On the Asymptotic Behaviour of Some Towers of Function Fields over Finite Fields*

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0. INTRODUCTION

Let $F/F$ be an algebraic function field of one variable, whose constant field is the finite field of cardinality $l$. Weil’s theorem states that the number $N = N(F)$ of places of degree one of $F/F$ satisfies the estimate

$$N \leq l + 1 + 2g \sqrt{l}, \quad (0.1)$$

where $g = g(F)$ denotes the genus of $F$. It is well known that for $g$ large with respect to $l$, the Weil bound (0.1) is not optimal; see [5, 9]. Drinfeld and Vladut [1] proved the following asymptotic result: Let

$$N_l(g) := \max \{ N(F) \mid F \text{ is a function field over } \mathbb{F}_l \text{ of genus } g \},$$

and

$$A(l) := \limsup_{g \to \infty} N_l(g)/g. \quad (0.2)$$

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Then one has the so-called Drinfeld–Vladut bound:

\[ A(l) \leq \sqrt{l} - 1. \]  

(0.3)

Note that the Weil bound (0.1) gives only the much weaker estimate 

\[ A(l) \leq 2 \sqrt{l}. \]  

If \( l = q^2 \) is a square, the inequality (0.3) is in fact an equality: Ihara [5] and Tsfasman et al. [10] proved that

\[ A(q^2) = q - 1. \]  

(0.4)

Their proof requires deep results from algebraic geometry and modular curves; one shows that certain modular curves have sufficiently many rational points over \( \mathbb{F}_{q^2} \). In a recent paper [3], we gave an explicit construction of function fields \( E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \) over \( \mathbb{F}_q \) such that \( \lim_{n \to \infty} N(E_n)/g(E_n) = q - 1 \), thereby giving a simpler and more elementary proof of Eq. (0.4).

For \( l \) being a non-square, the Weil bound (0.1) can be improved by the Serre bound

\[ N \leq l + 1 + g \cdot [2 \sqrt{l}], \]  

(0.5)

where \( [2 \sqrt{l}] \) denotes the integer part of \( 2 \sqrt{l} \). The exact value of \( A(l) \) is unknown in this case. Using classfield towers, Serre proved that \( A(l) \geq c \cdot \log l \) with a constant \( c > 0 \). The best known lower bounds for small values of \( l \) are \( A(2) \geq 2/9 \), \( A(3) \geq 1/3 \), and \( A(5) \geq 1/2 \), see [6–11].

In this paper, we study the asymptotic behaviour of towers of function fields over \( \mathbb{F}_q \); i.e., we consider sequences \( F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \) of function fields \( F_i/\mathbb{F}_q \), and we are interested in the behaviour of the ratios \( N(F_i)/g(F_i) \), for \( i \to \infty \). Clearly,

\[ \limsup_{i \to \infty} N(F_i)/g(F_i) \leq A(l), \quad \text{if} \quad g(F_i) \to \infty. \]

In Section 1, we will introduce notation and recall some facts from the theory of algebraic function fields. In Section 2, we show that the sequence \( (N(F_i)/g(F_i))_{i=1} \) is convergent, for any tower \( F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \) over \( \mathbb{F}_q \), with \( g(F_i) \to \infty \). Moreover, we give sufficient conditions for such a sequence to have zero as its limit, and also sufficient conditions implying a non-zero limit.

We give, in Section 3, a new explicit example of a tower of function fields over \( \mathbb{F}_q \) that attains the Drinfeld–Vladut bound (0.3). This new tower is in some sense simpler than the tower in [3].

Finally, in Section 4, two interesting towers over \( \mathbb{F}_q \) (\( l \) being a non-square) are discussed, for which the limit \( \lim_{i \to \infty} N(F_i)/g(F_i) \) turns out to be zero.
1. PRELIMINARIES

Throughout this paper, we will use the following notation:

- $\mathbb{F}_l$ the finite field of cardinality $l$.
- $E, F, F_1, \ldots$ algebraic function fields of one variable over $\mathbb{F}_l$.
- $g(F)$ the genus of $F/\mathbb{F}_l$.
- $\mathbb{P}(F)$ the set of places of $F/\mathbb{F}_l$.
- $N(F)$ the number of places $P \in \mathbb{P}(F)$ of degree one.
- $v_P$ the normalized discrete valuation associated with $P$.

For $x, y, z \in F$ and $P \in \mathbb{P}(F)$, we write

$$x = y + O(z) \text{ at } P,$$

if $x = y + t \cdot z$ with $v_P(t) \geq 0$. In particular, $x = y + O(1)$ means that $v_P(x - y) \geq 0$.

Let $E/F$ be a separable extension of function fields (over $\mathbb{F}_l$), $P \in \mathbb{P}(F)$, and $P' \in \mathbb{P}(E)$ be a place of $E$ lying above $P$. Then we denote

- $e(P' \mid P)$ the ramification index of $P' \mid P$.
- $d(P' \mid P)$ the different exponent of $P' \mid P$.
- $\text{Diff}(E/F)$ the different of $E/F$.

The Hurwitz genus formula states that in this situation,

$$2g(E) - 2 = [E : F] \cdot (2g(F) - 2) + \deg \text{Diff}(E/F).$$

We recall some well-known facts about Artin–Schreier extensions of function fields, cf. [9, Chap. III.7].

**Proposition 1.1.** Suppose that $F/\mathbb{F}_q^2$ is an algebraic function field over $\mathbb{F}_q^2$. Let $w \in F$ and assume there exists a place $P \in \mathbb{P}(F)$ such that $v_P(w) = -m$, $m > 0$, and $\gcd(m, q) = 1$.

Let $E = F(z)$, where $z$ satisfies the equation

$$z^q + z = w.$$ 

Then, the following holds:

(i) $[E : F] = q$, and $\mathbb{F}_q^2$ is algebraically closed in $E$. 

(ii) The place \( P \) is totally ramified in \( E \); i.e., there is exactly one place \( P' \in \mathbb{P}(E) \) lying above \( P \), and \( e(P' | P) = q \). Moreover, \( \deg P' = \deg P \), and the different exponent of \( P' | P \) is given by
\[
d(P' | P) = (q - 1)(m + 1).
\]

(iii) Let \( R \in \mathbb{P}(F) \) and assume that
\[
w = u^q + u + o(1)
\]
at \( R \), for some element \( u \in F \). Then, the place \( R \) is unramified in \( E/F \). In particular, this is the case if \( v_R(w) \) is non-negative.

(iv) Suppose that the place \( Q \in \mathbb{P}(F) \) is a zero of \( w = \# \), with \( \# \in \mathbb{F}_q \). The equation \( x^q + x = \# \) has \( q \) distinct roots \( x \in \mathbb{F}_{q^2} \), and for any such \( x \) there exists a unique place \( Q_x \in \mathbb{P}(E) \) such that \( Q_x \) lies above \( Q \), and \( Q_x \) is a zero of \( z - x \); in particular, the place \( Q \) splits completely in \( E \).

We will also need some results about ramification in composita of function fields:

**Proposition 1.2.** Let \( E/F \) be a separable extension of function fields over \( \mathbb{F}_q \). Assume that \( H_1, H_2 \) are intermediate fields of \( E/F \) such that \( E = H_1 \cdot H_2 \). For a place \( P' \in \mathbb{P}(E) \), let \( P_1 \in \mathbb{P}(H_1) \) be the restriction of \( P' \) to \( H_1 \) \((i = 1, 2)\), and let \( P \in \mathbb{P}(F) \) be the restriction of \( P' \) to \( F \). Suppose that \( e(P_1 | P) \) and \( e(P_2 | P) \) are relatively prime. The following hold:

(i) \( e(P' | P) = e(P_1 | P) \cdot e(P_2 | P) \).

(ii) If \( P_1 | P \) is tame (i.e., \( e(P_1 | P) \) is prime to the characteristic of \( \mathbb{F}_q \)), then
\[
d(P' | P_1) = e(P_1 | P) \cdot d(P_2 | P) - (e(P_1 | P) - 1)(e(P_2 | P) - 1).
\]

Proof. (i) This is a special case of Abhyankar's Lemma, see [9, Prop. III.8.9].

(ii) Since \( P_1 | P \) and \( P_2 | P \) are tame, one has \( d(P_1 | P) = d(P' | P_2) = e(P_1 | P) - 1 \). Using the transitivity of the different exponent (cf. [9, Cor. III.4.11]), one obtains
\[
d(P' | P) = d(P' | P_2) + e(P_2 | P) \cdot (e(P_1 | P) - 1)
\]
\[
= e(P_1 | P) - 1 + e(P_1 | P) \cdot d(P_2 | P).
\]
The assertion follows.
Definition 1.3. A tower of function fields over $\mathbb{F}_l$ is a sequence $\mathcal{F} = (F_1, F_2, F_3, \ldots)$ of function fields $F_i/\mathbb{F}_l$, having the following properties:

(i) $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$.

(ii) For each $n \geq 1$, the extension $F_{n+1}/F_n$ is separable of degree $[F_{n+1} : F_n] > 1$.

(iii) $g(F_j) > 1$, for some $j \geq 1$.

Note that condition (iii) and the Hurwitz genus formula imply that $g(F_n) \to \infty$ for $n \to \infty$.

Definition 1.4. Let $\mathcal{F} = (F_1, F_2, F_3, \ldots)$ be a tower of function fields over $\mathbb{F}_l$. Another such tower $\mathcal{E} = (E_1, E_2, E_3, \ldots)$ is said to be a subtower of $\mathcal{F}$ (written $\mathcal{E} \subset \mathcal{F}$), if there exists an embedding (over $\mathbb{F}_l$)

$$
i : \bigcup_{i \geq 1} E_i \to \bigcup_{i \geq 1} F_i.$$

In other words, for any $i \geq 1$ there is an index $m = m(i) \geq 1$ such that $i(E_i) \subseteq F_m$.

2. Remarks on the Asymptotic Behaviour

In this section, we put together some simple observations on the behaviour of the sequence $(N(F_i)/g(F_i))_{i \geq 1}$, where $\mathcal{F} = (F_1, F_2, F_3, \ldots)$ is a tower of function fields over $\mathbb{F}_l$.

Lemma 2.1. Let $E/F$ be a finite extension of function fields over $\mathbb{F}_l$. Assume that $g(F) > 1$. Then,

$$\frac{N(E)}{g(E) - 1} \leq \frac{N(F)}{g(F) - 1}.$$

Proof. There is an intermediate field $F \subseteq H \subseteq E$ such that $H/F$ is separable and $E/H$ is purely inseparable of degree $q = p^r$, where $p = \text{char}(\mathbb{F}_l)$ and $r \geq 0$. Then $H = E^{p^r}$ is isomorphic to $E$ (see [9, Prop. III.9.2]), so $N(H) = N(E)$ and $g(H) = g(E)$. The Hurwitz genus formula for $H/F$ gives

$$g(H) - 1 = [H : F] \cdot (g(F) - 1) + \frac{1}{2} \deg \text{Diff}(H/F)$$

$$\geq [H : F] \cdot (g(F) - 1).$$
Any place \( P \in \mathbb{P}(F) \) of degree one has at most \([H : F]\) extensions \( P' \in \mathbb{P}(H) \) of degree one, hence

\[
N(H) \leq [H : F] \cdot N(F).
\]

It follows that

\[
\frac{N(E)}{g(E) - 1} = \frac{N(H)}{g(H) - 1} \leq \frac{[H : F] \cdot N(F)}{(g(F) - 1)} = \frac{N(F)}{g(F) - 1}.
\]

**Corollary 2.2.** For any tower \( \mathcal{F} = (F_1, F_2, F_3, ... ) \) of function fields over \( \mathbb{F}_q \), the sequence

\[
(N(F_i)/g(F_i))_{i \geq 1}
\]

is convergent.

**Proof.** We can assume that \( g(F_i) > 1 \) for all \( i \) (see Definition 1.3 (iii)). By Lemma 2.1, the sequence

\[
(N(F_i)/(g(F_i) - 1))_{i \geq 1}
\]

is monotonously decreasing, hence convergent. Since \( g(F_i) \to \infty \) for \( i \to \infty \), the sequence \( (N(F_i)/(g(F_i) - 1))_{i \geq 1} \) is also convergent, and

\[
\lim_{i \to \infty} N(F_i)/g(F_i) = \lim_{i \to \infty} N(F_i)/(g(F_i) - 1).
\]

**Definition 2.3.** For a tower \( \mathcal{F} = (F_1, F_2, F_3, ...) \) of function fields over \( \mathbb{F}_q \), let

\[
\lambda(\mathcal{F}) := \lim_{i \to \infty} N(F_i)/g(F_i).
\]

The tower \( \mathcal{F} \) is said to be asynptotically good (resp. asymptotically bad) if \( \lambda(\mathcal{F}) > 0 \) (resp., \( \lambda(\mathcal{F}) = 0 \)).

It is obvious that \( \lambda(\mathcal{F}) \leq A(l) \) (see (0.2)); therefore we call the tower \( \mathcal{F} \) optimal if \( \lambda(\mathcal{F}) = A(l) \).

**Corollary 2.4.** Let \( \mathcal{F} \) be a tower of function fields over \( \mathbb{F}_q \), and let \( \mathcal{E} \subset \mathcal{F} \) be a subtower. Then, the following hold:

(i) \( \lambda(\mathcal{E}) \geq \lambda(\mathcal{F}) \).

(ii) If \( \mathcal{E} \) is asymptotically bad, then \( \mathcal{F} \) is also asymptotically bad.

(iii) If \( \mathcal{F} \) is optimal, then \( \mathcal{E} \) is also optimal.

**Proof.** Follows easily from Lemma 2.1. □
Proposition 2.5. Let $\mathcal{F} = (F_1, F_2, F_3, \ldots)$ be a tower of function fields over $\mathbb{F}_q$. Suppose that $(\rho_2, \rho_3, \rho_4, \ldots)$ is a sequence of real numbers with the following properties:

(a) $\rho_2 > 0$.
(b) $\rho_{n+1} \leq \deg \text{Diff}(F_{n+1}/F_n)$, for all $n \geq 1$.
(c) $\rho_{n+1} \geq [F_{n+1}: F_n] \cdot \rho_n$, for all $n \geq 2$.

Then one has:

(i) There is a constant $\rho > 0$ such that, for all $n \geq 1$,
\[
g(F_{n+1}) - 1 \geq [F_{n+1}: F_1](g(F_1) - 1 + \rho \cdot n).
\]

(ii) $\lambda(\mathcal{F}) = 0$; i.e., the tower $\mathcal{F}$ is asymptotically bad.

Proof. We abbreviate $D_{n+1} := \deg \text{Diff}(F_{n+1}/F_n)$. The assumption (c) implies, by induction, that $\rho_{n+1} \geq [F_{n+1}: F_n] \cdot \rho_n$. By transitivity (see [9, Cor. III.4.11]), the degree of the different of $F_{n+1}/F_1$ is given by
\[
D_{n+1} + [F_{n+1}: F_n] D_n + \cdots + [F_{n+1}: F_2] D_2.
\]

Now the Hurwitz genus formula for the extension $F_{n+1}/F_1$ yields
\[
2g(F_{n+1}) - 2 = [F_{n+1}: F_1] (2g(F_1) - 2) + \deg \text{Diff}(F_{n+1}/F_1)
\geq [F_{n+1}: F_1] (2g(F_1) - 2) + \sum_{i=1}^{n} [F_{n+1}: F_i] \rho_{i+1}
\geq [F_{n+1}: F_1] (2g(F_1) - 2) + \sum_{i=1}^{n} [F_{n+1}: F_2] \rho_2
= [F_{n+1}: F_1] (2g(F_1) - 2) + \frac{\rho_2}{[F_2: F_1]} [F_{n+1}: F_1] \cdot n.
\]

Setting $\rho := \rho_2/[F_2: F_1]$, one obtains the desired inequality
\[
g(F_{n+1}) - 1 \geq [F_{n+1}: F_1](g(F_1) - 1 + \rho n).
\]

Since $N(F_{n+1}) \leq [F_{n+1}: F_1] \cdot N(F_1)$, the assertion
\[
\lambda(\mathcal{F}) = \lim_{n \to \infty} N(F_{n+1})/(g(F_{n+1}) - 1) = 0
\]
follows immediately from (i).
**Remark 2.6.** The conclusions of Proposition 2.5 hold also if condition (c) is replaced by the slightly weaker condition

\[(c') \quad \rho_{n+1} \geq [F_{n+1}: F_2] \cdot \rho_2, \quad \text{for all} \quad n \geq 2.\]

The following simple criterion shows that, under certain conditions, the tower \( \mathcal{F} \) is asymptotically good.

**Proposition 2.7.** Let \( \mathcal{F} = (F_1, F_2, F_3, \ldots) \) be a tower of function fields over \( \mathbb{F}_1 \). Suppose that

\[
\deg \operatorname{Diff}(F_{n+1}/F_n) \leq \varepsilon \cdot [F_{n+1}: F_n] \cdot \deg \operatorname{Diff}(F_n/F_{n-1})
\]

holds for all \( n \geq 2 \), where \( \varepsilon \) is a constant satisfying

\[0 \leq \varepsilon < 1.\]

Moreover, suppose that there exists a non-empty set \( S \subseteq \mathbb{P}(F_1) \) of places of degree one of \( F_1/\mathbb{F}_1 \) such that any \( F \in S \) splits completely in all extensions \( F_n/F_1 \). Then, the tower \( \mathcal{F} \) is asymptotically good. More precisely, one has

\[
\lambda(\mathcal{F}) \geq \frac{2(1-\varepsilon)[F_2:F_1]}{\deg \operatorname{Diff}(F_2/F_1) + (1-\varepsilon)[F_2:F_1](2g(F_1) - 2)},
\]

if \( \deg \operatorname{Diff}(F_2/F_1) + (1-\varepsilon)[F_2:F_1](2g(F_1) - 2) > 0 \).

**Proof.** We set \( D_{n+1} := \deg \operatorname{Diff}(F_{n+1}/F_n) \), for all \( n \geq 1 \). Assume first that \( D_2 + (1-\varepsilon)[F_2:F_1](2g(F_1) - 2) \) is strictly positive. The assumption \( \rho_{n+1} \leq \varepsilon \cdot [F_{n+1}: F_n] \cdot D_n \) implies that the inequality

\[
D_{i+1} \leq \varepsilon^{i-1} \cdot [F_{i+1}: F_2] \cdot D_2
\]

holds for each \( i \geq 2 \). As in the proof of Proposition 2.5, one obtains therefore

\[
2g(F_{n+1}) - 2 = [F_{n+1}: F_1](2g(F_1) - 2) + \deg \operatorname{Diff}(F_{n+1}/F_1)
\]

\[
= [F_{n+1}: F_1](2g(F_1) - 2) + \sum_{i=1}^{n} [F_{n+1}: F_{i+1}] \cdot D_{i+1}
\]

\[
\leq [F_{n+1}: F_1](2g(F_1) - 2) + \sum_{i=1}^{n} [F_{n+1}: F_2] \varepsilon^{i-1} \cdot D_2
\]

\[
= [F_{n+1}: F_1] \left( 2g(F_1) - 2 + \frac{D_2}{[F_2:F_1]} \frac{1 - \varepsilon^n}{1 - \varepsilon} \right)
\]

\[
\leq [F_{n+1}: F_1] \left( 2g(F_1) - 2 + \frac{D_2}{(1-\varepsilon)[F_2:F_1]} \right).
\]
Since \( N(F_{n+1}) \geq [F_{n+1} : F_1] \cdot \# S \), it follows that
\[
\lambda(\mathcal{F}) \geq \frac{2 \cdot \# S}{2g(F_1) - 2 + \frac{D_2}{(1 - \varepsilon)[F_2 : F_1]}} = \frac{2(1 - \varepsilon)[F_2 : F_1] \cdot \# S}{D_2 + (1 - \varepsilon)[F_2 : F_1](2g(F_1) - 2)}.
\]

Now if \( D_2 + (1 - \varepsilon)[F_2 : F_1](2g(F_1) - 2) \leq 0 \), we replace \( \mathcal{F} \) by the subtower \( \mathcal{F}' := (F_j, F_{j+1}, F_{j+2}, \ldots) \), where \( j \) is chosen such that \( g(F_j) > 1 \). Applying the same arguments as in the beginning of the proof to the tower \( \mathcal{F}' \), one concludes that \( \mathcal{F}' \) and, a fortiori, the tower \( \mathcal{F} \) is asymptotically good.

Remark 2.8. In Proposition 2.7, the assumption \( D_{n+1} \leq \varepsilon \cdot [F_{n+1} : F_n] \cdot D_n \) can be replaced by the weaker condition \( D_{n+1} \leq \varepsilon^{-1} [F_{n+1} : F_n] \cdot D_n \), for all \( n \geq 2 \).

It is in general hard to find asymptotically good towers of function fields. For instance, if \( F = (F_1, F_2, F_3, \ldots) \) is a tower of abelian extensions of \( F_1 \) (i.e., all extensions \( F_n/F_1 \) are Galois with Abelian Galois groups \( \text{Gal}(F_n/F_1) \)), then \( \mathcal{F} \) is asymptotically bad, see \([2]\).

**Example 2.9 (Classfield Towers).** Classfield theory can be used to construct asymptotically good towers of function fields, see for example \([6, 7]\). We recall this method briefly: Let \( C_1/F_1 \) be a function field of genus \( g(C_1) > 1 \) and let \( S_1 \) be a non-empty set of places of degree one of \( C_1 \). Let \( C_2 \) be the maximal unramified abelian extension of \( C_1 \) such that all places \( P \in S_1 \) split completely in \( C_2 \subset C_1 \), and let \( S_2 \) denote the set of all places of \( C_2 \) lying above some place of \( S_1 \). Iterating this construction, one obtains a sequence \( C_1 \subset C_2 \subset C_3 \subset \cdots \). Assuming that \( C_{n+1} \neq C_n \) for all \( n \geq 1 \), Proposition 2.7 yields for the tower \( \mathcal{E} = (C_1, C_2, C_3, \ldots) \) that
\[
\lambda(\mathcal{E}) \geq \frac{\# S_1}{g(C_1) - 1}.
\]

A crucial step in this classfield tower construction is to show that for specific choices of \( C_1 \) and \( S_1 \), the assumption \( C_{n+1} \neq C_n \) does hold for each \( n \geq 1 \). Note that classfield towers are not “explicit”; i.e., one does not have an explicit description of the function fields in terms of generators and equations.

**Example 2.10.** In a previous paper \([3]\), we considered the tower \( \mathcal{E} = (E_1, E_2, E_3, \ldots) \) of function fields over \( \mathbb{F}_q \): Let \( E_1 := \mathbb{F}_q(x_1) \) be the rational function field and, for \( n \geq 1 \), let \( E_{n+1} := E_n(z_{n+1}) \), where \( z_{n+1} \) satisfies the equation
\[
z_{n+1}^q + z_{n+1} = x_n^q \quad \text{with} \quad x_n := z_n/x_{n-1} \quad \text{(for} \ n \geq 2 \text{).} \]
Letting \( E'_n := E_{2n-1} \), one obtains a tower \( \mathcal{E}' = (E'_1, E'_2, E'_3, \ldots) \) with the following properties (see [3]):
\[
[E'_{n+1} : E'_n] = q^2 \quad \text{and} \quad \deg \text{Diff}(E'_{n+1}/E'_n) = 2q^2(q + 2)(q - 1).
\]

Moreover, all places in the set
\[ S := \{ P \in \mathbb{P}(E'_1) \mid \deg P = 1, \text{ and } P \text{ is neither the zero nor the pole of } x_1 \} \]
split completely in \( E'_{n+1}/E'_1 \). Applying Proposition 2.7 (with \( \varepsilon := q^{-1} \)), one obtains
\[
\lambda(\mathcal{E}') \geq \frac{2(1 - q^{-1}) \cdot q^2 \cdot (q^2 - 1)}{2q(q + 2)(q - 1) + (1 - q^{-1}) \cdot q^2 \cdot (-2)} = q - 1.
\]

It follows from the Drinfeld–Vladut theorem that \( \lambda(\mathcal{E}) = q - 1 \). Since \( \mathcal{E} \ll \mathcal{E}' \), this implies that \( \lambda(\mathcal{E}) = q - 1 \), by Corollary 2.4.

3. A NEW TOWER ATTAINING THE DRINFELD–VLADUT BOUND OVER \( \mathbb{F}_{q^2} \)

Explicit examples of asymptotically good towers are of high interest for coding theory, since they can be used for the explicit construction of asymptotically good families of codes, cf. [9, 10]. So far, the only known explicit tower with \( \lambda(\mathcal{F}) > 0 \) is the tower \( \mathcal{E} \) given in Example 2.10. Now we introduce another example.

We consider here the tower \( \mathcal{F} = (T_1, T_2, T_3, \ldots) \) of function fields over \( \mathbb{F}_{q^2} \) given by \( T_n := \mathbb{F}_{q^2}(x_1, \ldots, x_n) \), with
\[
x_{i+1}^q + x_{i+1} = \frac{x_i^q}{x_i^q + 1}, \quad \text{for} \quad i = 1, \ldots, n - 1. \tag{3.1}
\]

The main result of this section is:

**Theorem 3.1.** The tower \( \mathcal{F} \) as defined in (3.1) attains the Drinfeld–Vladut bound over \( \mathbb{F}_{q^2} \); i.e.,
\[
\lambda(\mathcal{F}) = q - 1.
\]

Let
\[
\Omega := \{ x \in \mathbb{F}_{q^2} \mid x^q + x = 0 \} \quad \text{and} \quad \Omega^* := \Omega \setminus \{ 0 \} = \{ x \in \mathbb{F}_{q^2} \mid x^{q-1} = -1 \}.
\]

The tower \( \mathcal{F} \) considered here is in fact a subtower of the tower \( \mathcal{E} \) of Example 2.10; see Remark 3.11.
We will need the following lemma:

**Lemma 3.2.** Let $F = \mathbb{F}_q(y, z)$ be defined by the equation

$$z^q + z = \frac{y^q}{y^{q-1} + 1}.$$  \hspace{1cm} (3.2)

Then, the following assertions hold:

(i) $[F : \mathbb{F}_q(y)] = [F : \mathbb{F}_q(z)] = q$.

(ii) The function $y$ has a unique pole $P_\infty$ in $F$, and this place $P_\infty$ is totally ramified in $F/\mathbb{F}_q(y)$.

(iii) For any $\alpha \in \Omega^*$, the function $(y - \alpha)$ has a unique zero $P_\alpha$ in $F$, and this place $P_\alpha$ is totally ramified in $F/\mathbb{F}_q(y)$.

(iv) For any $\gamma \in \Omega$, there is a unique common zero $Q_\gamma$ of $y$ and $z - \gamma$ in $F$.

(v) The principal divisors in $F$ of the functions $(y - \alpha)$ and $(z - \gamma)$ (with $\alpha, \gamma \in \Omega$) are as follows:

$$(y) = \sum_{\gamma \in \Omega} Q_\gamma - qP_\infty,$$

$$(y - \alpha) = q - qP_\alpha, \quad \text{for } \alpha \in \Omega^*,$$

$$(z - \gamma) = qQ_\gamma - P_\alpha - \sum_{\alpha \in \Omega^*} P_\alpha, \quad \text{for } \gamma \in \Omega.$$

(vi) The places of $F$ that are ramified over $\mathbb{F}_q(y)$ are exactly the places $P_\infty$ and $P_\alpha$, with $\alpha \in \Omega^*$. Their different exponents with respect to the extension $F/\mathbb{F}_q(y)$ are

$$d(P_\infty) = d(P_\alpha) = 2(q - 1).$$

(vii) The places of $F$ that are ramified over $\mathbb{F}_q(z)$ are exactly the places $Q_\gamma$, with $\gamma \in \Omega$.

**Proof.** The assertions (i)--(vi) follow immediately from Proposition 1.1 and Eq. (3.2). In order to prove (vii), let $w := y^{-1}$. Then, $F = \mathbb{F}_q(z, w)$ with

$$w^q + w = \frac{1}{z^q + z} = \frac{1}{\prod_{\gamma \in \Omega} (z - \gamma)}.$$

By Proposition 1.1, the ramified places in $F/\mathbb{F}_q(z)$ are exactly the zeros of $z - \gamma$, for $\gamma \in \Omega$. \hfill \Box
From now on, we investigate the tower $F = (T_1, T_2, T_3, ...)$ over $F_q$ that is defined by Eq. (3.1).

**Lemma 3.3.**  
(i) $[T_n : F_q(x_i)] = q^{n-1},$ for $i = 1, ..., n.$  
(ii) Let $P \in P(T_n)$ be a pole of $x_i$ or a zero of $x_i - \alpha$, for some $\alpha \in \Omega^*.$ Then $P$ is a pole of $x_2, ..., x_n.$ The place $P$ is totally ramified in the extension $T_n/T_1,$ and it is unramified in $T_n/F_q(x_n).$ The different exponent $d(P)$ of $P$ with respect to the extension $T_n/T_{n-1}$ is given by $d(P) = 2(q-1).$  
(iii) Let $R \in P(T_n)$ be a place which is neither the pole of $x_1,$ nor a zero of $x_1 - \alpha$, for all $\alpha \in \Omega = \Omega^* \cup \{0\}.$ Then $R$ is unramified in $T_n/T_1.$

**Proof.** Consider a place $P \in P(T_n)$ which is either a pole of $x_1$ or a zero of $x_1 - \alpha,$ for some $\alpha \in \Omega^*.$ Then, the restriction of the place $P$ is ramified in $T_2/T_1$ with ramification index $q,$ and it is a simple pole of $x_2$ in $T_2$ (by Lemma 3.2(v)). The function $x_2$ has a unique pole in $F_q(x_2, x_3),$ and this place is a simple pole of $x_3,$ hence unramified in $F_q(x_2, x_3)/F_q(x_3)$ (by Lemma 3.2(v) again). Repeating this argument one sees that $P$ is a pole of the functions $x_2, x_3, ..., x_n,$ and that the ramification indices of the restrictions of $P$ in the extensions $F_q(x_1, x_i+1)$ over $F_q(x_i),$ resp. over $F_q(x_i+1),$ are as in Fig. 1.

The following assertions follow easily from this figure and Proposition 1.2:

(a) The ramification index of $P$ with respect to the extension $T_n/T_1$ is $q^{n-1}.$
(b) $[T_n : F_q(x_i)] = q^{n-1},$ for $i = 1, ..., n.$
(c) $P$ is unramified over $F_q(x_n).$

![Figure 1](image_url)
Now we determine the different exponent of $P$ in the extension $T_n/T_{n-1}$. For $n = 2$, this was done already in Lemma 3.2(vi), hence we assume that $n \geq 3$. Let $P'$ be the restriction of $P$ to $T_{n-1}$ and $\tilde{P}$ be the restriction of $P$ to $F_q(x_{n-1}, x_n)$. From assertion (c) above, the place $P'$ is unramified over $F_q(x_{n-1})$. By Lemma 3.2(vi), the different exponent of $\tilde{P}$ in $F_q(x_{n-1}, x_n)/F_q(x_{n-1})$ is $d(\tilde{P}) = 2(q-1)$, and it now follows from Proposition 1.2 that the different exponent of $P|P'$ is also equal to $2(q-1)$.

So we have proved parts (i) and (ii) of Lemma 3.3.

Next we consider a place $R \in P(T_n)$ which is neither the pole of $x_1$ nor a zero of $x_1$, for all $x \in \mathbb{F}_q$. It follows from Lemma 3.2(v), by induction, that $R$ is neither a pole of $x_i$ nor a zero of $x_i$, for $i = 1, \ldots, n$ and for all $x \in \mathbb{F}_q$. Now Lemma 3.2(vii) shows that the restrictions of $R$ are unramified in all extensions $F_q(x_i, x_{i+1})/F_q(x_i)$ and $F_q(x_i, x_{i+1})/F_q(x_{i+1})$; hence $R$ is unramified in $T_n/F_q(x_i)$, for $i = 1, \ldots, n$.

Our aim is to calculate the degree of the different Diff($T_n/T_{n-1}$), for all $n \geq 2$. By the previous lemma, it remains to investigate the behaviour of the zeros of $x_1$ in the extension $T_n/T_{n-1}$. From Lemma 3.2, one has the following possibilities for such a place $Q \in P(T_n)$:

(a) The place $Q$ is a common zero of the functions $x_1, \ldots, x_n$.

(b) There is some $t$, $1 \leq t < n$, such that
   (b1) $Q$ is a common zero of $x_1, \ldots, x_t$.
   (b2) $Q$ is a zero of $(x_t + 1)^*$.
   (b3) $Q$ is a common pole of $x_{t+2}, \ldots, x_n$.

(Note that condition (b2) implies both (b1) and (b3).)

In case (a), the places below $Q$ are unramified in the extensions $F_q(x_t, x_{t+1})/F_q(x_t)$, for $j = 1, \ldots, n-1$. This implies that $Q$ is unramified in $T_n/T_{n-1}$.

In case (b), the ramification indices for the restrictions of the place $Q$ in the extensions $F_q(x_t, x_{t+1})$ over $F_q(x_t)$ (resp. over $F_q(x_{t+1})$), for $t-1 \leq i \leq t+2$, are as in Fig. 2 (as follows from Lemma 3.2(v)).

<table>
<thead>
<tr>
<th>$q$</th>
<th>$q$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_q(x_{t-1})$</td>
<td>$F_q(x_t)$</td>
<td>$F_q(x_{t+1})$</td>
</tr>
<tr>
<td>$F_q(x_{t+2})$</td>
<td>$F_q(x_{t+3})$</td>
<td></td>
</tr>
</tbody>
</table>

$x_{t-1} = 0$, $x_t = 0$, $x_{t+1} = \alpha$, $x_{t+2} = \infty$, $x_{t+3} = \infty$

Figure 2
From this figure, one cannot deduce the ramification indices of $Q$ in the extension $T_n/F_q(x_j)$, for all $j$. For instance, what is the ramification index of the restriction of $Q$ in $F_q(x_1, x_2, x_j)/F_q(x_j)$? The determination of these ramification indices is done in the next lemma, which is central for the proof of Theorem 3.1.

**Lemma 3.4** (See Fig. 3). For $1 \leq k \leq t$, let $E_k := F_q(x_{i+1-k}, \ldots, x_{i+k})$ and $H_k := E_k(x_{i+1,k})$. Suppose that $Q \in \mathcal{P}(H_k)$ is a zero of $(x_{i+1} - \alpha)$, for some $\alpha \in \Omega^*$. Then, the place $Q$ is unramified in the extension $H_k/E_k$.

**Proof.** The circled one in Fig. 3 is the content of this lemma. The other ramification indices in Fig. 3 are then obtained, by diagram chasing, from the circled ones. Now, we start with the proof: For a place $Q$ as in the statement of the lemma, we will show by induction on $k$:

the place $Q$ is unramified in $H_k/E_k$,  

(3.3)  

and

at the place $Q$, one has $x_{i+k+1} = \alpha^{r+1} x_{i+1-k} + O(1)$.  

(3.4)

(Concerning the symbol $O(\ldots)$, see Section 1.)

---

**Figure 3**

```
\begin{tikzpicture}
  \node (top) at (0,0) {$H_2$};
  \node (e1) at (-1,-2) {$E_1$};
  \node (e2) at (1,-2) {$E_2$};
  \node (h1) at (0,-4) {$H_1$};
  \node (bottom1) at (-2,-6) {$F_p(x_{i+1})$};
  \node (bottom2) at (2,-6) {$F_p(x_{i+2})$};
  \node (bottom3) at (0,-6) {$F_p(x_{i+3})$};

  \draw (top) -- (e1) node[midway, above] {$1$};
  \draw (top) -- (e2) node[midway, above] {$1$};
  \draw (e1) -- (h1) node[midway, above] {$q$};
  \draw (e1) -- (bottom1) node[midway, above] {$q$};
  \draw (e2) -- (h1) node[midway, above] {$q$};
  \draw (e2) -- (bottom2) node[midway, above] {$q$};
  \draw (h1) -- (bottom3) node[midway, above] {$1$};

  \node at (-2,0) {$x_{i-1} = 0$};
  \node at (2,0) {$x_{i+2} = \infty$};
  \node at (0,0) {$x_{i+1} = \alpha$};
  \node at (0,-1) {$x_{i+3} = \infty$};
\end{tikzpicture}
```
For $k = 1$, one has $E_1 = F_q(x_t, x_{t+1})$ and $H_1 = E_q(x_{t+1})$. The place $Q$ is a common zero of $x_t$ and $(x_{t+1} - \alpha)$, and one has at such a place:

\[(x_{t+1} - \alpha)^q + (x_{t+1} - \alpha) = x_{t+1}^q + x_{t+1}^{q-1} + \mathcal{C}(x_t^q),\]

It follows that

\[x_{t+1} - \alpha = x_t^q(1 - x_t^{q-1} + \mathcal{C}(x_t^q)) - (x_{t+1} - \alpha)^q
\]

\[= x_t^q(1 - x_t^{q-1} + \mathcal{C}(x_t^q)),\]

and hence

\[\frac{1}{x_{t+1} - \alpha} = x_t^{-q}(1 + x_t^{q-1} + \mathcal{C}(x_t^q)) = x_t^{-q} + x_t^{-1} + \mathcal{C}(1).\]  

(3.5)

On the other hand, at the place $Q$ holds

\[x_t^q + x_{t+1} = \frac{(x_{t+1} - \alpha)^q + x_t^q}{x_{t+1}^q + 1} = \frac{x_t^q}{x_{t+1}^q + 1} + \mathcal{C}(1).\]  

(3.6)

Write $x_{t+1}^q + 1 = (x_{t+1} - \alpha) \cdot h(x_{t+1})$, where $h(x_{t+1})$ is a polynomial of degree $q - 2$. Differentiating this equation, one obtains that

\[-x_{t+1}^q = h(x_{t+1}) + (x_{t+1} - \alpha) \cdot h'(x_{t+1}),\]

and hence $h'(x) = -\alpha^{q-2}$. It now follows that

\[\frac{x_t^q}{x_{t+1}^q + 1} - \frac{x_t^q + 1}{x_{t+1} - \alpha} = x_t^q \frac{1 - \alpha \cdot h'(x_{t+1})}{x_{t+1}^q + 1} = \mathcal{C}(1)\]

(3.7)

at the place $Q$, since $1 - \alpha \cdot h'(x) = 1 - \alpha \cdot (-\alpha^{q-2}) = 1 + \alpha^{q-1} = 0$. From

(3.6), (3.7), and (3.5), one has (at $Q$):

\[x_t^q + x_{t+1} = \frac{x_t^q + 1}{x_{t+1} - \alpha} + \mathcal{C}(1)\]

\[= \frac{x_t^{q+1}}{x_t^q + 1} + \mathcal{C}(1).\]

Since $(x^{q+1})^q = x^{q+1}$, one concludes that

\[\left( x_{t+1} - \frac{x_t^{q+1}}{x_t} \right)^q + \left( x_{t+1} - \frac{x_t^{q+1}}{x_t} \right) = \mathcal{C}(1).\]  

(3.8)
As \( H_1 = E_1(x_{i+2}) \), it follows from (3.8) and Proposition 1.1(iii) that the place \( Q \) is unramified in the extension \( H_1/E_1 \). Moreover, the assertion (3.4) (for \( k = 1 \)) follows immediately from (3.8).

Let now \( k \geq 2 \). At a place \( Q \in \mathbb{V}(H_k) \) which is a zero of \( x_{i+1} - \alpha \) (with \( \alpha \in \Omega^\times \)), one has

\[
x_{i+1+k} + x_{i+k+1} = \frac{x_{i+k}^q}{x_{i+k}^{q+1} + 1} = \frac{x_{i+k}^q}{1 + (x_{i+k}^q)^q} = x_{i+k} + \epsilon(1) \quad (3.9)
\]

Hence it follows that

\[
x_{i+1+k} + x_{i+k+1} = x_{i+k} + \epsilon(1) \quad (3.9)
\]

(Note that the place \( Q \) is a zero of the function \( x_{i+k}^{-1} \).) On the other hand, one has also at such a place:

\[
x_{i+1+k} + x_{i+k+1} = \frac{x_{i+k}^q}{x_{i+k}^{q+1} + 1} = x_{i+k}^q(1 - (x_{i+k}^{-1})^q + \epsilon(x_{i+k}^{-q}))
\]

and hence

\[
x_{i+1+k} + x_{i+k+1} = x_{i+k}^q(1 - (x_{i+k}^{-1})^q + \epsilon(x_{i+k}^{-q}))
\]

(Note that the place \( Q \) is a zero of \( x_{i+k}^{-1} \).) Therefore,

\[
x_{i+1+k} = x_{i+1+k}^{-1}(1 + x_{i+k}^{-1} + \epsilon(x_{i+k}^{-1})).
\]

This means one has at \( Q \):

\[
x_{i+1+k}^{-1} = x_{i+1+k}^{-1} + x_{i+k}^{-1} + \epsilon(1). \quad (3.10)
\]

By induction hypothesis (i.e., formula (3.4) for \( k - 1 \)), one has

\[
x_{i+k} = \alpha^{q+1}x_{i+2+k} + \epsilon(1). \quad (3.11)
\]

Combining (3.9), (3.10), and (3.11), one obtains

\[
x_{i+1+k} + x_{i+k+1} = x_{i+k} + \epsilon(1)
\]

\[
= \alpha^{q+1}x_{i+2+k} + \epsilon(1)
\]

\[
= \alpha^{q+1}x_{i+1+k} + \alpha^{q+1}x_{i+k+1} + \epsilon(1).
\]
The conclusion is that, at such a place $Q$, the following holds:

$$(x_{t+k+1} - x_{t+1}^{-1} x_{t+1-k}) y + (x_{t+k+1} - x_{t+1}^{-1} x_{t+1-k}) = \mathcal{E}(1).$$

This finishes the proof of the lemma.

**Lemma 3.5.** Let $1 \leq t < n$ and $Q \in \mathbb{P}(T_n)$ be a place having the following properties:

- $Q$ is a common zero of $x_1, ..., x_t$;
- $Q$ is a zero of $x_{t+1} \alpha$ for some $\alpha \in \Omega^*$;
- $Q$ is a common pole of $x_{t+2}, ..., x_n$.

Then one has:

(i) If $n \leq 2t + 1$, then the place $Q$ is unramified in $T_n/T_{n-1}$.

(ii) For $2t + 1 < n$, the place $Q$ is totally ramified in $T_n/T_{2t+1}$, and for $2t + 1 \leq s \leq n$, the restriction of $Q$ to $T_s$ is unramified in the extension $T_s/F_{q^2}(x_s)$.

(iii) If $2t + 1 < n$, the different exponent $d(Q)$ of $Q$ in the extension $T_n/T_{n-1}$ is given by $d(Q) = 2(q-1)$.

**Proof.** The assertions in (i) and (ii) follow by “diagram chasing” from Figs 2 and 3 and Lemma 3.4, and assertion (iii) follows from (ii), Lemma 3.2 (vi), and Proposition 1.2 (cf. the proof of Lemma 3.3(ii)).

For $1 \leq t < (n-1)/2$ and $\alpha \in \Omega^*$, set

$$X_{t, \alpha} := \{Q \in \mathbb{P}(T_n) | Q \text{ is a zero of } x_{t+1} - \alpha \}$$

and

$$A_{t, \alpha} := \sum_{Q \in X_{t, \alpha}} Q.$$

As an immediate consequence of Lemma 3.4, Lemma 3.5(ii), and of the so-called fundamental equality \(\sum e_i f_i = n\), we have:

**Lemma 3.6.** Let $1 \leq t < (n-1)/2$ and $\alpha \in \Omega^*$. Then,

$$\deg A_{t, \alpha} = q^t.$$ 

**Lemma 3.7.** For $n \geq 2$, the degree of the different of the extension $T_n/T_{n-1}$ is given by

$$\deg \text{Diff}(T_n/T_{n-1}) = 2 \cdot (q-1) \cdot q^{[n/2]}.$$
Proof. One obtains, from Lemma 3.3 (parts ii, iii), Lemma 3.5, and Lemma 3.6, that
\[
\deg \text{Diff}(T_n/T_{n-1}) = q \cdot 2 \cdot (q-1) + \sum_{i=1}^{\lceil (n-2)/2 \rceil} q' \cdot 2 \cdot (q-1)
\]
\[
= 2q(q-1)(1+(q^{(n-2)/2}-1))
\]
\[
= 2 \cdot (q-1) \cdot q^{(n/2)}.
\]

Remark 3.8. One can easily determine the genus \(g(T_n)\), using Lemma 3.7 and the Hurwitz genus formula. The result is:
\[
g(T_n) = \begin{cases} 
(q^n - 1)^2, & \text{if } n \equiv 0 \mod 2, \\
(q^{(n+1)/2} - 1)(q^{(n-1)/2} - 1), & \text{if } n \equiv 1 \mod 2.
\end{cases}
\]
However, for the proof of Theorem 3.1 we will not need the precise value of \(g(T_n)\).

Next, we consider places of degree one in the function field \(T_n/F_q\). For \(\alpha \in F_q\), let \(R_\alpha \in \mathcal{P}(T_1)\) denote the zero of \(x_1 - \alpha\) in \(T_1\).

**Lemma 3.9.** Let \(S := \{ R_\alpha : \alpha \in \mathcal{P}(T_1) \mid \alpha \notin \Omega \}\). Then, any \(R \in S\) splits completely in all extensions \(T_n/T_1\).

**Proof.** Let \(R \in S\). The following assertions will be shown by induction on \(n\):
(a) \(R\) splits completely in \(T_n/T_1\).
(b) For any \(R' \in \mathcal{P}(T_n)\) lying above \(R\), there is some \(\alpha \in F_q \setminus \Omega\) such that \(R'\) is a zero of \(x_n - \alpha\). The case \(n = 1\) is trivial. Suppose now that (a) and (b) hold for \(n\). Let \(R' \in \mathcal{P}(T_n)\) be a place lying above \(R\), and let \(\alpha \in F_q \setminus \Omega\) be such that \(R'\) is a zero of \(x_n - \alpha\). One has \(T_{n+1} = T_n(x_{n+1})\), where
\[
x_{n+1} = x_n + x_{n+1} = \frac{x_n^2}{x_n^{2-1} + 1} = \frac{x_n^{x+1}}{x_n^x + x_n}.
\]
The residue class of the right-hand side of (3.12) at the place \(R'\) is equal to \(\gamma := x^{x+1}/(x^x + \alpha)\) (note that \(x^x + \alpha \neq 0\), as \(\alpha \notin \Omega\)). Since \(x^{x+1}\) (resp. \(x^x + \alpha\)) is the norm (resp. the trace) of \(x\) in the field extension \(F_q/F_p\), the element \(\gamma = x^{x+1}/(x^x + \alpha)\) is in \(F_q \setminus \{0\}\). It follows from Proposition 1.1(iv) that \(R'\) splits completely in the extension \(T_{n+1}/T_n\) and that, for any \(R' \in \mathcal{P}(T_{n+1})\) lying above \(R'\), there is some element \(\alpha' \in F_q \setminus \Omega\) such that \(R''\) is a zero of \(x_{n+1} - \alpha'\). This finishes the proof of the lemma. \(\square\)
Proof of Theorem 3.1. We replace the tower $\mathcal{F} = (T_1, T_2, T_3, \ldots)$ by its subtower $\mathcal{F}' = (T'_1, T'_2, T'_3, \ldots)$ where

$$T'_n := T_{2n-1}, \quad \text{for } n \geq 1.$$ 

Since $\mathcal{F}' < \mathcal{F} < \mathcal{F}'$ it follows from Proposition 2.5(i) that $\lambda(\mathcal{F}) = \lambda(\mathcal{F}')$. By the Drinfeld–Vladut bound (0.3) it is sufficient to show that

$$\lambda(\mathcal{F}') \geq q - 1. \quad (3.13)$$

Applying Proposition 2.7 to the tower $\mathcal{F}'$, one has (for $n \geq 1$) that

$$D_{n+1} := \deg \text{Diff}(T'_{n+1}/T'_n) = \deg \text{Diff}(T_{2n+1}/T_{2n-1})$$

$$= 2 \cdot (q - 1) \cdot q^{(2n+1)/2} + q \cdot 2 \cdot (q - 1) \cdot q^{(2n)/2}$$

$$= 2 \cdot (q^2 - 1) \cdot q^n,$$

as follows from Lemma 3.7 and the transitivity of the different, see [9, Cor. III.4.11]. The assumptions of Proposition 2.7 hold with $\varepsilon := q^{-1}$ and $\mathcal{S}$ as in Lemma 3.9, and one obtains the desired estimate (3.13):

$$\lambda(\mathcal{F}') \geq \frac{2 \cdot (1 - q^{-1}) \cdot q^2 \cdot (q^2 - q)}{2 \cdot (q^2 - 1) \cdot q + (1 - q^{-1}) \cdot q^2 \cdot (-2)} = q - 1.$$ 

Remark 3.10. The tower $\mathcal{F}$ of Theorem 3.1 is in a sense simpler than the tower $\mathcal{E}$ of Example 2.10; namely, the ramification structure of the extension $T_n/T_1$ (as described in Lemmas 3.3 and 3.5) is less complicated than the ramification structure of the extension $E_n/E_1$ (see [3, Sect. 2]).

Remark 3.11. After finishing the final version of this paper, we realized that the tower $\mathcal{F}$ considered in Theorem 3.1 is indeed a subtower of the tower $\mathcal{E}$ of Example 2.10. This can be shown as follows: with notation as in Example 2.10, one has

$$z_n^q + z_{n+1}^q = x_n^{q+1} = \frac{x_n^{q+1}}{x_n^{q+1} + z_n} = \frac{x_n^{q+1}}{x_n^{q+1} + z_n^q + 1}.$$ 

It follows that the subfield $\mathcal{F}_n(z_2, \ldots, z_{n+1}) \subseteq E_{n+1}$ is isomorphic to the field $T_n$ in the tower $\mathcal{F}$, and hence $\mathcal{F}$ is a subtower of $\mathcal{E}$.

This means that we have now two proofs that the tower $\mathcal{F}$ is optimal, i.e., it attains the Drinfeld–Vladut bound: the one presented here in Theorem 3.1 and the other deduced from the optimality of the tower $\mathcal{E}$ and Corollary 2.4(iii). Even though we decided to leave the paper unmodified, we believe that the arguments used in the proof of Theorem 3.1 (especially...
thinking of the tower $\mathcal{S}$ as a "pyramid," see Figs. 1–3) are more transparent and can be useful in dealing with other towers of function fields, cf. Example 4.1.

4. SOME BAD TOWERS

We do not know any explicit example of an asymptotically good tower of function fields over $\mathbb{F}_l$ when $l$ is a non-square. Hence we investigate now some towers over fields $\mathbb{F}_l$ that look promising because the function fields have “many” places of degree one and the defining equations have low degrees.

Example 4.1. The following tower $\mathcal{S} = (K_1, K_2, K_3, \ldots)$ over $\mathbb{F}_8$ was introduced by Feng and Rao (unpublished) when they attempted to construct an explicit family of asymptotically good codes over the field $\mathbb{F}_8$. Let $K_n := \mathbb{F}_8(u_1, \ldots, u_n)$, where

$$u_i^3 \cdot u_{i-1} + u_{i-1}^3 + u_i = 0,$$

(4.1)

for $i = 2, \ldots, n$. Feng and Rao call $K_n$ the function field of the “generalized Klein quartic,” since $K_2$ is the function field of the so-called Klein quartic. It is well-known that $g(K_2) = 3$ and $N(K_2) = 24 = 8 + 1 + 3 \cdot [2 \sqrt{8}]$; so $K_2$ attains the Serre bound (0.5), see [9, Example VI.3.8].

One can show that $[K_n : K_{n-1}] = 3$ holds for all $n \geq 2$, and that the places of $K_i = \mathbb{F}_8(u_1)$ which are zeros of $u_1 + x$, with $x \in \mathbb{F}_8 \setminus \{0\}$, split completely in $K_n / K_i$. Therefore $K_n / \mathbb{F}_8$ has many places of degree one.

We will prove that the tower $\mathcal{S}$ is asymptotically bad. By Corollary 2.4, it is sufficient to construct an asymptotically bad subtower $\mathcal{S} \subset \mathcal{S}$. We set $x_i := u_{i+1}^7 \cdot u_i$, for all $i \geq 1$.

From Eq. (4.1) it follows that

$$0 = (u_i^3 \cdot u_i + u_i^3 + u_{i-1}) \cdot u_{i+1}^7 = u_{i+1}^7 (1 + x_i) + x_i^3,$$

hence

$$u_{i+1}^7 = x_i^3 / (1 + x_i).$$

(4.2)

On the other hand,

$$0 = (u_{i+2}^3 u_{i+1} + u_{i+1}^3 + u_{i-2})^2 \cdot u_{i+1} = u_{i+1}^7 + x_i^3 + x_{i+1}.$$ 

(4.3)
Let \( Y = (L_1, L_2, L_3, \ldots) < X \) denote the subtower of \( X \) with \( L_n := F_a(x_1, \ldots, x_n) \). From (4.2) and (4.3) one has the relations

\[
x_i^3 + x_{i+1} = \frac{x_i^3}{1 + x_i}, \quad \text{for} \quad i = 1, \ldots, n - 1.
\]

We need the following lemma:

**Lemma 4.2.** Let \( L = \mathbb{F}(y, z) \) with \( \text{char}(\mathbb{F}) = 2 \) and

\[
z^3 + z = \frac{y^3}{1 + y}. \tag{4.5}
\]

Then \( [L : \mathbb{F}(y)] = [L : \mathbb{F}(z)] = 3 \), and the principal divisors in \( L \) of the functions \( y, y + 1, z, \) and \( z + 1 \) are as follows:

\[
(y) = P_0 + 2Q - 3P_{\infty}; \quad (y + 1) = 3P_1 - 3P_{\infty};
\]

\[
(z) = 3P_0 - P_1 - 2P_{\infty}; \quad (z + 1) = 3Q - P_1 - 2P_{\infty}.
\]

The different exponent of the place \( Q \) in the extension \( L/\mathbb{F}_a(y) \) is \( \Delta(Q) = 4 \).

**Proof.** Let \( Q \in \mathbb{P}(L) \) be a zero of \( z + 1 \). One gets from Eq. (4.5) that

\[
2v_Q(z + 1) = v_Q((z + 1)^2) + v_Q(z) = v_Q(z^3 + z)
\]

\[
= v_Q(y^3/(1 + y)) = 3v_Q(y).
\]

This implies \( v_Q(y) = 2 \) and \( v_Q(z + 1) = 3 \). Moreover, it follows that \( [L : \mathbb{F}_a(y)] = [L : \mathbb{F}_a(z)] = 3 \). In a similar way one determines the other zeros and the poles of \( y, y + 1, z, \) and \( z + 1 \) and thus the principal divisors of these functions.

Denote by \( df \) the differential of a function \( f \in L \). Since \( y^3 = (z^3 + z) \times (1 + y) \), one has

\[
y^2 dy = (z^3 + z) dy + (1 + y)(z^3 + 1) dz,
\]

and therefore

\[
dz = \frac{y^2 + (z^3 + z)}{(1 + y)(1 + z^2)} dy = \frac{y^2 + y^3/(1 + y)}{(1 + y)^2(1 + z^2)} dy = \frac{y^2}{(1 + y)^2(1 + z^2)} dy. \tag{4.6}
\]

The place \( Q \) is neither a pole of \( z \) nor a pole of \( y \), hence \( v_Q(dz) = v_Q(\Delta(L/\mathbb{F}_a(z))) \) and \( v_Q(dy) = v_Q(\Delta(L/\mathbb{F}_a(y))) = \Delta(Q) \). Also,
Consider the tower \( \mathcal{L} = (L_1, L_2, L_3, ...) \) as defined by the equations (4.4). Let \( P' \in \mathcal{P}(L_n) \) be a zero of \( x_n + 1 \). From Lemma 4.2, the place \( P' \) is then a zero of the functions \( x_{n-1}, ..., x_1 \). The ramification indices of the restrictions of \( P' \) are as in Fig. 4 (as follows from Lemma 4.2).

From this figure it follows (by diagram chasing) that the ramification index of the place \( P' \) in the extension \( L_n/\mathbb{F}_2(x_n) \) is \( 3^{n-1} \), and hence \( [L_n : \mathbb{F}_2(x_i)] = 3^{n-1} \) for \( i = 1, ..., n \). The restriction \( P_1 \in \mathcal{P}(L_{n-1}) \) of \( P' \) to \( L_{n-1} \) is totally ramified in the extension \( L_{n-1}/\mathbb{F}_2(x_{n-1}) \). The different exponent \( d(P'|P_1) = v_{P_1}(\text{Diff}(L_n/L_{n-1})) \) is then, from Proposition 1.2(ii) and Lemma 4.2, given by

\[
\text{Diff}(L_{n+1}/L_n) = 3^{n-2} \cdot 4 + (3^{n-2} - 1) \cdot (2 - 1) = 5 \cdot 3^{n-2} - 1. \tag{4.7}
\]

Now we apply Proposition 2.5 to the tower \( \mathcal{L} \), setting \( \rho_{n+1} := 3^n \) for \( n \geq 1 \). The assumptions (a) and (c) of Proposition 2.5 hold obviously, and the assumption (b), i.e.,

\[
\rho_{n+1} \leq \text{deg Diff}(L_{n+1}/L_n),
\]

follows immediately from (4.7). Thus, Proposition 2.5(ii) yields \( \lambda(\mathcal{L}) = 0 \), and, a fortiori, \( \lambda(\mathcal{K}) = 0 \). We have shown:

**PROPOSITION 4.3.** The tower \( \mathcal{K} = (K_1, K_2, K_3, ...) \) of function fields over \( \mathbb{F}_8 \) as given in Example 4.1 is asymptotically bad.

**Remark 4.4.** The tower \( \mathcal{K} \) defined by Eq. (4.1) can be considered as a tower of function fields over any finite field of characteristic 2. The proof of Proposition 4.3 shows that \( \mathcal{K} \) is asymptotically bad over all these fields.

\[
\begin{array}{cccccc}
L_2 & & & & & \\
\mathbb{F}_4(x_1) & \mathbb{F}_4(x_2) & \ldots & \mathbb{F}_4(x_{n-2}) & \mathbb{F}_4(x_{n-1}) & \mathbb{F}_4(x_n) \\
1 & 3 & 1 & 3 & 2 & 3 \\
x_1 = 0 & x_2 = 0 & x_{n-2} = 0 & x_{n-1} = 0 & x_n = 1 \\
\end{array}
\]

\[\text{Figure 4}\]
Serre suggested the following tower of function fields over $F_{q^3}$:

**Example 4.5.** Let $S = (S_1, S_2, S_3, ...)$ denote the tower of function fields over $F_{q^3}$, given by $S_1 = F_{q^3}(x_1)$ and $S_n = S_{n-1}(z_n)$, where

$$z_n^{q^2} + z_n^{q} + z_n = x_{n-1}^{q^2 + q + 1}, \quad \text{for } n \geq 2, \quad (4.8)$$

with the functions $x_n$ being defined inductively by

$$x_n = z_n / x_{n-1}, \quad \text{for } n \geq 2. \quad (4.9)$$

One easily sees that the pole of $x_1$ is totally ramified in the extension $S_n / S_1$ with ramification index $q^{2(n-1)}$, and this proves that the defining equation (4.8) for $z_n$ over $S_{n-1}$ is indeed absolutely irreducible, for all $n \geq 2$. Also, there is a unique place of $S_n$ which is a zero of $x_n$, namely the common zero of $x_1, z_2, z_3, ..., z_n$.

Let $N = q^2 + q + 1$ and let $P_n \in \mathbb{P}(S_{n+1})$ be the place of $S_{n+1}$ which is a common zero of $x_n$ and $(z_{n+1} - x)$, where $x \in F_{q^3} \setminus \{0\}$ satisfies $x^{q^2} + x^q + x = 0$. All power series expansions below should be considered at such a place $P_n$. Note that $x_1$ is a prime element at the place $P_n$. We have that

$$(z_{n+1} - x)^{q^2} + (z_{n+1} - x)^q + (z_{n+1} - x) = x_n^N$$

and hence

$$x_n \cdot x_{n+1} = z_{n+1} = x + x_n^N - x_{n+1} + \ldots.$$  

From this it follows that

$$x_{n+1} = x \cdot x_n^{-1} + x_n^{N-1} - x_{n+1} + \ldots,$$

so

$$x_n^{N-1} = \left( \frac{x}{x_n} \right)^N + x^{N-1} + \ldots. \quad (4.10)$$

Since $P_n$ is a common zero of $x_1, z_2, ..., z_n$, we have similarly

$$x_r = x_{r-1}^{N_r-1} - x_{r+1}^{N_{r+1}-1} + \ldots, \quad \text{for } r \leq n. \quad (4.11)$$

From the equations (4.11), one obtains

$$x_n = x_1^{(N-1)^{r-1}} + h(x_1^r) + (-1)^{n-1} x_1^{N_n} + \ldots, \quad (4.12)$$
where
\[ M_n = 1 = (Nq - 2) \cdot \frac{(N-1)^{n-1} - 1}{N-2}, \quad \text{for } n \geq 2. \]

In Eq. (4.12) above it is meant that \( M_n \) is the lowest exponent in the power series of \( x_n \) which is not a multiple of \( q \). From (4.12) it follows that
\[ x_n^N = x_1^{(N-1)^{n-1} \cdot (1 + g(x_1^n) + (-1)^{n-1} \cdot x_1 + \cdots)} \] (4.13)
and hence
\[ x_n^{-N} = x_1^{-N(N-1)^{n-1}} (1 + f(x_1^n) + (-1)^n \cdot x_1 + \cdots), \] (4.14)
where \( t = M_n - (N-1)^{n-1} \). Again, in the expressions (4.13) and (4.14) it is meant that \( t \) is the lowest exponent of the power series inside the brackets which is not a multiple of \( q \).

One sees that the highest pole-order, denoted here by \( m^{(n+2)} \), in the expansion (4.14) which is not a multiple of \( q \) is given by
\[ m^{(n+2)} = (N+1)(N-1)^{n-1} - M_n. \]

One checks easily that
\[ m^{(n+2)} > \frac{N(N-1)^{n-1}}{q}. \]

It then follows from Eq. (4.10) (see [4] or [9, Prop. III.7.10]) that the place \( P_n \) is totally ramified in the extension \( S_{n+2}/S_{n+1} \), and that the different exponent \( d_s^{(n+2)} \) of the unique place of \( S_{n+2} \) lying above \( P_n \) is
\[ d_s^{(n+2)} = (q^2 - 1)(m^{(n+2)} + 1). \]

One gets
\[ d_s^{(n+2)} = (N-2)(N+1)(N-1)^{n-1} - (Nq - 2)((N-1)^{n-1} - 1) \]
\[ d_s^{(n+1)} = (N-2)(N+1)(N-1)^{n-2} - (Nq - 2)((N-1)^{n-2} - 1), \]
and this implies that
\[ d_s^{(n+2)} = q^2 \cdot d_s^{(n+1)} = [S_{n+2} : S_{n+1}] \cdot d_s^{(n+1)}. \]

Now it follows from Proposition 2.5 that \( \lambda(\mathcal{S}) = 0. \) We have proved:

**Proposition 4.6.** The tower \( \mathcal{S} = (S_1, S_2, S_3, \ldots) \) of function fields over \( \mathbb{F}_q^U \) as given in Example 4.5 is asymptotically bad.
Remark 4.7. The function fields $S_n$ in the tower $S$ above have many places of degree one. Namely, it is easily seen that the zeros of $x_1 - \alpha$, with $0 \neq \alpha \in F_{q^r}$, split completely in the extension $S_n / S_1$.

Remark 4.8. The towers $S$ (from Example 4.5) and $E$ (from Example 2.10) are special cases of the following construction: Let $l = q^r$, with $r \geq 2$, and consider the polynomials

$$\sigma(X) = X^{q^r-1} + X^{q^r-2} + \cdots + X^q + X,$$

$$\nu(X) = X^{q^r-1} + q^r + \cdots + q + 1.$$ 

Note that $\sigma(\alpha)$ (resp. $\nu(\alpha)$) is the trace (resp., the norm) of $\alpha \in F_{q^r}$ with respect to the extension $F_{q^r} / F_q$. Define a tower $(S_n^{(r)}) = (S_1^{(r)}, S_2^{(r)}, S_3^{(r)}, \ldots)$ of function fields over the field $F_{q^r}$ by $S_1^{(r)} = F_{q^r}(x_1)$ and $S_n^{(r)} = S_{n-1}^{(r)}(z_n)$, where

$$\sigma(z_n) = \nu(x_{n-1})$$

and $x_n = z_n / x_{n-1}$, for $n \geq 2$.

For $r = 2$ one gets the tower $(S_2^{(2)}) = E$, and for $r = 3$ one has $(S_3^{(3)}) = S$. It can be shown that for any $r \geq 3$, the tower $(S_n^{(r)})$ over $F_{q^r}$ is asymptotically bad (although the function fields $S_n^{(r)}$ have "many" places of degree one: the zeros $P \in \overline{F}(S_n^{(r)})$ of $x_1 - \alpha$, with $\alpha \in F_{q^r} \setminus \{0\}$, split completely in the extensions $S_n^{(r)} / S_1^{(r)}$).

REFERENCES


