A Semi-Analytical Approximation to the Block Error Rate in Nakagami-$m$ Block Fading Channels

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Abstract

One of the main performance measures of any packet based communication system is the success rate of the packets. A packet is successfully received if the arrival of the packet is detected correctly, header is received successfully, and payload is decoded without errors. At the moment, the probability of failure to decode the payload, i.e., block error rate (BLER), is often obtained by numerical methods, such as Monte-Carlo simulations due to shortage of analytical expressions for BLER.

In this work, an approximation method for estimating the BLER in Nakagami-\(m\) block-fading channels is proposed. Furthermore, an analytical expression with a single parameter is derived for Nakagami-\(m\) channels. This parameter can be obtained by single simulation at a single signal to noise ratio (SNR) in Rayleigh fading channel. Our simulation results demonstrates that the proposed methods is highly accurate is estimating the BLER of many important wireless technologies.
1 Introduction

Block fading channels introduced in [1] are channels where several symbols (a block of symbols) are exposed to the same channel gain. This channel model is applicable in systems based on orthogonal frequency division multiplexing (OFDM) within a time-frequency grid where the channel is constant. The size of this time-frequency grid mainly depends on the mobility of the nodes. This channel model is also relevant in systems with slow time-frequency hopping such as Bluetooth or GSM. Furthermore, the fading channel in many wireless sensor networks (WSNs) where the network consists of low mobility sensor nodes equipped with a narrow band radio can be accurately modelled as a block fading channel.

This work is focused on a family of block fading channels in which the variations of the channel gain between different blocks, in the time scale of interest, can be accurately modelled by the Nakagami-\(m\) probability distribution function (pdf). The Nakagami-\(m\) (\(m \geq 0.5\)) is chosen as it has been shown to be a good fit to a wide variety of empirical data [2,3]. By varying the \(m\) parameter various fading conditions can be generated. For instance, for \(m = 1\) the Nakagami channel simplifies to a Rayleigh fading channel. The channels with larger \(m\) have less severe fading conditions. In the extreme case, the Nakagami channel reduces to AWGN channel (no fading) as \(m\) goes to infinity.

In this work, the problem of finding an analytical approximation for the BLER in Nakagami-\(m\) block fading channels is considered. Although the analytical solution to the BLER has been found in few special cases such as non-coherent FSK modulated linear block codes in block fading Rayleigh channels [4], for reasons which will be explained in Sec. 2, there is a lack of analytical expressions for the BLER for a wide range of modulation, coding, decoding methods, and channel models. As a result, numerical methods such as Monte-Carlo simulations are often used with much added cost and computational complexity. The number of required simulations is equal to all the different combination of the physical layer parameters (i.e. modulations, codings, decoding methods, and block sizes), SNR points, and channel models. Thus, the number of required simulations can be potentially prohibitive, specially in the design phase where large set of physical layer parameters and channel models are considered. Hence, there is a need for methods that can reduce both the number of required simulations and their complexity [5–7]. In addition, storing, processing, and accessing these large sets of numerical data is not trivial. These issues gain in importance when BLER is incorporated into larger problems such as
link level performance optimization where it becomes highly advantageous to have an analytical formula for the BLER.

The analytical approximation method employed here is based on a threshold method introduced in [8,9] which maps the instantaneous BLER to 0 or 1 based on a threshold SNR. This approximation method, explained in details in Sec. 2, results in a simple analytical formula for the BLER estimate in many channel models. In this report, the validity of this method is examined for various block and convolutional codes in Nakagami-m block fading channels.

The rest of this report is organized as follows. In Sec. 2, the approximation method is explained and an analytical formula is derived for Nakagami-m channels. A heuristic method to find the SNR threshold is discussed in Sec. 3. In Sec. 4, the accuracy of the proposed method is examined by extensive simulations. The conclusion is presented in Sec. 5.

2 BLER approximation

The block diagram of the system considered in this work is shown in Fig. 1. A block of \( k \) information bits, \( X \), is encoded with block codes or convolutional codes. The resulting \( n \) coded bits are interleaved using a random block interleaver and modulated by an \( M \)-ary Gray-coded QAM-modulator (\( M = 2, 4, 16, 64 \)). The resulting block of \( q \) symbols is transmitted through a Nakagami-m block fading channel. A block of received information bits, \( \hat{X} \), is generated at the receiver by demodulating, deinterleaving and decoding the received symbols. A block is declared as an error if \( X \neq \hat{X} \).
In block fading channels, the SNR is fixed during the \( q \) consecutive transmitted symbols. Therefore, for each block, the channel can be modelled as an AWGN channel with the SNR adjusted by the current channel gain. Let \( E_b \) represents the average received energy per information bit and \( N_0/2 \) represents the noise spectral density. For notational convenience, let the received average (information bit) SNR, \( E_b/N_0 \) be denoted by \( \overline{\gamma} \). Now, the instantaneous SNR (per block) denoted by \( \gamma \) is given by

\[
\gamma = \kappa^2 \overline{\gamma}
\]

(1)

where \( \kappa \) is a unit-power Nakagami-\( m \) distributed fading gain (channel gain). As a result, \( \gamma \) is distributed according to the Gamma distribution with the probability distribution function, denoted by \( f_\gamma(\gamma; \overline{\gamma}, m) \), which is parameterized by \( \overline{\gamma} \) and \( m \) and is given by

\[
f_\gamma(\gamma; \overline{\gamma}, m) = \frac{1}{\Gamma(m)} \left( \frac{m}{\overline{\gamma}} \right)^m \gamma^{(m-1)} \exp(-m\gamma/\overline{\gamma}) \]

(2)

where \( \Gamma(m) \) is the Gamma function. Consequently, if the BLER in AWGN channel is represented by \( P_a(\gamma) \), the BLER in Nakagami-\( m \) block fading channel is given by

\[
P_f(\overline{\gamma}; m) = \Pr\{\hat{X} \neq \hat{X} \} = \int_0^\infty P_a(x) f_\gamma(x; \overline{\gamma}, m) dx
\]

(3)

where \( P_f(\overline{\gamma}; m) \) is the BLER in Nakagami-\( m \) block fading channel at given \( \overline{\gamma} \), parameterized by \( m \). Note that (3) is valid regardless of how fast the fading varies from a block to block as long as distribution of \( \gamma \) (during the time scale of interest) measures up with the Gamma distribution.

As mentioned in the introduction, (3) has been solved analytically only in a few special cases. One reason is that (3) becomes mathematically intractable for many combinations of physical layer parameters and channel models. For instance, \( P_a(\gamma) \) for a block of binary modulated coded bits capable of correcting \( t \) coded-bit errors with a hard-decision bounded distance decoder [10] is given by

\[
P_a(\gamma) = 1 - \sum_{i=0}^{t} \binom{n}{i} p(\gamma)^i [1 - p(\gamma)]^{n-i}
\]

(4)

where \( n \) is the block size in bits and \( p(\gamma) \) is the coded-bit error probability. In case of binary modulations, (3) is solved only in the case of non-coherent FSK in block Rayleigh fading channels [4]. Solving (3) in a more general channel models is still an open problem. In addition, non-binary modulation complicates the problem even
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Further as often different bit positions in one modulated symbol have different bit error probabilities. For instance, a Gray-coded 16-QAM symbol has two different bit error probabilities [11].

Secondly, there is lack of exact analytical expressions for \(P_a(\gamma)\) for a large number of modulation, coding, and decoding methods [12]. More often, only an analytical upper bound on \(P_a(\gamma)\) is known. In these cases, finding the exact analytical solution to (3) is not feasible.

One way to be able to obtain an analytical expression for \(P_f(\tau; m)\) is to replace \(P_a(\gamma)\) with a simplified approximation. In this report, \(P_a(\gamma)\) is replaced with a simple approximation based on a threshold method defined as

\[
\hat{P}_a(\gamma; \Theta) = \begin{cases} 
1, & \text{if } \gamma \leq \Theta \\
0, & \text{if } \gamma > \Theta 
\end{cases} \tag{5}
\]

where \(\Theta\) is the SNR threshold and \(\hat{P}_a(\gamma; \Theta)\) is the approximation of \(P_a(\gamma)\) parameterized by \(\Theta\). Thus, the approximation of BLER is given by

\[
\hat{P}_f(\tau; m, \Theta) = \int_0^\infty \hat{P}_a(x; \Theta) f_\gamma(x; \tau, m) dx = \int_0^\Theta f_\gamma(x; \tau, m) dx = F_\gamma(\Theta; \tau, m) \tag{6}
\]

where \(F_\gamma(x; \tau, m)\) is the cumulative distribution function (CDF) of \(\gamma\) and is given by

\[
F_\gamma(\gamma; \tau, m) = \frac{\Gamma(m, m\gamma/\tau)}{\Gamma(m)} \tag{7}
\]

Here, \(\Gamma(m, x)\) is the lower part incomplete Gamma function which is defined as

\[
\Gamma(m, x) = \int_0^x t^{m-1} e^{-t} dt \tag{8}
\]

Hence, by combining (6) and (7), \(\hat{P}_f(\tau; m, \Theta)\) can be written as

\[
\hat{P}_f(\tau; m, \Theta) = \frac{\Gamma(m, m\Theta/\tau)}{\Gamma(m)} \tag{9}
\]

A simpler expression for \(\hat{P}_f(\tau; m, \Theta)\) can be obtained for the special case of integer \(m\) where \(\Gamma(m, x)\) can be simplified to

\[
\Gamma(m, x) = \Gamma(m) \left[1 - e^{-x} \sum_{k=0}^{m-1} \frac{x^k}{k!}\right] \tag{10}
\]

In this case, we have

\[
\hat{P}_f(\tau; m, \Theta) = 1 - e^{-\frac{m\Theta}{\tau}} \sum_{k=0}^{m-1} \frac{(\frac{m\Theta}{\tau})^k}{k!} \tag{11}
\]
In a given Nakagami-\(m\) channel, the only parameter needed for this approximation is \(\Theta\) which depends on the physical layer parameters. In the next section, a method for finding \(\Theta\) is discussed.

## 3 How to find \(\Theta\)

In work by Rodrigues et al. [8], \(\Theta\) is set as the iterative decoder convergence threshold. This method, however, is only applicable to turbo-coded blocks. In general, the optimum value of \(\Theta\) in a given range of \(\tau\), e.g., \(\tau_a \leq \tau \leq \tau_b\), can be found by minimizing an error measure. One example of such an error measure is

\[
\varepsilon(\Theta) = \max_{\tau_a \leq \tau \leq \tau_b} \left| \frac{P_f(\tau; m) - \hat{P}_f(\tau; m, \Theta)}{P_f(\tau; m)} \right|
\]  

(12)

which measures the maximum absolute relative deviation of the approximation from the correct value within a given range of \(\tau\). The optimum \(\Theta\) based on this error definition is given by

\[
\Theta_{opt} = \min_{\Theta} \max_{\tau_a \leq \tau \leq \tau_b} \left| \frac{P_f(\tau; m) - \hat{P}_f(\tau; m, \Theta)}{P_f(\tau; m)} \right|
\]  

(13)

As (13) demonstrates, finding \(\Theta_{opt}\) could be achieved only if \(P_f(\tau; m)\) is known which renders \(\hat{P}_f(\tau; m, \Theta)\) redundant. Thus for the proposed approximation method to be beneficial, a simple method of finding \(\Theta\) is required. We propose a heuristic suboptimal method to adjust \(\Theta\) by forcing \(P_f(\tau; m)\) and \(\hat{P}_f(\tau; m, \Theta)\) to have a common point. Therefore, \(\Theta\) is found as a solution to

\[
\hat{P}_f(\tau_c; m, \Theta) = P_f(\tau_c; m)
\]  

(14)

where \(P_f(\tau_c)\) is obtained by simulation at the common point, \(\tau_c\). Choosing \(\tau_c\) optimally is, again, a complex optimization problem which depends on many parameters. Heuristically, however, one can simply choose \(\tau_c\) in the middle of the range of interest.

## 4 Simulation results

In this section, the accuracy of the proposed approximation is examined by comparing the BLER estimates with results obtained from extensive simulations. To ensure the accuracy of \(P_f(\tau; m)\), 100,000 random blocks were generated at each \(\tau\) point. Both block codes and convolutional codes are considered. The details of the simulation environment and the results for the block codes and convolutional codes are presented respectively in Sec. 4.1 and 4.2. Simulated scenarios are presented in Table 1. These scenarios are chosen such that a variety of modulations, error correction capabilities, and block sizes are examined.
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Table 1: The simulation scenarios for both block codes (BC) and convolutional codes (CC).

<table>
<thead>
<tr>
<th>Sce.</th>
<th>(n)</th>
<th>Modulation</th>
<th>Coding method</th>
<th>(R = k/n)</th>
<th>(\Theta) [dB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1024</td>
<td>BPSK</td>
<td>BC ((t = 0))</td>
<td>1</td>
<td>7.22</td>
</tr>
<tr>
<td>2</td>
<td>3200</td>
<td>16-QAM</td>
<td>BC ((t = 20))</td>
<td>0.95</td>
<td>5.53</td>
</tr>
<tr>
<td>3</td>
<td>4800</td>
<td>QPSK</td>
<td>BC ((t = 50))</td>
<td>0.92</td>
<td>1.28</td>
</tr>
<tr>
<td>4</td>
<td>2400</td>
<td>64-QAM</td>
<td>BC ((t = 100))</td>
<td>0.75</td>
<td>5.78</td>
</tr>
<tr>
<td>5</td>
<td>1024</td>
<td>16-QAM</td>
<td>CC (23, 35), (d_{\text{free}} = 7)</td>
<td>1/2</td>
<td>1.40</td>
</tr>
<tr>
<td>6</td>
<td>3200</td>
<td>BPSK</td>
<td>CC (133, 171), (d_{\text{free}} = 10)</td>
<td>1/2</td>
<td>2.24</td>
</tr>
<tr>
<td>7</td>
<td>4800</td>
<td>QPSK</td>
<td>CC (133, 171, 165), (d_{\text{free}} = 15)</td>
<td>1/3</td>
<td>-1.02</td>
</tr>
<tr>
<td>8</td>
<td>2400</td>
<td>64-QAM</td>
<td>CC (51, 55, 67, 77), (d_{\text{free}} = 18)</td>
<td>1/4</td>
<td>2.32</td>
</tr>
</tbody>
</table>

4.1 Block Codes

In this section, the accuracy of our proposed approximation is examined for several different block codes with different error correction capabilities. Only hard-decision decoding methods are considered as the low-complexity decoding methods for classical block codes, such as Hamming, BCH, and Reed-Solomon codes, are almost always based on hard bits [12]. For nonbinary modulations, hard bits are obtained by making a hard-decision on the soft bits outputs of the Log-MAP algorithm [12]. For a given \(t\) and \(n\), the number of information bits per block, \(k\), (and the code rate \(k/n\)) is found as a largest integer value that satisfies the Hamming bound [12]. That is to say \(k\) is given by

\[
k = \left\lfloor \log_2 \left( \frac{2^n}{\sum_{i=0}^{t} \binom{n}{i}} \right) \right\rfloor \tag{15}
\]

where \([x]\) is the largest integer that is smaller or equal to \(x\). While the block code for these values of \(n\), \(k\), and \(t\) might not necessarily exist, this method provides a reasonable approximation to a good block code at the given \(n\) and \(k\). At the receiver, a block is considered to be successfully received if the number of coded-bit errors is less than or equal to \(t\). The block diagram of the simulated system is shown in Fig. 2.

Each of simulation scenarios given in Table 1 were simulated in different Nakagami-\(m\) block fading channels for \(m\) from 1 to 7. To find \(\Theta\), (14) is solved at \(\gamma_c = 15\) dB. Interestingly, our results indicated that there is little variations in the value of \(\Theta\) as a function
of $m$. Therefore, $\Theta$ obtained in Rayleigh channels ($m = 1$) is used to evaluate and plot $\hat{P}_f(\tau; m, \Theta)$ for all other values of $m$. These values of $\Theta$ are also given in Table 1. For block codes, results are presented in Fig. 3-6. As it can be observed from these figures, there is a good match between $P_f(\tau; m)$ and $\hat{P}_f(\tau; m, \Theta)$.
4.2 Convolutional Codes

In this section, the accuracy of the $\hat{P}_f(\tau; m, \Theta)$ is examined for the convolutional codes in Table 1. For each of these codes, which have maximum free distance and good distance spectrum properties, see [13], the generator polynomials and the free distance are given. In accordance to common practice, the generator polynomials are
presented in octal format. The soft-decision Viterbi decoder is used. At the encoder, enough zero tail bits are added to ensure that the trellis ends at all-zero state. The soft bits required by the Viterbi are generated at the receiver by the Log-MAP algorithm. The interleaver is a random block interleaver with the size equal to $n$. The simulations, see Fig. 7, were performed in C++ with the help of the IT++ library [14].

![Figure 7: Block diagram of the C++ based simulator for convolutional codes.](image)

The results for scenarios 5 to 8 are presented in Fig. 8-11. Sim-
ilar to block codes, Θ exhibited little sensitivity to m. Therefore, Θ obtained from a single simulation at $\tau_c = 15$ dB in a Rayleigh fading channel is used in all the figures. As these figures clearly demonstrate, $\hat{P}_f(\tau; m, \Theta)$ provides a highly accurate approximation also in the case of convolutional codes.

![Figure 8: $\hat{P}_f(\tau; m, \Theta)$ (solid line) compared to simulations (circles) for scenario 5 in Nakagami-m channels.](image)

![Figure 9: $\hat{P}_f(\tau; m, \Theta)$ (solid line) compared to simulations (circles) for scenario 6 in Nakagami-m channels.](image)
4 Simulation results

Figure 10: $\hat{P}_f(\tau; m, \Theta)$ (solid line) compared to simulations (circles) for scenario 7 in Nakagami-$m$ channels.

Figure 11: $\hat{P}_f(\tau; m, \Theta)$ (solid line) compared to simulations (circles) for scenario 8 in Nakagami-$m$ channels.

4.3 Discussion

The fact that $\Theta$ is weakly correlated to $m$ offers a possibility for a significant reduction in number of required simulations. For instance, for a given set of physical layer parameters, a single simulation in a Rayleigh fading channel is sufficient to have access to the analytical approximation of the BLER in a wide range of
Nakagami-\(m\) channels.

It is also notable that \(\hat{P}_f(\tau; m, \Theta)\) becomes less accurate with an increase in \(m\). This is expected as in the limit (\(m \to \infty\)), Nakagami-\(m\) tends to AWGN channel where the proposed method would provide a poor estimate (only 0 or 1).

5 Conclusion

In this work, the difficulties of obtaining an easy-to-use analytical expression for the BLER in block fading Nakagami channels are discussed. In addition, it was argued that obtaining the BLER using Monte-Carlo method requires large number of simulations with much added complexity. Furthermore, the storing and accessing these numerical results are not trivial.

To address these shortcomings, a method to obtain a single parameter analytical approximation is examined. This single parameter depends on physical layer parameters such as modulation, coding, or block size. A heuristic method of finding this parameter with a single simulation is proposed and its accuracy is examined by extensive simulations. These results demonstrated that the proposed method is highly accurate in Rayleigh block fading channels. In Nakagami-\(m\) channels, the accuracy of the BLER estimate reduces with an increase in \(m\) but our simulations indicates that it is highly accurate for a wide range of parameters \(m\). In addition, this method offers a substantial reduction in number of required simulation for obtaining the BLER estimate.

References


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References


