Effect of thermal stresses on stability and frequency response of a capacitive microphone

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ABSTRACT
This article deals with effects of thermal stresses on stability and frequency response of a fully clamped circular microplate, which acts as the diaphragm of a capacitive MEMS microphone. Static and dynamic pull-in phenomena limit the stable regions of a capacitive MEMS microphone. The results show that the non-dimensional static pull-in voltage of the studied case is about 5.23 (38.6 V). On the other hand, according to the results, the non-dimensional dynamic pull-in of the diaphragm is about 4.74 (34.98 V), which is as low as 90.63% of the static pull-in threshold. Because of the thermal expansion coefficient, diaphragm temperature increment leads to compressive thermal stresses and conversely, decrement of the diaphragm temperature creates tensile thermal stresses. The effect of temperature on the pull-in parameters is given by a design-correcting factor.

As results demonstrate, the deflection of the diaphragm subjected to a given electrostatic force can be controlled by means of the temperature changes. In the absence of electrostatic force, as the results show, although temperature changes do not create any deflection, but for a critical temperature increment the diaphragm stiffness vanishes and the buckling phenomenon takes place. Effects of the electrostatic force and the temperature variation on the frequency response of the microphone subjected to a sound pressure wave are investigated. As the results illustrate, increment of the electrostatic force or increment of the diaphragm temperature increases the output level and sensitivity of the microphone and decreases the fundamental frequency of the microphone, limiting the upper band of its bandwidth. It is obvious that decrement of the diaphragm temperature acts conversely.

In addition, the results show that in the presence of the electrostatic force sensitivity of the output level of the diaphragm to the temperature change increases.

1. Introduction

A microphone is a transducer that converts unsteady pressure inputs into electrical signals. The microphones are widely used in communications, hearing-aids devices and noise and vibration control [1]. Generally, microphones are categorized as dynamic, condenser and optical microphones [2]. A capacitive or condenser microphone couples acoustical, mechanical and electrical domains. Condenser microphones are the most common type of silicon microphone to date because of their high sensitivity and low noise level [1]. Traditional microphones, such as Bruel and Kjær condenser microphones, offer excellent performance, but they are costly and currently not suitable for miniaturization [3].

Microelectromechanical systems (MEMS) technology has been rapidly growing since its beginning in 1980s. With the development of MEMS technology, hundreds or thousands of devices can be fabricated together on a single silicon wafer. This leads to the production of MEMS microphones that can approach the performance of traditional microphones with lower cost and smaller size [3].

A condenser microphone conceptually consists of a thin vibrating diaphragm and a rigid ground plate that are separated by a small air gap. The relative movement of the diaphragm to the ground plate is solely due to the applied acoustic pressure on the diaphragm, so that the ground plate has no movement as the reference [4].

Mechanical behavior of MEMS microphones has been studied recently in several investigations. Stoffel et al. [5] represented a new miniature microphone concept in which the sound wave enters a gap between the microphone membrane surface and a cover plate through a slot for excitation. They used an analytical...
description to calculate the frequency response as a function of the gap height and compared it to experimental results. Li et al. [6] developed a micro-machined condenser microphone with a novel single deeply corrugated diaphragm for sound sensing in order to improve the mechanical sensitivity by both releasing the initial stress and decreasing the mechanical stiffness of the diaphragm. Chen et al. [7] introduced a novel single chip condenser miniature microphone with a circular corrugated diaphragm for releasing the residual stresses. Abuelma'atti [8] represented a mathematical model for the open-circuit output voltage of a micro-machined silicon condenser microphone with a single deeply corrugated diaphragm as a function of the applied acoustic pressure. Chen et al. [9] developed a nodal analysis model of a condenser microphone in an IC design environment. Their model is capable of dealing with the multi-field coupling effects including acoustical, mechanical and electrostatic behavior. Quaegebeur and Chaigne [10] studied mechanical and electrical nonlinearity effects on the electroacoustical behavior of electrodynamic transducers. Liu et al. [3] presented the nonlinear identification of a capacitive dual-backplate MEMS microphone developing a nonlinear lumped element model of the coupled electromechanical microphone dynamics.

Static and dynamic pull-in instability phenomena are important issues in capacitive microphones and all electrostatically driven microstructures. Nathanson first has analyzed static pull-in phenomenon [11]. This phenomenon is a discontinuity related to the interplay of the elastic and electrostatic force while the electrostatic force is related to the applied voltage. The specific applied voltage that leads to static pull-in instability is called “static pull-in voltage”. Indeed, when the actuating voltage exceeds the critical pull-in value, no stable equilibrium exists and the structure collapses on the actuating electrode. Therefore, this phenomenon is encountered as a basic static instability mechanism limiting the operational range of these kinds of devices in terms of deflections and voltage [12]. According to the static nature of this phenomenon, the inertia and damping related forces are neglected in calculations [12–14]. On the other hand, dynamic pull-in instability phenomenon is usually investigated for the case of a step applied bias DC voltage for microstructures and considers the inertial effect [15]. Therefore, to determine the proper value of applied bias DC voltage for a condenser microphone, it is necessary to determine the static and dynamic stability regions through investigating the static and dynamic pull-in voltages. Nevertheless, in previous works, these instabilities for condenser microphones with circular diaphragm were not investigated sufficiently.

One of the most important characteristics of microphones is their frequency response. Frequency response refers to the way a microphone responds to different frequencies. Indeed, a frequency response curve represents the output level or sensitivity of a microphone over its operating range from the lowest to the highest frequency. Therefore, it is quite necessary to study the frequency response for microphones.

The operating temperature of the flexible part of the MEMS devices can be changed. These changes can be occurred owing to change of environmental temperature or due to heat generation because of intrinsic damping or power dissipation [16–18]. Any change in the operating temperature of microphone diaphragm creates thermal stresses due to the thermal expansion coefficient of the diaphragm. Therefore, a full thermo-electro-mechanical analysis is required to investigate the behavior of flexible circular diaphragm of condenser microphones, which has not been investigated in previous works. Of course, the temperature changes can alter the spectral density of thermal noise through affecting the random motion of the molecules in the solid lattice of the diaphragm material and in the air [19–21], which is not the interest of this paper.

Therefore, in this paper, effects of temperature variation on the stable region and the frequency response of a circular condenser microphone are investigated using a mathematical model having distributed parameters. The considered microphone has a high fundamental resonant frequency in order to have a wide bandwidth and flat response in the sound bandwidth and have capability to filter low frequency mechanical shock. The static and dynamic pull-in voltages are determined and presented in the non-dimensional form and the effect of the temperature changes on the pull-in parameters of the microphone is studied introducing a design-correcting coefficient. Applying a Galerkin based reduced order model frequency response of the microphone is studied and effects of the temperature changes on the amplitude of the response and the fundamental frequency, limiting the microphone bandwidth, are studied.

2. Model description and mathematical modeling

Condenser or capacitive MEMS microphones are generally modeled as two circular microplates with a voltage between them as shown in Fig. 1. The upper plate is a thin deformable elastic circular plate with thickness \( h \) \((-h/2 \leq z \leq h/2)\), radius \( R \) \((-R \leq r \leq R)\) and isotropic with Young’s modulus \( E \) and Poisson’s ratio \( v \) that are held fixed along its boundary and acts as the diaphragm. The lower plate must be thick enough, as it has no movement as the reference. The space between these plates is filled with a dielectric substance like air. The diaphragm vibrates when struck by sound waves. Changing the distance between the two plates and therefore changing the capacitance is possible applying a bias DC voltage to the plates. The cylindrical coordinate system \((r, \theta, z)\) with an origin located at the center point of the plate is utilized to study the transverse vibration of the thin circular diaphragm as it is obvious in Fig. 1.

Since the diaphragm is assumed to be subjected to a bias DC voltage, an electrostatic force can be represented as follows [12]:

\[ F(w, V) = \frac{\varepsilon_0 V^2}{2(g_0 - w(r, t))} \]

(1)

where \( \varepsilon_0 \) is the dielectric (permittivity) of the air, \( V \) is the applied voltage, \( g_0 \) is the initial gap between the diaphragm and the ground plate, \( t \) is the physical time and \( w(r, t) \) is the deflection of the diaphragm, defined to be positive downward.

A sound pressure wave can be considered as a random time dependent function having frequencies from 40 Hz to 20 KHz. Random sound pressure waves can be expressed as a sum of

![Fig. 1. Schematic view of a capacitive microphone with parallel circular plates.](https://via.placeholder.com/150)
simple pure single frequency sounds applying Fourier series. On the other hand, the sound pressure waves, which are not periodic time functions, can be represented by Fourier integrals in which the periods are extended to a large value approaching infinity. In this paper, we consider a simple sinusoidal pure sound as the actuation of the microphone diaphragm as the following:

\[ P(\omega, t) = P_0\sin(\omega t) \]

in which \( P_0 \) and \( \omega \) are the amplitude and frequency of the sound pressure wave, respectively.

When the thickness-to-diameter ratio of a circular plate \((h/2R)\) is less than 1/20, it can be considered as a thin plate and then the Kirchhoff thin plate theory can be utilized for analyzing the mechanical behavior of the plate [22]. According to the Hook’s law, the stress–strain relations for a plate in the cylindrical coordinate system can be expressed as the following [23]:

\[
\begin{bmatrix}
\sigma_r \\
\sigma_\theta \\
\tau_{r\theta}
\end{bmatrix} =
\begin{bmatrix}
E & 0 & 0 \\
0 & E & 0 \\
0 & 0 & G
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u}{\partial r} \\
\frac{\partial v}{\partial \theta} \\
\frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial u}{\partial \theta}
\end{bmatrix}
\]

Based on Kirchhoff thin plate theory the relationship between the displacement components along the radial, circumferential and transversal directions can be expressed, respectively, as follows:

\[
u(r, \theta, z, t) = -r \frac{\partial w(r, \theta, t)}{\partial \theta}, \quad v(r, \theta, z, t) = -2 \frac{\partial u(r, \theta, t)}{\partial r}, \quad w(r, \theta, z, t) = w(r, \theta, t)\]

According to Eq. (4) the strain components can be achieved as

\[
\epsilon_\theta = \frac{\partial u}{\partial r} = -2 \frac{\partial^2 w}{\partial r^2} \]

\[
\epsilon_\theta = u + \frac{1}{r} \frac{\partial v}{\partial \theta} = -2 \left( \frac{\partial w}{\partial \theta} + \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \right) \]

\[
\gamma_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} = -2z \frac{\partial}{\partial (1/r)} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} \right) \]

Substituting Eqs. (5), (6) and (7) into Eq. (3) leads to the following stress components:

\[
\sigma_r = \frac{-E}{1-\nu^2} \left[ 2 \frac{\partial^2 w}{\partial r^2} + 2 \left( \frac{\partial w}{\partial \theta} + \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \right) \right] \]

\[
\sigma_\theta = \frac{-E}{1-\nu^2} \left[ 2 \left( \frac{\partial w}{\partial \theta} + \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \right) + \frac{\partial^2 w}{\partial r^2} \right] \]

\[
\tau_{r\theta} = G\gamma_{r\theta} = -2Gz \frac{\partial}{\partial (1/r)} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} \right) \]

Assuming the deflection of the plate is symmetrical relative to circumferential coordinate \((\partial w/\partial \theta = 0)\), stress components take the following forms:

\[
\sigma_r = \frac{-EZ}{1-\nu^2} \left[ \frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r} \right] \]

\[
\sigma_\theta = \frac{-EZ}{1-\nu^2} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} \]

\[
\tau_{r\theta} = G\gamma_{r\theta} = 0 \]

Bending and twisting moments per unit length of the circular plate can be calculated as follows:

\[
M_r = \int_{-h/2}^{h/2} \sigma_r zdz = -\frac{Eh^3}{12(1-\nu^2)} \left[ \frac{\partial^2 w}{\partial r^2} + 2 \frac{\partial w}{\partial r} \right] \]

\[
M_\theta = \int_{-h/2}^{h/2} \sigma_\theta zdz = -\frac{Eh^3}{12(1-\nu^2)} \left[ \frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r} \right] \]

\[
M_{r\theta} = \int_{-h/2}^{h/2} \tau_{r\theta} zdz = 0 \]

The equation of transverse motion for a circular plate considering electrostatic force and sound pressure wave can be expressed as [24]

\[
\rho h \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 M_r}{\partial r^2} + \frac{\partial}{\partial r} \left( \frac{\partial M_{r\theta}}{\partial \theta} \right) - \frac{\partial}{\partial r} \left( \frac{\partial M_\theta}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial M_{r\theta}}{\partial \theta} \right) + F(w, V) + P(\omega, t) \]

Substituting Eqs. (1), (2), (14), (15) and (16) into Eq. (17), the equation of transverse motion for a circular microplate subjected to nonlinear electrostatic force and sound pressure wave takes the following form:

\[
D \frac{\partial^4 w}{\partial r^4} + \frac{\rho h^3}{12} \frac{\partial^2 w}{\partial r^2} + \frac{\rho h^3}{2} \frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r} + \frac{\partial w}{\partial r} = \frac{Eh^3}{12(1-\nu^2)} \]

in which \( \nabla^4 \) and \( \nabla \) are the biharmonic operator in the polar coordinate system for the axisymmetric circular plate and the flexural rigidity, respectively:

\[
\nabla^4 = \nabla^2 \nabla^2 = \frac{\partial^4}{\partial r^4} + 2 \frac{\partial^3}{\partial r^3} + 1 \frac{\partial^2}{\partial r^2} + 1 \frac{\partial}{\partial r} D = \frac{Eh^3}{12(1-\nu^2)} \]

Note that \( \nabla^2 \) is the Laplace operator:

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + 1 \frac{\partial}{\partial r} \]

2.1. Thermal stresses effect

Because of the thermal expansion coefficient \((\alpha_T)\) of the diaphragm, temperature variation \((\Theta)\) of the diaphragm creates following thermal stresses:

\[
\sigma_r = \sigma_\theta = (1 + \nu) \frac{Ez_T \Theta}{1-\nu^2} = \frac{Ez_T \Theta}{1-\nu^2} \]

in which \( \sigma_r \) and \( \sigma_\theta \) represent the thermal stresses in radial and circumferential directions, respectively, and \( \Theta \) is the temperature change with respect to a reference temperature.

Since the deflection of the plate is assumed independent of \( \Theta \), so the thermal stress in circumferential direction has no component on the transverse direction; therefore, the transverse motion equation of the diaphragm considering thermal stresses takes the following form:

\[
D \nabla^4 w + \frac{Ez_T \Theta}{1-\nu^2} \nabla^2 w + \rho h^3 \frac{\partial^2 w}{\partial r^2} = \frac{\epsilon_0 V^2}{2(\delta_0 - W_1)} + P_0 \sin(\omega t) \]

Eq. (22) is a nonlinear equation, which represents the transverse deflection of the diaphragm due to a sound pressure wave and a voltage applied to the parallel plates.

On the other hand, small vibrations about a static equilibrium position, which obtained through applying a bias DC voltage to the parallel plates, can be studied introducing the dynamic deflection of the diaphragm about the static equilibrium position.
(\(w_r(t)\)) as \(w_0(t)\); therefore, the total transverse deflection can be expressed as

\[
\mathbf{w}(r,t) = w_r(t) + w_0(t) \tag{23}
\]

Considering \(w_0 \ll w_r\), the nonlinear electrostatic force \(F(w,V)\) using Calculus of Variation Theory and Taylor series expansions can be written as [12]

\[
F(w,V) = \frac{\varepsilon_0 V^2}{2(g_{so} - w_0)^2} + \frac{\varepsilon_0 V^2}{(g_{so} - w_0)^2} w_0 \tag{24}
\]

Substituting Eqs. (23) and (24) into Eq. (22) leads to a nonlinear static and a linear small dynamic deflection equations of the diaphragm as follows:

Nonlinear equation of static deflection:

\[
D \nabla^4 w_1 + \left(\frac{E_0 \varepsilon^2}{1 - \nu} \right) h \nabla^2 w_0 = \frac{\varepsilon_0 V^2}{2(g_{so} - w_0)^2} \tag{25}
\]

Linear equation of small dynamic motion about static equilibrium position:

\[
D \nabla^4 w_d + \left(\frac{E_0 \varepsilon^2}{1 - \nu} \right) h \nabla^2 w_d + \frac{\rho h^2 \omega^2}{c^2} w_d - \frac{\varepsilon_0 V^2}{(g_{so} - w_0)^2} w_d = p_0 \sin(\omega, t) \tag{26}
\]

For convenience the following dimensionless parameters are defined to transform Eqs. (22), (25) and (26) into non-dimensional forms:

\[
\tilde{w} = \frac{w}{g_{so}; \tilde{w}_0 = \frac{w_0}{g_{so}; \tilde{r} = \frac{r}{R}; \tilde{t} = \frac{t}{R}; \tilde{c} = R^2 \sqrt{\frac{\rho h}{D} \omega}; \tilde{\omega} = \frac{\omega}{\omega_b}; \tilde{r} = \frac{r}{r_b} = 1} \tag{27}
\]

According to Eqs. (19) and (20), the non-dimensional Laplace and biharmonic operators can be achieved as

\[
\tilde{\nabla}^2 = \frac{1}{R^2} \nabla^2, \quad \tilde{\nabla}^4 = \frac{1}{R^4} \nabla^4 \tag{28}
\]

Applying Eqs. (27) and (28) into Eq. (25), non-dimensional static deflection equation arisen from statically applying bias DC voltage takes the following form:

\[
\tilde{\nabla}^4 \tilde{w}_1 + \Gamma_\mu \Gamma_\phi \tilde{\nabla}^2 \tilde{w}_1 = \frac{\tilde{V}_n^2}{2(1 - \tilde{w}_0)^2} \tag{29}
\]

Nonlinear equation of transverse static deflection owing to an applied bias DC voltage and linear small dynamic motion due to the sound pressure wave about the static equilibrium position, using Eqs. (27) and (28), take the following non-dimensional forms, respectively:

\[
\tilde{\nabla}^4 \tilde{w}_d + \Gamma_\mu \Gamma_\phi \tilde{\nabla}^2 \tilde{w}_d + \frac{\tilde{c}^2 \tilde{w}_d}{\tilde{c}^2} = \frac{\tilde{V}_n^2}{2(1 - \tilde{w}_0)^2}, \quad \Gamma_\phi = \frac{\varepsilon_0 R^4}{\varepsilon_0 g_{so} D^2} \tag{30}
\]

\[
\tilde{\nabla}^4 \tilde{\nabla}^2 \tilde{w}_d + \Gamma_\mu \Gamma_\phi \tilde{\nabla}^2 \tilde{w}_d + \frac{\tilde{c}^2 \tilde{w}_d}{\tilde{c}^2} = \frac{\tilde{V}_n^2}{2(1 - \tilde{w}_0)^2} \tag{31}
\]

in which

\[
\Gamma_\mu = \frac{E h R^2}{D(1 - \nu)}, \quad \Gamma_\phi = \varepsilon_0 R^4 \varepsilon_0 g_{so} D^2; \quad \tilde{r}_s = \tilde{R}^2 \sqrt{\frac{\rho h}{D} \omega}; \quad \tilde{t}_s = \tilde{T}^2 \sqrt{\frac{\rho h}{D} \omega}; \quad \tilde{\omega} = \frac{\omega}{\omega_b}; \quad \tilde{r} = \frac{r}{r_b} = 1 \tag{32}
\]

Note that we call \(\Gamma_\phi\) and \(V_n\) as non-dimensional effective temperature change (NETC) and non-dimensional voltage, respectively.

3. Numerical solutions

3.1. Nonlinear equation of static deflection

Static deflection of a circular diaphragm due to an applied DC voltage can be obtained through solving Eq. (29). Due to the nonlinearity of Eq. (29), the solution is complicated and time consuming. Therefore, it is proper to linearize it. Because of the sensitivity of the value of \(\tilde{w}_0\) with respect to the initial gap, especially when the applied DC voltage increases, the linearization with respect to the initial position may cause considerable errors. Therefore, to minimize these errors, a step-by-step increase of applied DC voltage is used and the governing equation is linearized at each step [12]. Assume that \(\tilde{w}_0\) is the static deflection of the diaphragm due to the applied DC voltages \(V_n\), according to Eq. (29), it can be achieved through the following equation:

\[
\hat{\nabla}^4 \hat{w}_1 + \Gamma_\mu \Gamma_\phi \hat{\nabla}^2 \hat{w}_1 = \frac{\hat{V}_n^2}{2(1 - \hat{w}_0)^2} \tag{33}
\]

By increasing the applied voltages to

\[
(V_{n,h+1} - V_{n,h} + \delta V_n) \tag{34}
\]

we have

\[
\hat{w}_{n+h} = \hat{w}_{n,h} + \hat{\psi}\hat{r} \tag{35}
\]

Substituting Eqs. (34) and (35) into Eq. (29), using Calculus of Variation Theory and Taylor series expansion about \(\hat{w}_0\) and truncating its higher orders, Eq. (29) can be written as follows:

\[
\hat{\nabla}^4 \hat{w}_{n+1}^2 + \hat{\psi} + \Gamma_\mu \Gamma_\phi \hat{\nabla}^2 \hat{w}_{n+1} = \frac{\hat{V}_{n+1}^2}{2(1 - \hat{w}_{n+1})^2} \tag{36}
\]

Subtracting Eq. (33) from Eq. (36) leads to the following linear equation:

\[
\hat{\nabla}^4 \hat{\psi} + \Gamma_\mu \Gamma_\phi \hat{\nabla}^2 \hat{\psi} - \frac{\hat{V}_{n+1}^2}{2(1 - \hat{w}_{n+1}^2)} \hat{\psi} = \frac{\hat{V}_{n+1}^2 - \hat{V}_{n+1}^2}{2(1 - \hat{w}_{n+1}^2)} \tag{37}
\]

A Galerkin weighted residual method is utilized to solve Eq. (37). In this method, an approximate solution is considered as

\[
\hat{\psi}_{N}(\hat{r}) = \sum_{n=1}^{N} a_n \phi_n(\hat{r}) \tag{38}
\]

where the considered approximate solution converges to \(\hat{\psi}(\hat{r})\) in the mean, for a proper number of used shape functions [25].

\[
\int_{-1}^{1} (\hat{\psi}(\hat{r}) - \hat{\psi}_{N}(\hat{r}))^2 d\hat{r} \leq \epsilon^2
\]

where \(\phi_n(\hat{r})\) are linearly independent trial or shape functions and \(a_n\) are unknown multipliers, which must be determined in the solution. Note that \(\phi_n(\hat{r})\) are chosen in a way that they satisfy all boundary conditions of a fully clamped circular plate.

Substituting the approximated solution into Eq. (37) leads to the following residual:

\[
\sum_{n=1}^{N} a_n \hat{\nabla}^4 \phi_n + \Gamma_\mu \Gamma_\phi \sum_{n=1}^{N} a_n \hat{\nabla}^2 \phi_n - \frac{\hat{V}_{n+1}^2}{2(1 - \hat{w}_{n+1}^2)} \sum_{n=1}^{N} a_n \phi_n = \frac{\hat{V}_{n+1}^2}{2(1 - \hat{w}_{n+1}^2)} \tag{39}
\]

Based on the Galerkin weighted residual method \(a_n\) can be determined. Therefore, \(a_n\) can be calculated from the following set of \(N\) algebraic equations:

\[
\int_{0}^{1} \phi f R_1 f \hat{r} = 0 \quad f = 1, \ldots N \tag{40}
\]

Applying Eq. (40) leads to

\[
\sum_{n=1}^{N} K_{n} a_n = F_f \quad f = 1, \ldots N(Ka = F) \tag{41}
\]
in which
\[ K_{m} = \frac{1}{\int_{0}^{1} \phi_{f} \left[ \phi_{m}^{4} + \frac{\partial M \partial \phi_{m}}{\partial \phi_{m}} \right] \phi_{m}^{2} - \frac{(V_{m} h_{2})^{2}}{(1 - \omega_{1}^{2})} \phi_{m}^{2} \, df = 0 } \]

\[ F_{f} = \frac{1}{\int_{0}^{1} \phi_{m}^{2} (1 - \omega_{1}^{2})^{2} \phi_{m} \, df = 0 } \quad f = 1, \ldots, N \] (42)

The multipliers are achieved through Eq. (41) and then the \( \psi_{n}(t) \) can be calculated. Therefore, static deflection at a given bias DC voltage can be calculated by increasing the applied DC voltage and calculating the deflection in each step.

3.2. Nonlinear equation of transverse motion

Since owing to the nonlinear term (electrostatic force) existing in Eq. (30) creating a reduced order model applying a Galerkin based reduced order model is difficult, therefore existing nonlinear term is considered as forcing term and integration of this term over the diaphragm domain is done at each step of time. Then the value of the nonlinear term can be corrected using an iterative process that leads to a proper convergence result.

The transverse deflection of the diaphragm can be approximated in terms of linear combinations of finite number of suitable shape functions with time dependent coefficients:

\[ \hat{w}_{m}(\hat{r}, \hat{t}) = \sum_{m=1}^{M} \hat{u}_{m}(\hat{r}) \phi_{m} \] (43)

where approximate solution \( \hat{w}_{m}(\hat{r}, \hat{t}) \) converges to \( \hat{w}(\hat{r}, \hat{t}) \) for a proper number of used shape functions. Substituting Eq. (43) into Eq. (30) leads to the following residual:

\[ \sum_{m=1}^{M} \hat{u}_{m} \partial_{r}^{2} \phi_{m} + \frac{\partial M \partial \phi_{m}}{\partial \phi_{m}} \sum_{m=1}^{M} \hat{u}_{m} \partial_{r}^{2} \phi_{m} + \sum_{m=1}^{M} \hat{u}_{m} \phi_{m} - \mathcal{F}(\hat{r}, \hat{w}, V_{a}) = R_{z} \] (44)

Using \( \mathcal{F}(\hat{r}, \hat{w}, V_{a}) = \frac{V_{a}^{2}}{2(1-\omega_{1}^{2})} \)

According to the Galerkin method:

\[ \int_{0}^{1} \phi_{g} \partial_{r}^{2} \phi_{r} \, df = 0 \quad g = 1, \ldots, M \] (46)

Using Eq. (46), Eq. (44) takes the following form:

\[ \sum_{m=1}^{M} M_{gm} \hat{u}_{m} + \sum_{m=1}^{M} K_{gm} \hat{u}_{m} = \mathcal{N}_{g} \hat{u}_{m} \quad g = 1, \ldots, M(M \hat{u} + Ku = \mathcal{N}) \] (47)

in which

\[ M_{gm} = \int_{0}^{1} \phi_{g} \phi_{m} \, df, \quad K_{gm} = \int_{0}^{1} \phi_{g} \partial_{r}^{2} \phi_{m} + \frac{\partial M \partial \phi_{m}}{\partial \phi_{m}} \partial_{r}^{2} \phi_{m} \, df = 0, \]

\[ \mathcal{N}_{g} \hat{u}_{m} = \int_{0}^{1} \phi_{g} \mathcal{F}(\hat{r}, \hat{w}, V_{a}) \, df = 0 \quad g = 1, \ldots, M \] (48)

The order of Eq. (47) can be reduced by introducing the following parameter:

\[ u_{m} = T_{m}, \quad u_{m} = H_{m} \quad m = 1, \ldots, M \] (49)

The following system of equations is derived by substituting Eq. (49) into Eq. (47):

\[ \frac{dT}{df} = \mathcal{H} \] (50)

\[ \frac{dT}{df} = M^{-1}(N - Ku) \] (51)

Eqs. (50) and (51) can be expressed in following vector form:

\[ \begin{bmatrix} T \\ H \end{bmatrix}_{2M+1} = \begin{bmatrix} M^{-1} N \quad -Ku \end{bmatrix}_{2M+1} \] (53)

Indeed, Eq. (52) represents the transverse deflection behavior of the diaphragm and can be integrated over the time using a proper integration method.

3.3. Linear equation of small dynamic motion about static equilibrium position

Eq. (31) is a linear equation that can be solved applying a Galerkin based reduced order model. To this end, we assume the dynamic deflection with the following finite series:

\[ \hat{w}_{d}(\hat{r}, \hat{t}) = \sum_{k=1}^{p} \hat{q}_{k}(\hat{r}) \phi_{k}(\hat{r}) \] (54)

Substituting Eq. (54) into Eq. (31), some residual is obtained as

\[ \sum_{k=1}^{p} \hat{q}_{k} \partial_{r}^{2} \phi_{k} + \frac{\partial M \partial \phi_{k}}{\partial \phi_{k}} \sum_{k=1}^{p} \hat{q}_{k} \partial_{r}^{2} \phi_{k} + \sum_{k=1}^{p} \hat{q}_{k} \phi_{k} - \frac{V_{a}^{2}}{2(1-\omega_{1}^{2})} \sum_{k=1}^{p} \hat{q}_{k} \phi_{k} = \mathcal{R}_{2} \] (55)

Using \( \phi_{k}(l = 1, \ldots, \) p) as weighting functions and applying Galerkin method leads to

\[ \sum_{k=1}^{p} M_{k} \hat{q}_{k} + \sum_{k=1}^{p} K_{k} \hat{q}_{k} = P_{l} \quad l = 1, \ldots, p(M \hat{q} + K \hat{q} = P) \] (56)

in which

\[ M_{k} = \int_{0}^{1} \phi_{k} \phi_{l} \, df, \quad K_{k} = \int_{0}^{1} \phi_{k} \partial_{r}^{2} \phi_{l} + \frac{\partial M \partial \phi_{l}}{\partial \phi_{l}} \partial_{r}^{2} \phi_{l} \, df = 0 \]

\[ P_{l} = \mathcal{P} \sin[\mathcal{W}(\hat{r}, \hat{t})] \int_{0}^{1} \phi_{l} \, df = 0 \quad l = 1, \ldots, p \] (57)

According to the harmonic sound pressure excitation, the stationary dynamic response is also harmonical; therefore sound pressure and stationary dynamic response can be present as follows:

\[ P(t) = P_{0} e^{j \gamma_{p}}, \quad Q = q_{0} e^{j \gamma_{q}} \text{ with } |Q| = |P_{0}|, \quad \gamma_{p} = \arg(Q) \] (58)

in which \( \gamma_{p} \) and \( \gamma_{q} \) are the phase angles with respect to the input sound pressure and the complex amplitude of the response, respectively. Applying Eq. (58) into Eq. (56) leads to

\[ Q = (K - \omega_{0}^{2} M)^{-1} P_{0} \] (59)

Through Eq. (59), the dynamic response of the diaphragm about the static equilibrium position can be calculated.
4. Numerical results

The diaphragm investigated in this paper is a silicon microplate, which has the material and geometrical properties listed in Table 1.

Also this study uses the following shape functions, which satisfy all boundary conditions for the circular microplate. It must be noted that axisymmetric shape functions are utilized due to axisymmetry of the studied problem.

\[ \phi_1(\bar{r}) = (\bar{r} - 1)^2(\bar{r} + 1)^2; \]
\[ \phi_2(\bar{r}) = \phi_2(\bar{r}) = (\bar{r} - 1)^2(\bar{r} + 1)^2(\bar{r} - 0.8)(\bar{r} + 0.8) \]
\[ \phi_3(\bar{r}) = (\bar{r} - 1)^2(\bar{r} + 1)^2(\bar{r} - 0.7)(\bar{r} + 0.7)(\bar{r} - 0.9)(\bar{r} + 0.9) \]
\[ \phi_4(\bar{r}) = (\bar{r} - 1)^2(\bar{r} + 1)^2(\bar{r} - 0.7)(\bar{r} + 0.7)(\bar{r} - 0.8)(\bar{r} + 0.8)(\bar{r} - 0.9)(\bar{r} + 0.9) \]  

Number of used shape functions in each case are considered based on the expected accuracy.

4.1. Stable region of the diaphragm due to gradual application of a bias DC voltage

Application of a bias DC voltage to a condenser microphone decreases the equivalent stiffness of the microplate and then increases the sensitivity of it, but pull-in instability phenomenon limits the range of this applied bias DC voltage because this phenomenon leads the diaphragm to collapse on the ground plate. Therefore, it is necessary to calculate the static pull-in threshold. Applying finite difference method, the static pull-in voltage of a circular microplate has been investigated in [22]. The following table compares our numerical results of static pull-in voltage to the finite difference results reported in [22], for different applied voltage steps. As shown in Table 2 there is a good agreement between the results.

Note that \( \Delta \) is the difference percentage in comparison of the results. As shown in Fig. 2, neglecting the thermal stress effects, the diaphragm with properties listed in Table 1 is in the stable region for non-dimensional bias DC voltages less than 5.23 (38.6 V).

### Table 1

<table>
<thead>
<tr>
<th>Geometrical and material properties of diaphragm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R ) (radius)</td>
</tr>
<tr>
<td>( h ) (thickness)</td>
</tr>
<tr>
<td>( E ) (Young’s modulus)</td>
</tr>
<tr>
<td>( v ) (Poisson’s ratio)</td>
</tr>
<tr>
<td>( \rho ) (density)</td>
</tr>
<tr>
<td>( \varepsilon_f ) (permittivity of air)</td>
</tr>
<tr>
<td>( g_0 ) (initial gap)</td>
</tr>
</tbody>
</table>

### Table 2

Comparison of obtained results to the available results in [22].

<table>
<thead>
<tr>
<th>Value of the step of applied voltage ( \delta V ) (V)</th>
<th>3</th>
<th>1</th>
<th>0.10</th>
<th>0.05</th>
<th>0.025</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pull-in voltage (V)</td>
<td>Obtained results (( N = 1 ))</td>
<td>381.00</td>
<td>378.00</td>
<td>375.60</td>
<td>375.45</td>
</tr>
<tr>
<td>Obtained results (( N = 2 ))</td>
<td>339.00</td>
<td>334.00</td>
<td>332.00</td>
<td>331.80</td>
<td>331.800</td>
</tr>
<tr>
<td>Obtained results (( N = 3 ))</td>
<td>327.00</td>
<td>323.00</td>
<td>320.30</td>
<td>320.15</td>
<td>320.150</td>
</tr>
<tr>
<td>Obtained results (( N = 4 ))</td>
<td>324.00</td>
<td>319.00</td>
<td>317.10</td>
<td>317.50</td>
<td>316.875</td>
</tr>
<tr>
<td>Numerical results of [22]</td>
<td>321.00</td>
<td>316.00</td>
<td>313.80</td>
<td>313.70</td>
<td>313.700</td>
</tr>
<tr>
<td>( \Delta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.2. Influence of the NETC on the stable region of the diaphragm for gradual application of the bias DC voltage

The NETC can change the static threshold parameters of the microphone, because it affects the stiffness of the diaphragm due to creating thermal axial stresses. As shown in Fig. 3, positive and negative NETC decreases and increases the static pull-in voltage, respectively.

![Fig. 2. Static pull-in phenomenon for the diaphragm.](image)

![Fig. 3. Static pull-in variation due to the NETC for gradual application of a DC voltage.](image)
4.3. Stable region of the diaphragm due to application of a step DC voltage

When a step DC voltage is applied to the condenser microphone, the threshold of the DC voltage is more limited due to displacement dependent of the electrostatic force. The value of this threshold DC voltage is known as the dynamic pull-in voltage. Of course, there are several investigations, which have shown that at the critical point of dynamic pull-in, the velocity of the movable electrode is equal to zero [26–29]; this means that the dynamic pull-in phenomenon is a saddle node bifurcation and is a kind of static instability.

The dynamic pull-in voltage is as low as 90–92% of static pull-in voltage [15]. Applying fourth-order Runge Kutta method, Eq. (52), the equation of dynamic motion can be integrated over the time. As shown in Fig. 4 the obtained pull-in voltage for the microphone is 90.63% of the static pull-in voltage in the absence of thermal stress effects.

4.4. Influence of the NETC on the stable region of the diaphragm with application of a step DC voltage

As shown in Fig. 5, dynamic pull-in can be affected through the NETC. Positive NETC decreases the pull-in voltage threshold; this means that positive NETC reduces the value of the applicable step DC voltage and the negative NETC vice versa.

4.5. Design thermal correcting factor for the pull-in voltage

Static and dynamic pull-in voltages alter through changing thermal stresses as shown in Figs. 3 and 5. A design thermal correcting factor (\(\beta_{T,\Theta}\)) is introduced as follows, which indicates the effect of NETCs on the threshold values of the microphone.

\[
V_{\text{pull}} = \beta_{T,\Theta} V_{\text{pull, in}}
\]

Eq. (61)

Fig. 6 demonstrates the values of \(\beta_{T,\Theta}\) for various NETCs for both static and dynamic pull-in voltages.

4.6. Influence of the NETC on the deflection of the diaphragm for a given applied DC voltage

Fig. 7 shows when the diaphragm is subjected to an electrostatic force, positive and negative NETCs increase and decrease the diaphragm deflection, respectively.

As shown in Fig. 7, in the presence of a bias DC voltage, increase in temperature changes, due the displacement dependent of the electrostatic force, can lead the microphone to an unstable position with pulling down the substrate in a threshold temperature, wherein the absence of the electrostatic force occurs Euler buckling phenomenon.
4.7. Influence of the bias DC voltages on the frequency response of the condenser microphone

As shown in Fig. 8, application of a bias DC voltage increases the output level of the microphone and raises the microphone sensitivity. Note that in this figure we assume the static equilibrium position as the reference of the calculated dynamic deflection amplitude.

Therefore, it is important to consider the increment of the frequency response curve due to application of bias DC voltages. In addition, it is obvious that applied bias DC voltages also decrease the fundamental frequency and limit the upper band of the microphone bandwidth.

4.8. Influence of the NETC on the frequency response of the condenser microphone for different applied bias DC voltages

Figs. 9 and 10 show that positive and negative NETCs increase and decrease the output level and sensitivity of the microphone. In addition, it is obvious that positive NETCs also decrease the fundamental frequency, limit the upper band of the microphone bandwidth, and negative NETCs vice versa. For the case in which \( V = 0 \), the effect of positive and negative NETCs is shown in Fig. 9.

In Fig. 10 the effects of positive and negative NETCs on the output level of the diaphragm are shown in the presence of an electrostatic force \( (V = 3.38) \).

Positive NETCs create compressive thermal stresses and soften the diaphragm. On the other hand, the electrostatic force depends on both the value of applied DC voltage and the deflection of the diaphragm. Therefore, according to the presented results in Fig. 6, in the presence of a DC voltage, a positive NETC increases the deflection; therefore as shown in Fig. 10, positive NETCs decrease the stiffness in the presence of the electrostatic force, and negative NETCs vice versa. Hence, it can be concluded that the sensitivity to the temperature changes in the presence of the bias DC voltage increases.

4.9. Design thermal correcting factor of frequency response

As it was concluded in Sections 4.7 and 4.8, variations of the NETC and the bias DC voltage can both affect the output level of the microphone. For convenience, we introduce a correcting factor...
In which $A_0$ is the design thermal correcting factor of the frequency response, which can be achieved through Fig. 11 for various NETCs and bias voltages.

5. Conclusions

We presented an investigation into the effect of the thermal stresses on the stable region of a capacitive microphone with a thin circular diaphragm. The effects of thermal stresses and applied bias DC voltages on the frequency response of the microphone were studied. In the absence of the thermal stresses' effect, the results showed that the non-dimensional static and dynamic pull-in thresholds of the studied case are 5.23 (38.6 V) and 4.74 (34.98 V), respectively. Study of the thermal stresses' effects on the static and dynamic pull-in instabilities illustrated that the decrement of the diaphragm temperature, which leads to a tensile thermal stress in the diaphragm, increases the stable region of the capacitive diaphragm, and increment of the diaphragm temperature vice versa. The results showed that for a given electrostatic force, the deflection of the diaphragm is changed by variation of the diaphragm temperature. Decrement of the diaphragm temperature decreases the deflection of the diaphragm and increase in the diaphragm temperature raises the deflection of the diaphragm, and for a given temperature, the buckling phenomenon takes place and it is shown that the value of this temperature decreases by increasing the applied DC voltage. In addition, the results showed that for the case in which there is no initial electrostatic force, increase or decrease in the temperature of the diaphragm does not cause any deflection although it leads to the buckling phenomenon for a given increased temperature. The results illustrated that the applied DC voltage and thermal stresses affect output level, sensitivity and the fundamental frequency of the microphone. The diaphragm temperature increment and applied DC voltage increase the output level and sensitivity and decrease the fundamental frequency of the microphone, which limits the upper band of the bandwidth. It is clear that the temperature decrement acts conversely. In addition, it is concluded that increment of the applied DC voltages increases the sensitivity to the temperature changes. Therefore, for study of frequency response of a capacitive microphone, temperature variation is more important for the cases with applied DC voltages especially close to the pull-in voltage. At least the effects of the temperature changes on the pull-in parameters and output level are given by design-correcting factors in order to simplify temperature effects' consideration in the design procedure.

References


