Revenue in Contests with Many Participants

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Abstract

We show that in a contest with many participants and a single prize, the expected effort (or resources) made by a participant whose evaluation of the prize is ranked as the $k^{th}$ highest valuation obeys an exponential rule in the limit and is equal to $1/2^k$ of the total expected effort made by all of the participants. Thus, even if the contest’s organizer can recover only $k$ of the highest effort, the total losses caused in the limit by the lost efforts is $1/2^{k+1}$ of all of the efforts. We also extend our results to contests with $m$ prizes and with risk averse participants.

Keywords: contest, all-pay auction, revenue, risk aversion.

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1 Introduction

We study a contest with $m$ identical prizes where the number of participants approaches infinity. Each participant’s valuation of the prize is private information arrived at independently according to the same common knowledge distribution function. In contest all participants make unrecoverable efforts such as money or resources. According to the Revenue Equivalence Theorem, the sum of the expected efforts is identical to the expected revenue in any auction mechanism such as first- or second-price auctions (see Myerson 1981, Riley and Samuelson 1981). However, a completely different situation arises in contests described in the literature as all-pay auctions. First-price auctions or other selling mechanisms occur when a seller actually sells an object and the winner pays the final price dictated by the auction’s rules. However, contests are used in many situations in which the participants’ efforts are effort or other non-recoverable resources, and the contest’s organizer may not benefit from the sum of the efforts made by all of the participants. If the contest’s organizer cannot recover all of the efforts, what portion of the total resources invested by all of the participants is recoverable? The current study shows that when the number of participants is large (we consider the limit when the number of participants approaches infinity), the expected payment made by the participants obeys an exponential rule. When there is a single prize in the contest, the participant with the highest effort generates $1/2$ of the total expected efforts. The second generates $1/4$, the third $1/8$ etc. Note that the sum of all of the expected efforts approaches 1 when the number of participants approaches infinity. Thus, if the organizer can recover only $k$ of the highest
efforts, the losses generated by the efforts that cannot be recovered is just $1/2^{k+1}$ of the total expected efforts. It follows that even if only a few of the highest efforts can be recovered, the losses are minor. We also extend our results to contests with $m$ prizes and with risk averse participants. Although the results in this paper are new to the literature, it worth noting that others have also studied the effect on the contest’s revenue of increasing the number of participants. One example is the study by Paul and Gutierrez (2004) on second-price auctions.

2 The Model

We consider a contest with $m$ identical prizes and $n$ risk neutral participants, each one of whom has a unit demand. Each participant has a private valuation $v$ for a prize that has been arrived at independently from a continuously differentiable distribution function $F(v)$ over the support $[0,1]$ with a strictly positive density $F' = f \geq 0$. Each participant $i$ makes an effort $b_i$ (e.g., resources, effort, etc.) independently of other participants. Each of the $m$ participants with the highest effort wins a single prize, but all of the $n$ participants pay their effort. To find the symmetric Bayesian equilibrium effort function, we follow the standard arguments (see, for example, Krishna 2002). Assume that there exists a symmetric and monotonic equilibrium effort function $b(v)$. The expected payoff for a participant with value $v$ when he is playing as a participant with value $\hat{v}$ and all other $n - 1$ participants are playing according to the equilibrium effort strategy $b(v)$ is given by
\[ u(\hat{v}; v) = v_i G(\hat{v}) - b_i(\hat{v}). \]

\( G(v) \) is the probability that in equilibrium, participant \( i \) will win one of the \( m \) prizes if his valuation of the prize is \( v \). Since the equilibrium effort function is monotonic with respect to \( v \), \( G(v) \) is the probability that the value \( v \) is one of the \( m \) highest valuations among the \( n \) participants, and is given by

\[
G(v) = \sum_{j=0}^{m-1} \binom{n-1}{j} F^n - j(v)(1 - F(v))^j. \tag{1}
\]

Thus, \( G \) is a distribution function of the \( m^{th} \) highest valuation and its density is given by

\[
G'(v) = \frac{(n-1)!}{(m-1)!(n-m-1)!} F^{n-m-1}(v)(1 - F(v))^{m-1} f(v). \tag{2}
\]

Following standard arguments, the equilibrium effort function is found by solving

\[
\frac{\partial}{\partial v} u(\hat{v}; v) \bigg|_{\hat{v} = v} = 0, \text{ which yields}
\]

\[
b(v) = vG(v) - \int_0^v G(s)ds. \tag{3}
\]

Let \( Y_{k,n} \) denote the distribution of the \( k^{th} \) highest value of \( n \) participants (i.e., the \( k \)-th order statistics). The distribution of \( Y_{k,n} \) is given by\(^1\)

\[
F_{Y_{k,n}}(v) = \sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}(v)(1 - F(v))^i. \tag{4}
\]

\(^1\)The distributions \( G(v) \) and \( F_{Y_{k,n}} \) are both order statistics. However, later on, we use a different notation to simplify the presentation.
3 The Main Result

The organizer’s expected revenue generated by the $k^{th}$ highest valuation participant is given by

$$R_k = \int_0^1 b(v) dF_{Y_{k,n}}(v)$$  \hspace{1cm} (5)

**Proposition 1** Let $R_k^\infty = \lim_{n \to \infty} R_k$. Then,

$$R_k^\infty = 1 - \sum_{i=0}^{k-1} \binom{m+i-1}{i} \frac{1}{2^{m+i}}, \quad k = 1, 2, ...$$  \hspace{1cm} (6)

In particular, if $m = 1$, then

$$R_k^\infty = \frac{1}{2^k}, \quad k = 1, 2, ...$$

Observe that if $m = 1$, then $\sum_{k=1}^\infty R_k^\infty = 1$. Thus, the contribution of the $k^{th}$ highest participant to the organizer’s expected revenue is identical to the total contributions of all of the participants ranked from the $k^{th} + 1$ to infinity. In other words,

$$R_k^\infty = \sum_{j=k+1}^\infty R_j^\infty.$$

An additional interesting observation is that $R_k$ is not necessarily monotonic increasing with $n$. For example, considering $F(v) = v^6$ and $n = 2$ gives $R_1 > 1/2$. However, it is well known (e.g., Krishna 2002 for the monotonicity of the expected revenue in auctions) that $\sum_{j=1}^n R_j^n$ which is the sum of all efforts is monotonic increasing with $n$.

**Proof:** Integrating (5) by parts and substituting (2,4) yields
\[ R_k = b(1) - \frac{(n - 1)!}{(m - 1)! (n - m - 1)!} \int_0^1 v f(v) \sum_{i=0}^{k-1} \binom{n}{i} F^{2n-i-m-1}(v)(1 - F(v))^{i+m-1} dv = \]
\[ = b(1) - \frac{(n - 1)!}{(m - 1)! (n - m - 1)!} \sum_{i=0}^{k-1} \binom{n}{i} \beta(2n - m - i, m + i) \times \]
\[ \times \int_0^1 v f(v) F^{2n-i-m-1}(v)(1 - F(v))^{i+m-1} \frac{1}{\beta(2n - m - i, m + i)} dv \]
\[ = b(1) - \frac{(n - 1)!}{(m - 1)! (n - m - 1)!} \sum_{i=0}^{k-1} \binom{n}{i} \beta(2n - m - i, m + i) E(Y_{i+m,2n-1}) \] (7)

where \( \beta(2n - m - i, m + i) \) is the beta function with parameters \( 2n - m - i \) and \( m + i \) given by

\[ \beta(2n - m - i, m + i) = \frac{(2n - m - i - 1)! (m + i - 1)!}{(2n - 1)!} \] (8)

and \( E(Y_{i+m,2n-1}) \) is the expectation of the random variable \( Y_{i+m,2n-1} \) defined above as the \( i + m \) order statistics of \( 2n - 1 \) iid random variables with distribution \( F(v) \). First, observe that by (3) \( \lim_{n \to \infty} b(1) = 1 \). Recall that \( F(v) \) is defined over the support \([0, 1]\). Then, for fixed \( i + m \), when \( n \to \infty \) the random variable \( Y_{i+m,2n-1} \) is the 100% percentile. Thus, \( \lim_{n \to \infty} E(Y_{i+m,2n-1}) = 1 \) and we find that

\[ R_k^\infty = 1 - \lim_{n \to \infty} \frac{(n - 1)!}{(m - 1)! (n - m - 1)!} \sum_{i=0}^{k-1} \binom{n}{i} \frac{(2n - m - i - 1)! (m + i - 1)!}{(2n - 1)!} \]
\[ = 1 - \sum_{i=0}^{k-1} \frac{(m + i - 1)!}{(m - 1)! i!} \lim_{n \to \infty} \frac{(n - m)(n - m + 1) \cdots (n - 1) \cdot (n - i + 1)(n - i) \cdots n}{(2n - m - i)(2n - m - i + 1) \cdots (2n - 1)} \]
\[ = 1 - \sum_{i=0}^{k-1} \frac{(m + i - 1) i}{(2n)^{m+i}} = 1 - \sum_{i=0}^{k-1} \binom{m + i - 1}{i} \frac{1}{2^{m+i}}. \]
4 A Contest with Many Risk Averse Participants

Assume that every participant has an identical utility function $U(x)$ where $U(0) = 0$ and $U' > 0, U'' < 0$. Let $R_{ra}^k$ be the $k$th highest participant’s expected effort and let $R_{ra,\infty}^k = \lim_{n \to \infty} R_{ra}^k$. In this section we consider the case in which $m = 1$. Fibich and Gavious (2010) show that $R_{\infty} > R_{ra,\infty}$ where $R_{\infty}, R_{ra,\infty}$ are the total expected efforts made by all participants in the case of risk neutral and risk averse participants respectively. Namely, $\sum_{k=0}^{\infty} R_{k}^\infty > \sum_{k=0}^{\infty} R_{k}^{ra,\infty}$. Thus, in a contest with many participants, the contest’s organizer prefers risk neutral participants. We will show that this result is much stronger and that $R_{k}^\infty \geq R_{k}^{ra,\infty}$ for every $k$. Observe that our result that $R_{k}^\infty \geq R_{k}^{ra,\infty}$, together with Fibich and Gavious’ (2010) result that $\sum_{k=0}^{\infty} R_{k}^\infty > \sum_{k=0}^{\infty} R_{k}^{ra,\infty}$, implies that for some $k$’s, a strict inequality holds. In other words, $R_{k}^\infty > R_{k}^{ra,\infty}$.

Proposition 2 The expected effort made by the $k$th highest valuation participant satisfies

$$R_{k}^{ra,\infty} \leq \frac{1}{2k}, \quad k = 1, 2, ...$$

Proof: We modify the technique used by Fibich and Gavious (2010). A participant with valuation $v$’s expected utility in equilibrium with $m = 1$ prizes is given by

$$V(v) = F^{n-1}(v)U(v - b(v)) + (1 - F^{n-1}(v))U(-b(v)).$$

Observe that $V(v) \geq 0$, because otherwise the participant would make an effort of zero to avoid losses. Thus, we have

$$0 \leq \int_0^1 V(v) dF_{Y_{k,n}}(v) = U'(0)A^k_n - U'(0)R_{k}^{ra} - C_n$$

(9)
where

\[ C_n = U'(0) \int_0^1 [(v - b(v))F_{n-1} - (1 - F_{n-1})b(v)]dY_{k,n}(v) - \int_0^1 V(v)dY_{k,n}(v), \]

\[ A_n^k = \int_0^1 vF_{n-1}dY_{k,n}(v), \]

and

\[ R^{ra}_k = \int_0^1 b(v)dY_{k,n}(v). \]

By (9) we get

\[ R^{ra}_k \leq A_n^k - \frac{C_n}{U''(0)}. \]

First, we will prove that

\[ \lim_{n \to \infty} A_n^k = \frac{1}{2^k}, \quad k = 1, 2, \ldots \]

From (2) we know that

\[ dY_{k,n}(v) = \frac{(n)!}{(k-1)!(n-k)!} F_{n-k}(v)(1 - F(v))^{k-1} f(v)dv. \]

Thus, we determine that

\[ A_n^k = \int_0^1 \frac{(n)!}{(k-1)!(n-k)!} F_{2n-k-1}(v)(1 - F(v))^{k-1} f(v)dv = \]

\[ = \frac{(n)!}{(k-1)!(n-k)!} \beta(2n-k,k)E(Y_{k,2n-1}). \]

Using the same technique as in Proposition 1 shows that \( \lim_{n \to \infty} A_n^k = \frac{1}{2^k} \). All that remains is to show that \( C_n \geq 0 \). Then, by expanding \( U(v-b) \) and \( U(-b) \) near zero, we get

\[ C_n = -\int_0^1 \left[ F_{n-1} \left( U(v-b) - (v-b)U'(0) \right) + (1 - F_{n-1}(v)) \left( U(-b(v)) + bU'(0) \right) \right]dY_{k,n}(v) = \]

\[ = -\int_0^1 \left[ F_{n-1} \frac{(v-b)^2}{2} \phi_1(v) + (1 - F_{n-1}(v)) \frac{b^2}{2} \phi_2(v) \right]dY_{k,n}(v) \]

where \( 0 < \phi_1(v) < v-b(v) \) and \( -b(v) < \phi_1(v) < 0 \). Given that \( U'' < 0 \), we determine that \( C_n \geq 0 \). Thus, \( \lim_{n \to \infty} C_n \geq 0 \), thereby completing the proof. \( \Box \)
References


