carry out the analysis, for \( n = 6 \), in Section 4.

3. The general theory

Let \( n \equiv 2 \pmod{4} \), and in \( S_n \), set
\[
\begin{align*}
  x &= (12)(34) \cdots (n-1,n), \\
  y &= (12)(56)(78) \cdots (n-1,n), \\
  z &= (34)(56) \cdots (n-1,n), \\
  b &= (23)(56)(78) \cdots (n-1,n), \\
  c &= (23)(45) \cdots (n-2,n-1), \\
  d &= (12)(34) \cdots (n-3,n-2), \\
  e &= (12)(45)(67) \cdots (n-2,n-1).
\end{align*}
\]

Thus \( x \) is odd, but \( y, z, b, c, d, \) and \( e \) are all even. Let \( a \) be an irreducible (that is, no
subword for consecutive letters corresponds to the identity element in \( S_n \) palindromic
word in \( x, y, z, b, c, d, \) and \( e \) of length \( 4n-1 \) having its middle symbol in \( A_n \) and
determining a treble-dodging lead with the property:

(*) For each half-lead, and for each position, the two rows when the treble is in
that position are of opposite parity (one each from \( A_n \) and from \( S_n - A_n \)).

Note that, due to the palindromic nature of \( a \), if (*) holds for one half-lead, it will
hold for the other.

Let \( m \) be an odd integer, \( 3 < m < n-1 \), with \( P = ap(p \in A - \{x, a, b\}) \) and \( B = ab \), and
suppose the word \( W(P, B) \) describes a hamiltonian cycle in the Schreier right coset
graph \( S_d(A_{n-1}/Z_m) \), where \( A = \{a, b, p\} \), such that \( W(P, B) \) has order \( m \) in \( A_{n-1} \). Then
we claim that \( [W(P, B)]^m \) gives a Treble Dodging extent on \( n \) bells.

Just as for our example in Section 2, we have a 'score' for connecting \( (n-1)!/4 \) leads
of \( 4n \) changes each into an aggregate of \( n!(+1) \) changes; clearly conditions (i) and (iii)
of Section 1 are met. Our task is to verify condition (ii); that is, we must show that no
change is repeated (other than rounds at beginning and end), that there is no falsity (as
the ringers would say). For the optional conditions, we remark that (iv) depends upon
a suitable selection for \( a \); (v) will hold, provided \( p \) is chosen so that \( P \) has order \( n-1 \):
(vi) is already guaranteed by our description of \( a \).

We begin by noting that, since the middle symbol of \( a \) is even and \( a \) is palindromic,
it follows that \( a \) is even. Thus if the lead begins with an even row (which we require it
to do), it will end with an even row. Then the palindromic nature of $a$ guarantees that corresponding rows (row $i$ and row $4n+1-i$, for $1 \leq i \leq 2n$) are in the same right coset of $A_{n-1}$ in $S_n$; let $g, h \in S_n$, with $g$ corresponding to row $i$ and $h$ to row $4n+1-i$. Let the treble be in position $w(1 \leq w \leq n)$ in both rows. (The palindromic nature of $a$ guarantees that $w$ exists.) Thus $g(w) = h(w)$, so that $w = g^{-1}(1) = h^{-1}(1)$. Hence $gh^{-1}(1) = 1$, so $gh^{-1} \in A_{n-1}$ (clearly $g$ and $h$ have the same parity), and $A_{n-1}g = A_{n-1}h$. (We identify $S_{n-1}$ with $(S_n)_1$.)

Now we invoke property (1), to show that the rows of each half-lead form a right transversal to $A_{n-1}$ in $S_n$. We need only show that the $2n$ rows represent $2n$ distinct cosets. Let two distinct rows correspond to $g$ and $h$ in $S_n$. If $g(k) = h(k) = 1$ (i.e. if the treble is in position $k$ in both rows), then $gh^{-1}(1) = 1$, so $gh^{-1} \in S_{n-1}$. But by (1), $gh^{-1} \in S_{n-1} - A_{n-1}$ so $A_{n-1}g \neq A_{n-1}h$. On the other hand, if $g(k) = 1 = h(l)$, with $k \neq l$, then $gh^{-1}(1) = g(l) \neq 1$, so that $gh^{-1} \notin S_{n-1}$ and hence $gh^{-1} \notin A_{n-1}$; i.e. $A_{n-1}g \neq A_{n-1}h$. Thus in either case the cosets are distinct. Moreover, since each lead is described by $a$, all leads represent the cosets in the same order. That is, if $\alpha$ and $\beta$ are two lead heads (so $\alpha, \beta \in A_{n-1}$) and if $g$ is any initial subword of $a$ (if $a = a_1a_2\cdots a_{4n-1}$, then $g = a_1a_2\cdots a_k$, where $0 \leq k \leq 4n-1$; if $k = 0$, $g$ is the identity element), then $A_{n-1}ag = A_{n-1}bg$, since $xg(bg)^{-1} = x\beta^{-1} \in A_{n-1}$.

We have shown that, under the given conditions:

1. In any given lead, for $1 \leq i \leq 2n$, row $i$ and row $4n+1-i$ are in the same right coset of $A_{n-1}$ in $S_n$;

2. For $1 \leq i \leq 4n$, in any two leads, the two rows $i$ are in the same right coset of $A_{n-1}$ in $S_n$.

Recalling that $a$ is palindromic and that each lead begins and ends with elements of $A_{n-1}$ (connected to each other by $a$ internally, and to the adjacent leads by either $b$ and $p$), we see that the $2n$ right cosets of $A_{n-1}$ in $S_n$ are precisely described by $A_{n-1}g$, as $g$ ranges over the initial subwords of $a$ of length $k$, $0 \leq k \leq 2n-1$. Now, since $a$ is irreducible, each lead consists of $4n$ distinct changes. Thus any falseness in our composition must arise from two distinct (in their order of appearance) rows representing the same element of $S_n$, say $u = v$, but appearing in different leads. But clearly these two rows are in the same right coset. Thus, by the above discussion, there are two distinct elements $\alpha$ and $\beta$ of $A_{n-1}$ and an initial subword $g$ of $a$ having length at most $2n-1$ such that $u = xg$ and $v = \beta g$. But then $xg = \beta g$ and $\alpha = \beta$, a contradiction. Therefore, $[W(P, B)]^m$ is a Treble Dodging extent.

4. Applying the theory

In the book *Diagrams* [12], plain and bob leads are given for twenty-nine Treble Dodging Minor methods. For each one, we use the approach developed in Section 3 to find all compositions of the form $[W(P, B)]^m$, where $3 \leq m \leq 5$. (Recall that $W(P, B)$ is a word in $P$ and $B$ which is replicated, $m$ times in all, to give the extent.) For each value of $m$, our choice of an $m$-cycle, to form the Schreier right coset graph $S_d(A_5/Z_m)$, is arbitrary; by an easy modification of the proof of Theorem 2 of [10], all such graphs
are isomorphic. Since 4-cycles are odd in symmetric groups, \( m \neq 4 \). Exhaustive inspection finds (for each of the twenty-nine methods) no hamiltonian cycles of order 5 in the \( S_3(A_5/Z_3) \), thus \( m \neq 5 \). We therefore restrict our attention to \( m = 3 \). (Another exhaustive inspection rules out \( m = 2 \), for products of two disjoint transpositions.)

These 'exhaustive' inspections are not as arduous as might at first appear. The twenty-nine methods are partitioned into seven cases, depending upon the value of \( a \) in \( A_5 \) and the choice of \( p \) (\( b \) does not vary). Each of these seven cases gives one Schreier right coset graph to analyze. Six of these coalesce into three pairs of isomorphic graphs. ((1) pairs with (1r), (2) with (2r), and (3) with (3r); the pairing is by 'reverse' — fix 4, but exchange 2 with 6 and 3 with 5; note that the bob \( b \) is fixed by this process). Thus the claims above and the results below are obtained by analyzing four Schreier right coset graphs \( S_4(A_5/Z_3) \) (the first of these is displayed in Fig. 1; this graph is planar, whereas the other three have genus one) for hamiltonian cycles corresponding to group elements of order three. It is routine to check that each of the twenty-nine words for \( a \) satisfies property (\( * \)), and the requirements preceding property (\( * \)), so that the theory developed in Section 3 does apply.

For each of these seven cases, we give the number and name (all have 'Minor' as the last part of the nomenclature) as in Diagrams, the exact formulation of \( a \) (we give \( a \) by specifying the first twelve letters \( a' \); the remaining eleven follow by symmetry), and the common word or words \( W(P,B) \) leading to the extents. (The 3-bob words — i.e. 9-bob compositions for the extent — give what is called the standard calling.) Throughout, we have:

\[
\begin{align*}
x &= (12)(34)(56), \\
y &= (12)(56), \\
z &= (34)(56), \\
b &= (23)(56), \\
c &= (23)(45), \\
d &= (12)(34), \\
e &= (12)(45).
\end{align*}
\]

(1) \( a = (34)(56), \ p = c = (23)(45): \)

\[
\begin{align*}
44 \; \text{Kent Treble Bob} & \quad a' = yxye(xxzxc)^2, \\
45 \; \text{Oxford Treble Bob} & \quad a' = xxye(xxzxc)^2, \\
58 \; \text{Norwich Surprise} & \quad a' = xxybxzxexxyc, \\
P^4B^2P^3B.
\end{align*}
\]
(1) \( a = (23)(45), \quad p = z = (34)(56): \)

51 Morning Star Treble Bob  \( a' = xyxcxzxcxyxc, \)
67 York Surprise  \( a' = xexbxzxbxbxe, \)
68 Carlisle Surprise  \( a' = yxebxzxexxbd, \)
69 Chester Surprise  \( a' = yxebxzxzxbxyd, \)
71 Durham Surprise  \( a' = xexbxzxexbxc, \)
72 Westminster Surprise  \( a' = xexbxzxzexxe, \)

\[ P^4 B^2 P^3 B. \]

(2) \( a = (26)(35), \quad p = c = (23)(45): \)

46 Killamarsh Treble Bob  \( a' = yxyxzzxcxyxe, \)
47 Snowdon Treble Bob  \( a' = yxyxzxzxyxe, \)
48 London Scholars Pleasure Treble Bob  \( a' = yxyxzzxxyxd, \)
49 Sandal Treble Bob  \( a' = yxyxzzxzxd, \)
59 Annable’s London Surprise  \( a' = xexbxzxexbxe, \)

\[ P^4 B^2 P^2 BP. \]

(3) \( a = (25)(46), \quad p = c = (23)(45): \)

55 Oswald Delight  \( a' = yxycxzxexxbd, \)
56 Kentish Delight  \( a' = yxycxzxzxbxd, \)
60 Primrose Surprise  \( a' = xexbxzxexbd, \)

\[ P^4 B^2 P^3 B \quad \text{or} \quad P^2 B^4 P^3 B. \]

(3r) \( a = (24)(36), \quad p = z = (34)(56): \)

52 Old Oxford Delight  \( a' = xyxbxzxxzxd, \)
53 College Bob IV Delight  \( a' = xyxbxzxxxyd, \)
54 Southwark Delight  \( a' = exebzxzcbxe, \)
64 Ipswich Surprise  \( a' = xexbxzxxbxc, \)
65 London Surprise  \( a' = exebzxzxbxe, \)
66 Wells Surprise  \( a' = exebzxzxbxc, \)

\[ P^4 B^2 P^3 B \quad \text{or} \quad P^2 B^4 P^3 B. \]
(4) \(a=(24)(36), p=c=(23)(45)\):

\[P^4BP^2BPB \text{ or } PB^4PB^2PB.\]

(There are no extents of this form in [12].)

(4r) \(a=(25)(46), p=z=(34)(56)\):

57 Francis Genius Delight \(a'=xexczxexbxc,\)
61 Cambridge Surprise \(a'=xexbxzexbxd,\)
62 Beverley Surprise \(a'=xexbxzexbxyd,\)
63 Surfleet Surprise \(a'=xexbxzexbxzd,\)

\[P^4BP^2BPB \text{ or } PB^4PB^2PB.\]

5. Extending the analysis

It is curious that the 'reverse' version of category (4) appears in Diagrams [12], but that no category (4) itself is required in Section 4. However, in [13] we find 152 Treble Dodging Minor methods in all, including all 29 of [12]. These include 29 Treble Bob, 77 Delight, and 46 Surprise methods, and they are grouped that way in [13]. (The distinction arises from the number of cross sections — the treble crossing from positions 1 and 2 over to 3 and 4 or from 3 and 4 to 5 and 6 — in a half lead having internal places made: zero, one, and two times respectively.) In Table 2 we count the total number of these methods in each of our eight categories. Note now that category (4) is represented. Thus the analysis of Section 4 produces all possible callings of the form studied for the additional 123 extents as well. Of course, what differentiates the methods in each category is the choice of the word of length 23 that produces the fixed

<table>
<thead>
<tr>
<th>Category</th>
<th>Number of methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>7</td>
</tr>
<tr>
<td>(1r)</td>
<td>17</td>
</tr>
<tr>
<td>(2)</td>
<td>24</td>
</tr>
<tr>
<td>(2r)</td>
<td>31</td>
</tr>
<tr>
<td>(3)</td>
<td>16</td>
</tr>
<tr>
<td>(3r)</td>
<td>20</td>
</tr>
<tr>
<td>(4)</td>
<td>12</td>
</tr>
<tr>
<td>(4r)</td>
<td>25</td>
</tr>
<tr>
<td>Total</td>
<td>152</td>
</tr>
</tbody>
</table>

Table 2
Counting the Treble Dodging Minor methods in each category
a for that category. We do not trouble to list the 123 additional words here, but they (and their category) can be readily ascertained from [13].

We consider instead the following problem: what other variants, if any, of Treble Dodging Minor methods might be possible? Two questions arise naturally in this connection:

(1) So far, the bob has always been \( b=(23)(56) \) (making fourth place); might some other bob be used?

(2) So far, exactly five of the 60 elements of \( A_5 \) have been used for \( a \); are others possible?

To study these questions, we begin by recalling (from Theorem 2.1 of [6]) that the number of possible generating transitions for \( n \) bells is \( F(n+1)-1 \), where \( F \) gives the Fibonacci sequence; for \( n=6 \), we see that twelve transitions are possible. Established practice (as in [13]) prohibits single transpositions for Treble Dodging methods in general (this makes it easier to meet condition (iv) of Section 2), so only the seven generators, \( x, y, z, b, c, d, e \) of Section 4 remain for Treble Dodging Minor methods. The generators \( p \) and \( b' \) (here we temporarily allow the possibility that \( b' \neq b \)) that produce the plain and bob leads \( P=ap \) and \( B=ab' \) must both be in \( A_5 \), regarded as a subgroup of the stabilizer \( (S_5)_1 \). Thus \( \{ p, b' \} \subseteq \{ z, b, c \} \).

Next, we write \( a=wmw^{-1} \), where \( w \) is a word in \( x, y, z, b, c, d, \) and \( e \) of length eleven, and \( m \) (the middle transition) must fix the treble in the sixth position; that is \( w(6)=1 \) and \( m-6 \). Thus \( m \in \{ c, d, e \} \). (We check that \( a(1)=wmw^{-1}(1)=1 \), as required.) But from \( a=wmw^{-1} \), we see that \( a \) is conjugate to \( m \) and must therefore have the same cycle structure: \( (-\_)(\_\_\_) \). In \( A_5 \), there are exactly \( 5 \cdot (4^2)/2-15 \) elements of this form; these are the candidates for \( a \).

Once we determine \( \Delta = \{ a, p, b' \} \), an analysis of the Schreier right coset graph \( S_4(A_5/Z_3) \) will find any and all extents of the form \( [W(P, B)]^3 \). Technically, a bob changes the 'coursing order' of exactly three (working) bells. We illustrate this for Oxford Treble Bob Minor in Table 3, by comparing the plain lead \( ap=ac \) with the bob lead \( ab'=ab \) and another lead (non existent for this method) \( az \). (Each lead here has commenced with rounds.) We see that, with respect to the unaltered order \( 4, 2, 6, 3, 5 \) in the continuing plain course, the bob \( b \) has affected bells 3, 5 and 6. However, \( z \) has affected (again with respect to the plain course) all five working bells and hence is unsuitable as a bob. (Moreover, we could never take \( b'=a \), for then \( \Delta = \{ a, p \} \) would not generate a connected Schreier right coset graph.) Now, it is easy to check that \( b \) can serve as a bob for either \( p=c \) or \( p=z \), as in Section 4, regardless of

| 124365 | 124365 | 124365 |
| 142635 | 142356 | 123456 |

Table 3
The nature of a bob
the value of \(a\). (In each such case, if \(b\) is a bob for \(p\), then also \(p\) is a bob for \(b\), and any extent \([W(P, B)]\) becomes an extent \([W(B, P)]\) if \(ab\) has order 5) and conversely (if \(ap\) has order 5); this possibility is allowed — and realized — in \([14]\) and \([15]\), but not in \([12]\) or \([13]\).) It is also easy to check that if \(p = c\), \(z\) is not a bob; if \(p = z\), \(z\) is not a bob. Thus, by convention, we cannot take \(\{p, b'\} = \{z, c\}\). It is interesting to note that for each of the five values of \(a\) employed in Section 4 (representing the 29 methods in \([12]\) and, in fact, all 152 in \([13]\)), the Schreier right coset graph \(S_5(A_5/Z_2)\), with \(\Delta = \{a, z, c\}\), consists of two (cubic) components of order ten. Thus no extent exists. As is often the case, what at first glance seems an arbitrary requirement of change-ringing lore (here requiring a bob to affect three bells, not five) has real mathematical significance to the composition being attempted.

Thus it is with very good reason that we continue to fix \(b' = b = (23)(56)\). Necessarily, then, \(p \in \{z, c\}\). Now we turn our attention to the word \(a\).

We see from Table 2 that for each choice of \(a\) from \(S = \{(34)(56), (25)(46), (26)(35), (24)(36), (23)(45)\}\), we find extents using \(\Delta = \{a, p, b\}\) for \(p \in \{z, c\}\) — except that we cannot take \(p = a\). Since both \(c\) and \(z\) are in \(S\), we find exactly \(5 \cdot 2 - 2 = 8\) suitable ordered couples \((a, p)\); these are the eight categories, paired by reverse, that we have been studying. In each case \(ap\) has order five (that is, is a 5-cycle); this insures that condition (v) of Section 1 will be met. Of the ten remaining candidates for \(a\), three of them \((36)(45), (23)(56),\) and \((25)(34)\) do not produce 5-cycles with either \(z\) or \(c\). (Of course, \((23)(56)\) is also ruled out by the requirement that \(a \neq b\).) The other seven produce four pairs (by reverse) of ordered couples \((a, p)\) as candidates for generating extents (along with the fixed \(b\)). We can represent the pairs by fixing \(p = c\); then \(a,c = \{(35)(46), (24)(56), (25)(36), (26)(34)\}\). The other four couples \((a, p)\) are the reverse of these. (One value of \(a\), \((25)(36)\), appears twice in the eight couples, as it is self-reverse and thus pairs with both \(z\) and \(c\) (which are reverse with each other).) We naturally ask the following questions:

1. Is there a mathematical reason prohibiting extents of Treble Dodging Minor methods using the eight ordered couples \((a, p)\) just described?

2. Is there a musical reason prohibiting such extents?

We provide a negative answer to (1) by constructing, for each of the eight pairs \((a, p)\), both a word \(wmw^{-1} = a\) having all the required properties and, utilizing the appropriate Schreier right coset graph, all possible callings \([W(P, B)]\) that will produce an extent of Treble Dodging Minor. The same four graphs already employed suffice for these eight pairs as well: the graphs for \((1), (2), (3), (4)\) (and their respective reverses) in Section 4 are isomorphic to those used, respectively, for \((6), (7), (5), (8)\) (and their respective reverses) below. We use the same notation as in Section 4 (noting that \(a' = wm\))

\[a = (35)(46), \quad p = c = (23)(45):\]

\[
\text{Delight} \quad a' = dxycxzxbxz, \\
P^4B^2P^3B \text{ or } P^2B^3PB^4.
\]
We remark that there are really ten compositions above, as the word $a'$ for (7) can also be used for (7r) and vice versa.

There are no doubt many other formulations of the word for $a$ for each of these eight categories. Here we are content to settle question (1) above. Each of the ten compositions is guaranteed to satisfy conditions (i), (ii), and (iii) of Section 1, by the theory of Section 3. We have constructed $a = wmw^{-1}$ to meet condition (vi), and we have chosen $p$ so as to satisfy (v). Thus only (iv) remains in doubt. For our ten
compositions, (iv) holds uniformly, except that: sixth place is made three times at the plain course connection for (5), (6), (7), and (7) using $a'$ from (7r).

In fact, all six of these compositions satisfying (iv) appear in [15], which describes all 2400 Treble Dodging Minor methods satisfying (iv) — including some using single transpositions as transitions, some using $p=b$, and some using $d$ at other than the half-lead (all are prohibited in [13]). The compositions (5r), (6r), (7r), and (8r) above are, respectively. Bradpole Delight, Longwood Treble Bob, Gloucester Old Spot Treble Bob, and Dearn Valley Surprise. Compositions (7r) — using $a'$ from (7) — and (8) are, respectively, Nos. 2292 and 1498 in [15]; both are as yet un-named. Thus the answer to question (2) above would appear to be negative.

However, ringers prefer that the bells stay close to their natural coursing order (see below), as in Plain Bob Minor (the most basic method on six bells). This gives a plain course regarded as being musical and also simplifies matters for the conductor; it will occur if the lead heads in the plain course (of a Treble Dodging Minor method) agree with those of Plain Bob Minor. The mathematical expression of this condition is that $a$ be in the subgroup of $A_5$ (regarded as the stabilizer of 1 in $S_6$) generated by $c=(23)(45)$ and $z=(34)(56)$._labelling the vertices of a regular pentagon by $2, 4, 6, 5, 3$ successively (this is called the coursing order for the working bells in the plain course of Plain Bob Minor), we see that $(c, z) = D_5$, the dihedral group of order 10 giving all symmetries of the pentagon. (The product $cz$ generates the plain course lead heads of Plain Bob Minor.) This group contains exactly five involutions (the five reflections), readily seen to be $c=(23)(45)$, $z=(34)(56)$, $cz=(25)(46)$, $zcz=(24)(36)$, and $zczcz=czczc=(26)(35)$. These are exactly the five values of $a$ we encountered in Section 4! Thus, perhaps, the answer to question (2) is affirmative, after all.

We note that there are exactly eight additional categories (as determined by the pairs $(a, p)$) — making twenty four in all — obtained by taking $p=b$ (as discussed above; in six cases this is actually an exchange of $p$ and $b$, keeping a fixed, and requires that $ab$ (as well as $ap$) be a 5-cycle). This occurs for the previous categories (3), (3r) (4) and (3r) now agree, as do (4r) and (3)), (5), (5r), (8), and (8r). In each case, the second word $W(P, B)$ given would then be interpreted as the standard calling (after the interchange). This gives six new categories, say (9), (9r), (10), (10r), (11), and (11r) respectively. The final two categories, say (12) and (12r), occur for $p=b$ (still) and $a=(36)(45)$ or $(25)(34)$, respectively. Recall that these two values of $a$ were prohibited for $p=c$ or $z$, since then $ap$ is not a 5-cycle; however $ab$ is a 5-cycle in either case. Bob $b = c$ is usually employed here, but $b = z$ will serve also (see [15]). In [15] we find ample instances of all twenty four categories, except none of categories (9), (9r), (10r) or (11r) — and only some of categories (10) and (11) — meeting the Central Council of Church Bell Ringers' requirement (see [15]) that a plain lead using $p=b$ not be a bob lead for $p=c$ or $z$. (This is apparently relaxed, if $p=z$ does not produce a 5-cycle and $p=c$ leads to a violation of condition (iv).)

It is natural to ask if the twenty four categories exactly coincide with the twenty four 5-cycles possible in $A_5$, regarded as the stabilizer of 1 in $S_6$, via the 5-cycle produced
by \( ap \) (the second lead head of the plain course (the first is rounds)) in each case. But this is \textit{not} the situation. In fact, the four 5-cycles generated by \((23456)\) \textit{never} occur as lead heads in any plain course. Two other 5-cycles ((24356) and (26453)) appear as lead heads \((ap)^3\), but never as \( ap \).

We close by observing that a composition can only be named by the band who first ring it to an extent and by giving, in Table 4, the first lead of the (un-named) Delight Minor extent corresponding to \((8)\).

### Acknowledgements

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References