Monochromatic and zero-sum sets of nondecreasing diameter

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Abstract

For positive integers $m$ and $r$ define $f(m, r)$ to be the minimum integer $n$ such that for every coloring of $\{1, 2, \ldots, n\}$ with $r$ colors, there exist two monochromatic subsets $B_1, B_2 \subseteq \{1, 2, \ldots, n\}$ (but not necessarily of the same color) which satisfy: (i) $|B_1| = |B_2| = m$; (ii) The largest number in $B_1$ is smaller than the smallest number in $B_2$; (iii) The diameter of the convex hull spanned by $B_1$ does not exceed the diameter of the convex hull spanned by $B_2$. We prove that $f(m, 2) = 5m - 3$, $f(m, 3) = 9m - 7$ and $12m - 9 \leq f(m, 4) \leq 13m - 11$. Asymptotically, it is shown that $c_1 mr \leq f(m, r) \leq c_2 m r \log_2 r$, where $c_1$ and $c_2$ are positive constants. Next we consider the corresponding questions for zero-sum sets and we generalize some of our results in the sense of the Erdős–Ginzburg–Ziv theorem. Moreover, stronger versions are derived when the group under consideration is cyclic of prime order.

1. Introduction

While serious progress has been made in the last 20 years in the determination of generalized Ramsey numbers for many families of graphs, not much of a similar progress has been made with Ramsey type problems concerning colorings of the integers. Till recently this area has been dominated by the trial to estimate the growth of the van der Waerden numbers and other general problems which developed from Rado’s dissertation [11], for a recent survey see e.g. [9]. Recently, in [6], exact Rado numbers for certain families of equations were calculated. Along a different line monochromatic configurations which take into account the metric structure of the integers were considered in [1, 5]. In this paper we pose a Ramsey type problem for colorings of the integers which involves three parameters. Assuming one of the

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parameters is 2, we achieve some exact values as well as lower and upper bounds for the general situation. Moreover, our results have a natural generalization in the new developing area of zero-sum Ramsey theorems, see [2, 7] for surveys on this topic.

First we introduce some notation. Let \( \mathbb{N} \) be the set of positive integers. For finite subsets \( X, Y \subseteq \mathbb{N} \), the diameter of \( X \), \( \text{diam}(X) \), is defined by \( \text{diam}(X) = \max(X) - \min(X) \). Moreover, we denote \( X <_p Y \) if and only if \( \max(X) < \min(Y) \). A mapping \( A : X \to C \) is called a coloring and we refer to \( C \) as the set of colors. For a nonempty subset \( Y \subseteq X \) let \( A(Y) \) denote the set \( \{ A(y) | y \in Y \} \). We say that a subset \( Y \subseteq X \) is monochromatic if and only if \( A(y) = A(y^*) \) for all \( y, y^* \in Y \). For \( c \in C \), where \( A^{-1}(c) \neq \emptyset \), we denote first \((c) = \min \{ x \in X | A(x) = c \} \) and last \((c) = \max \{ x \in X | A(x) = c \} \). Moreover, colorings \( A : \{1, 2, \ldots, n\} \to C \) will be identified with the strings \( A(1)A(2)\ldots A(n) \), and we use \( x^i \) to denote the string \( xx\ldots x \) of length \( i \). Finally, let \([a, b]\) denote the set of integers \( \{ n \in \mathbb{N} | a \leq n \leq b \} \).

**Definition 1.** Let \( m, r, t \) be positive integers. Define \( n = f(m, r, t) \) to be the least positive integer \( n \) such that for every coloring \( A : \{1, n\} \to \{1, r\} \) there exist \( t \) monochromatic subsets \( B_1, B_2, \ldots, B_t \subseteq \{1, n\} \) (but elements from different sets \( B_i \) and \( B_j \) may be colored differently), which satisfy:

(i) \( |B_1| = |B_2| = \cdots = |B_t| = m \), and

(ii) \( B_1 <_p B_2 <_p \cdots <_p B_t \), and

(iii) \( \text{diam}(B_1) \leq \text{diam}(B_2) \leq \cdots \leq \text{diam}(B_t) \).

We note that the existence of the numbers \( f(m, r, t) \) follows immediately from van der Waerden’s theorem on arithmetic progressions [13], but the upper bounds implied are certainly very crude, cf. [12]. In this paper we are mainly concerned with \( f(m, r, 2) \) which will be abbreviated as \( f(m, r) \). Section 2 deals with exact values of \( f(m, r) \) for small \( r \)'s and we give asymptotic bounds for \( f(m, r) \). In Sections 3 and 4 we will investigate the corresponding questions for zero-sum sets. Let \( \mathbb{Z}_m \) be the cyclic group of residues modulo \( m \). Let \( A : X \to \mathbb{Z}_m \) be a coloring. A subset \( Y \subseteq X \) is called zero-sum if and only if \( \sum_{y \in Y} A(y) \equiv 0 \mod m \). The motivation for our investigations of zero-sum sets is the following theorem of Erdős, Ginzburg and Ziv.

**Theorem 1** (Erdős et al. [10]). Let \( X \) be a finite set with \( |X| = 2m - 1, m \in \mathbb{N} \). Then for every coloring \( A : X \to \mathbb{Z}_m \) there exists an \( m \)-element subset \( Y \subseteq X \) which is zero-sum.

In Section 5 we extend this result and generalize Theorem 9 of Section 4 for \( m \) prime.

2. Monochromatic sets

In this section we will study the numbers \( f(m, r) \). First we consider the case \( r = 2 \) i.e. two-colorings. To do so we use the following lemma.
**Lemma 1.** Let \( m \) be a positive integer, \( m \geq 2 \). Let \( \Delta : [1, 3m - 2] \rightarrow \{1, 2\} \) be a coloring. Then the following holds:

(i) either there exists a monochromatic \( m \)-element subset \( B \subseteq [1, 3m - 2] \) with \( \text{diam}(B) \geq 2m - 2 \),

(ii) or there exist monochromatic \( m \)-element subsets \( B_1, B_2 \subseteq [1, 3m - 2] \) with \( B_1 < B_2 \) and \( \text{diam}(B_1) = \text{diam}(B_2) = m - 1 \).

**Proof.** Let \( \Delta : [1, 3m - 2] \rightarrow \{1, 2\} \) be a coloring. Let w.l.o.g. \( \Delta(1) = 1 \). If \( |\Delta^{-1}(1)| < m \), then \( |\Delta^{-1}(2)| \geq 2m - 1 \geq m \), hence \( \text{diam}(\Delta^{-1}(2)) \geq 2m - 2 \) and (i) follows.

Therefore, we can assume that \( |\Delta^{-1}(1)| \geq m \). If \( \text{last}(1) \geq 2m - 1 \), then (i) follows again, as \( \Delta(1) = 1 \). Thus let \( \text{last}(1) \leq 2m - 2 \), in particular, \( \Delta(i) = 2 \) for \( i = 2m - 1, 2m, \ldots, 3m - 2 \).

If \( \text{first}(2) \leq m \), we infer that \( \text{diam}(\Delta^{-1}(2)) \geq 2m - 2 \), and (i) follows, as \( |\Delta^{-1}(2)| \geq m \). Otherwise, if \( \text{first}(2) > m \), then the sets \( B_1 = [1, m] \) and \( B_2 = [2m - 1, 3m - 2] \) satisfy (ii).

Observe that the string \( 12^{m-2}2^{m-1} \) defines a coloring \( \Delta : [1, 3m - 3] \rightarrow \{1, 2\} \), such that neither (i) nor (ii) in Lemma 1 hold.

**Theorem 2.** For each positive integer \( m \geq 2 \),

\[ f(m, 2) = 5m - 3. \]

**Proof.** The colorings \( \Delta : [1, 5m - 4] \rightarrow \{1, 2\} \) given by the strings

\[ (21)^{m-1}2^{2m-2m-1} \quad \text{or} \quad 21^{m-1}2^{m-1}1^{m-1}2^{2m-2} \]

show the lower bound \( f(m, 2) \geq 5m - 4 \).

To see that \( f(m, 2) \leq 5m - 3 \), consider an arbitrary coloring \( \Delta : [1, 5m - 3] \rightarrow \{1, 2\} \). By the pigeon hole principle the set \([1, 2m - 1]\) contains a monochromatic \( m \)-element subset \( B_1 \) with \( \text{diam}(B_1) \leq 2m - 2 \). Next consider the restriction of the coloring \( \Delta \) to the set \([2m, 5m - 3]\) which is isometric to the set \([1, 3m - 2]\) and apply Lemma 1.

We consider next the case \( r = 3 \) of three-colorings. For proving the upper bound on \( f(m, 3) \) we use the following two lemmas.

**Lemma 2.** Let \( m \geq 2 \) be an integer. Let \( M \) be a subset of \([1, 6m - 5]\) with \( |M| \geq 5m - 4 \) and let \( \Delta : M \rightarrow \{1, 2\} \) be a coloring. Then

(i) either there exists a monochromatic \( m \)-element subset \( B \subseteq M \) with \( \text{diam}(B) \geq 3m - 3 \),

(ii) or there exist monochromatic \( m \)-element subsets \( B_1, B_2 \subseteq M \) with \( B_1 < B_2 \) and \( \text{diam}(B_1) = \text{diam}(B_2) = m - 1 \).

**Proof.** Let \( M \subseteq [1, 6m - 5] \) with \( |M| \geq 5m - 4 \) and let \( \Delta : M \rightarrow \{1, 2\} \) be given. Let \( P \) and \( Q \) denote the first \( m \) elements of \( M \) and the last \((2m - 1)\) elements of \( M \), respectively.
If the set $P \cup Q$ contains a monochromatic $(m+1)$-element subset $T$ with $T \cap P \neq \emptyset$ and $T \cap Q \neq \emptyset$, then
\[
\text{diam}(T) \geq (m+1) + (2m-3) - 1
\]
\[= 3m - 3
\]
and (i) follows.

Otherwise, as $P \cup Q$ is two-colored, each of the sets $P$ and $Q$ is monochromatic. Call an element $x \in [\min P, \max P]$ but with $x \notin P$ a hole of $P$. As $P$ has at most $(m-1)$ holes, we infer that $\text{diam}(P) \leq 2m-2$, hence $B_1 = P$ and $B_2$ satisfy (ii), where $B_2$ is an $m$-element subset of $Q$ with $\text{diam}(B_2) = \text{diam}(Q)$.

\textbf{Lemma 3.} Let $m$ be a positive integer, $m \geq 2$. Let $A : [1, 6m-5] \rightarrow [1, 3]$ be a coloring. Then

(i) either there exists a monochromatic $m$-element subset $B \subseteq [1, 6m-5]$ with $\text{diam}(B) \geq 3m - 3$,

(ii) or there exist monochromatic $m$-element subsets $B_1, B_2 \subseteq [1, 6m-5]$ with $B_1 < B_2$ and $\text{diam}(B_1) \leq \text{diam}(B_2)$.

\textbf{Proof.} Let $A : [1, 6m-5] \rightarrow [1, 3]$ be given. If one color occurs at most $(m-1)$ times, say, $|A^{-1}(3)| \leq m - 1$, then the set $M = A^{-1}(1) \cup A^{-1}(2)$ is two-colored and satisfies $|M| \geq 5m - 4$. In this case we are done by applying Lemma 2.

We assume in the following that each of the colors 1, 2, 3 occurs at least $m$ times. Let w.l.o.g. $A(1) = 1$. Then $\text{last}(1) \leq 3m - 3$, as otherwise (i) would follow. Therefore, the set $X = [3m-2, 6m-5]$ is colored by colors 2 and 3 only, and certainly contains a monochromatic $m$-element subset $B_2$.

If $A(1) = A(2) = \cdots = A(m) = 1$, then $B_1 = [1, m]$ and $B_2$ would satisfy (ii). Hence, we can assume that there exists some integer $j$ with $2 \leq j \leq m$ such that, say, $A(1) = A(2) = \cdots = A(j-1) = 1$ and $A(j) = 2$. Then $\text{last}(2) \leq 3m + j - 4$, as otherwise (i) would follow. Hence, the set $A = [3m+j-3, 6m-5]$ is monochromatic in color 3 and satisfies $|A| = 3m-1-j \geq 2m-1$. As $\text{last}(3) = 6m-5$, we infer that $\text{first}(3) \geq 3m-1$, as otherwise (i) would follow. But then the set $[1, 2m-1]$ is two-colored and contains a monochromatic $m$-element subset $B_1$ with $\text{diam}(B_1) \leq 2m-2$. Let $B_2$ be an $m$-element subset of $A$ with $\text{diam}(B_2) = \text{diam}(A)$, then (ii) follows, which finishes the proof of Lemma 3.

\textbf{Theorem 3.} For positive integers $m \geq 2$, $f(m, 3) = 9m - 7$.

\textbf{Proof.} The coloring $A : [1, 9m-8] \rightarrow [1, 3]$ given by the string
\[
31^{m-1}2^{m-1}3^{m-1}4^{m-1}5^{m-1}6^{m-1}7^{m-1}8^{m-1}9^{m-1}2
\]
shows that $f(m, 3) \geq 9m - 8$. 

To see that \( f(m, 3) \leq 9m - 7 \), let \( A : \left[1, 9m - 7\right] \rightarrow [1, 3] \) be a coloring. By the pigeon hole principle the subset \( \left[1, 3m - 2\right] \) contains a monochromatic \( m \)-element subset \( B \) with \( \text{diam}(B) \leq 3m - 3 \). By replacing the interval \( [3m - 1, 9m - 7] \) by \( [1, 6m - 5] \) and applying Lemma 3, the proof is finished. \( \square \)

**Lemma 4.** Let \( M \) be a subset of \( [1, 9m - 8] \) with \(|M| \geq 8m - 7\) and let \( A : M \rightarrow [1, 3] \) be a coloring. Then

(i) either there exists a monochromatic \( m \)-element subset \( B \subseteq M \) with \( \text{diam}(B) \geq 4m - 4 \),

(ii) or there exist monochromatic \( m \)-element subsets \( B_1, B_2 \subseteq M \) with \( B_1 \subsetneq B_2 \) and \( \text{diam}(B_1) \leq \text{diam}(B_2) \).

**Proof.** Let \( M \subseteq [1, 9m - 8] \) with \(|M| \geq 8m - 7\) be given and let \( A : M \rightarrow [1, 3] \) be a coloring. Let \( P \) and \( Q \) denote the first \((2m - 1)\) elements of \( M \) and the last \((3m - 2)\) elements of \( M \), respectively.

If the set \( P \cup Q \) contains a monochromatic \((m + 1)\)-element subset \( T \subseteq P \cup Q \) with \( P \cap T, Q \cap T \neq \emptyset \), then \( \text{diam}(T) \geq 4m - 4 \), and (i) follows. Indeed, one can say more. Assuming that (i) does not hold, there exists essentially at most one \( m \)-element subset \( T \subseteq P \cup Q \) with \( P \cap T, Q \cap T \neq \emptyset \), namely, \( T \) consists of an initial segment of \( Q \) and a final segment of \( P \).

Therefore, we assume in the following that

\[ |T| \leq m \tag{1} \]

for every monochromatic subset \( T \subseteq P \cup Q \) with \( P \cap T, Q \cap T \neq \emptyset \), and equality can hold essentially at most once in (1), where in this case \( T \) has the shape as given above.

First assume that all three colors occur in \( P \). As \( Q \) contains clearly a monochromatic \( m \)-element subset, we obtain a monochromatic \((m + 1)\)-element subset of \( P \cup Q \) with \( P \cap T, Q \cap T \neq \emptyset \), which contradicts (1). Hence, the set \( P \) is colored by at most 2 colors. Now suppose that all three colors occur in \( Q \). As \( P \) is two-colored, \( P \) contains a monochromatic \( m \)-element subset, and we obtain again a monochromatic \((m + 1)\)-element subset in \( P \cup Q \) with \( P \cap T, Q \cap T \neq \emptyset \), which contradicts (1) again. Therefore, we assume in the following that \( P \) and \( Q \) are each colored by at most two colors.

We distinguish three cases:

**Case a:** Assume first that there exists a color, say color 1, which occurs in \( P \) as well as in \( Q \), i.e. \( |A^{-1}(1) \cap P| = x > 0 \) and \( |A^{-1}(1) \cap Q| = y > 0 \). By (1) we have \( x + y \leq m \).

The set \( B_1^* = P \setminus A^{-1}(1) \) is monochromatic and satisfies \( |B_1^*| = 2m - 1 - x \geq m \) as \( x \leq m - 1 \). Since there are at most \((m - 1)\) holes in \( M \), and as color 1 occurs \( x \) times in \( P \), there exists an \( m \)-element subset \( B_1 \subseteq B_1^* \) such that

\[ \text{diam}(B_1) \leq m + (m - 1) + x - 1 \]

\[ = 2m + x - 2. \]
Moreover, the set $Q \setminus \Delta^{-1}(1)$ contains a monochromatic $m$-element subset $B_2$ with
\[ \text{diam}(B_2) \geq 3m - 3 - y. \]

For $x + y < m$, $\text{diam}(B_1) > \text{diam}(B_2)$ implies $2m + x - 2 > 3m - 3 - y$ or equivalently $x + y > m - 1$, which is a contradiction. On the other hand, for $x + y = m$ by the remarks following (1) we always have that $\text{diam}(B_1) \leq 2m - 2 \leq 3m - 3 - y \leq \text{diam}(B_2)$, as $y \leq m - 1$.

**Case b:** Assume that the set $P$ is monochromatic. As $|P| = 2m - 1$ and as $P$ has at most $(m - 1)$ holes there exists in $P$ an $m$-element subset $B_1$ with at most $\lfloor (m - 1)/2 \rfloor$ holes, hence
\[ \text{diam}(B_1) \leq m + \left\lfloor \frac{m - 1}{2} \right\rfloor - 1. \]

On the other hand, $Q$ contains a monochromatic $\lceil (3m - 2)/2 \rceil$-element subset $B_2$, thus
\[ \text{diam}(B_2) \geq \frac{3m - 2}{2} - 1, \]
which implies $\text{diam}(B_1) \leq \text{diam}(B_2)$ and hence (ii) is fulfilled, as $|B_1|, |B_2| \geq m$.

**Case c:** Assume that the set $Q$ is monochromatic. The set $P$, $|P| = 2m - 1$, is two-colored and, hence, contains a monochromatic $m$-element subset $B_1$ with
\[ \text{diam}(B_1) \leq 3m - 3, \]
as $P$ contains at most $(m - 1)$ holes.

Moreover, $Q$ contains a monochromatic $m$-element subset $B_2$ with
\[ \text{diam}(B_2) \geq 3m - 3, \]
as $Q$ is monochromatic. Thus $\text{diam}(B_1) \leq \text{diam}(B_2)$ and (ii) is fulfilled. \(\square\)

**Lemma 5.** Let $m \geq 2$ be a positive integer. Let $\Delta : [1,9m - 8] \rightarrow [1, 4]$ be a coloring. Then

(i) either there exists a monochromatic $m$-element subsets $B \subseteq [1,9m - 8]$ with $\text{diam}(B) \geq 4m - 4$,

(ii) or there exist monochromatic $m$-element subsets $B_1, B_2 \subseteq [1,9m - 8]$ with $\text{diam}(B_1) \leq \text{diam}(B_2)$.

**Proof.** Let $\Delta : [1,9m - 8] \rightarrow [1, 4]$ be given. If one color, say, color 4, occurs at most $(m - 1)$ times, i.e. $|\Delta^{-1}(4)| \leq m - 1$, then the set $M = \Delta^{-1}(1) \cup \Delta^{-1}(2) \cup \Delta^{-1}(3)$ is three-colored and satisfies $|M| \geq 8m - 7$. By Lemma 4 the assumption follows.

Hence we can assume in the following that each color occurs at least $m$ times, i.e. $|\Delta^{-1}(i)| \geq m$ for $i = 1, 2, 3, 4$. Let w.l.o.g. $\Delta(1) = 1$. If $\Delta(i) = 1$ for $i = 1, 2, \ldots, m$, then the set $[m + 1, 9m - 8]$ clearly contains a monochromatic subset of cardinality at least $2m - 2 \geq m$, in which case (ii) is satisfied.
Therefore, we can assume that $A(1) = A(2) = \cdots = A(j-1) = 1$ and $A(j) = 2$ for some $j$, with $2 \leq j \leq m$. We infer that
\[
\text{last}(1) \leq 4m - 4
\]
\[
\text{last}(2) \leq 4m - 5 + j
\]
\[
\leq 5m - 5,
\]
as otherwise (i) follows. In particular, the set $A_1 = [5m-4, 9m-8]$ is colored by colors 3 and 4 only. Then $A_1$ contains a monochromatic $(2m-1)$-element subset. Therefore, we can assume that the set $A_2 = [1, 2m-1]$ does not contain any monochromatic $m$-element subset, i.e. $A_2$ is colored by at least three colors, say 1, 2 and 3. But then first (3) implies that last(3) \leq 6m - 6, as otherwise (i) follows. Hence the set $A_3 = [6m-5, 8m-6]$ is monochromatic with $|A_3| = 2m$, and (ii) follows.

**Theorem 4.** For integers $m \geq 2$,
\[
12m - 9 \leq f(m, 4) \leq 13m - 11.
\]

**Proof.** The coloring $A : [1, 12m-11] \rightarrow [1, 2, 3, 4]$ given by the string
\[
41^{m-1}2^{m-1}3^{m-1}4^{m-1}1^{m-1}2^{m-1}3^{m-1}4^{m-1}1^{m-1}2^{m-1}3^{m-1}4^{m-1}1^{m-1}2^{m-1}3^{m-1}4^{m-1}1^{m-1}2^{m-1}3^{m-1}4^{m-1}1^{m-1}2^{m-1}3^{m-1}4^{m-1}1^{m-1}2^{m-1}3^{m-1}4^{m-1}1^{m-1}2^{m-1}3^{m-1}4^{m-1}1^{m-1}2^{m-1}3^{m-1}4^{m-1}
\]
shows that $f(m, 4) > 12m - 10$.

To see the upper bound, let $A : [1, 13m-11] \rightarrow [1, 4]$ be a coloring. By the pigeon hole principle the subset $[1, 4m-3]$ contains a monochromatic $m$-element subset $B$ with $\text{diam}(B) \leq 4m - 4$. By replacing the interval $[4m-2, 13m-11]$ by $[1, 9m-8]$ the proof is finished by Lemma 5.

We believe that $f(m, 4) = 12m - 9$. Our attempts to show this led to tedious considerations involving case analysis. The determination of the exact value of $f(m, 4)$ might give some insight concerning the exact growth of $f(m, r)$. Here we prove the following result.

**Theorem 5.** For positive integers $m \geq 2$ and $r \geq 4$,
\[
3r(m-1) + 3 \leq f(m, r) < ((2m-2) \cdot r + 1) \cdot (2 + \log_2 r) - 1.
\]

**Proof.** The lower bound follows by considering the $r$-coloring $A : [1, 3r(m-1)+2] \rightarrow [1, r]$ given by the string
\[
r_1^{m-1}2^{m-1} \cdots r^{m-1}1^{m-1}2^{m-1} \cdots (r-1)^{m-1}1^{m-1}2^{m-1} \cdots (r-1)^{m-1}r^{2m-1}.
\]

Concerning the upper bound, let $A : [1, n] \rightarrow [1, r]$ be a coloring such that there do not exist monochromatic subsets $B_1, B_2 \subseteq [1, n]$ with $B_1 \subseteq_B B_2$, $|B_1| = |B_2| = m$ and $\text{diam}(B_1) \leq \text{diam}(B_2)$. Partition the set $[1, n]$ into consecutive intervals
\[
I_j = [(2m-2) \cdot r \cdot (j-1) + j, (2m-2) \cdot r \cdot j + j]
\]
for \( j = 1, 2, \ldots, l \) and

\[
I_{i+1} = [(2m-2) \cdot r \cdot l + l + 1, n],
\]

where \( 0 \leq |I_{i+1}| \leq (2m-2) \cdot r \). We will show that \( l \leq 1 + \log_2 r \).

By the pigeon principle, as \(|I_j| = (2m-2)r + 1\), each interval \( I_j, j = 1, 2, \ldots, l \), contains elements \( a^{(j)}_1, a^{(j)}_2, \ldots, a^{(j)}_{2m-1} \) with \( a^{(j)}_1 < a^{(j)}_2 < \cdots < a^{(j)}_{2m-1} \), where \( A(a^{(j)}_1) = A(a^{(j)}_i) \) for \( 1 \leq i_1, i_2 \leq 2m-1 \). Set

\[
d_j = \min \{a^{(j)}_1 - a^{(j)}_j, a^{(j)}_{2m-1} - a^{(j)}_1\}
\]

for \( j = 1, 2, \ldots, l \).

We claim that

\[
d_{j+1} < \frac{d_j}{2}
\]

for \( j = 1, 2, \ldots, l \). Otherwise, if \( d_{j+1} \geq \frac{1}{2} d_j \) for some \( j \), then \( B_1 = \{a^{(j)}_1, a^{(j)}_2, \ldots, a^{(j)}_m\} \) or \( B_1 = \{a^{(j)}_m, a^{(j)}_{m+1}, \ldots, a^{(j)}_{2m-1}\} \) and \( B_2 = \{a^{(j+1)}_1, a^{(j+1)}_2, \ldots, a^{(j+1)}_m, a^{(j+1)}_{m-1}, a^{(j+1)}_{2m-1}\} \) would be two monochromatic \( m \)-element sets with \( \text{diam}(B_1) \leq \text{diam}(B_2) \).

Hence, as \( d_1 \geq m-1 \), we infer that

\[
d_1 \cdot \frac{1}{2^{l-1}} > m - 1,
\]

that is

\[
l < 1 + \log_2 \left( \frac{d_1}{m-1} \right)
\]

\[
\leq 1 + \log_2 r \quad \text{as} \quad d_1 \leq (m-1) \cdot r
\]

and thus

\[
n < ((2m-2)r + 1) \cdot (2 + \log_2 r) - 1. \quad \square
\]

The results given above support the possibility of proving the following upper bound:

\[
f(m, r) \leq crm, \quad \text{where} \ c \ \text{is a constant.}
\]

The lower bound for \( f(m, r) \) given in Theorem 4 can be improved for larger \( r \), for example \( f(2, 9) \geq 31 \) as can be seen from the following string:

9123456789 123456 123456 789 789 99

which, by replacing each digit \( x \) but the first and last by the string \( x^{m-1} \), implies \( f(m, 9) \geq 28(m-1) + 3 \). It is quite natural to investigate the functions \( f(m, 2, t) \) and \( f(2, r, t) \) as well. Using methods similar to those above we can prove the following proposition.
Proposition 1.

\[ f(m, 2, t) \leq cm^2, \quad \text{where } c \text{ is a constant.} \]

The first case that should be resolved is the evaluation of \( f(m, 2, 3) \). At the moment we can prove that

\[ 8m - 4 \leq f(m, 2, 3) \leq 10m - 6. \]

It seems to us that in general the following might be true:

\[ f(m, r, t) \leq cmrt, \quad \text{where } c \text{ is a constant.} \]

Bollobás, Erdős and Jin investigated the function \( f^*(2, r, t) \) (\( f^*(m, r, t) \) is a slight modification of \( f(m, r, t) \); namely, in (iii) of Definition 1 the ‘\( \leq \)’ is replaced by ‘<’), clearly \( f(m, r, t) \leq f^*(m, r, t) \). They obtained the following two results.

Theorem 6 (Bollobás et al. [4]). For each positive integers \( k \) and \( r \):

(i) \( 4r - \log_2 r + 1 \leq f^*(2, r^2) \leq 4r + 1 \).

(ii) If \( r = 2^k \), then \( f^*(2, r^2) = 4r + 1 \).

Theorem 7 (Bollobás et al. [4]). For each integer \( t \geq 3 \),

\[ (t - 1 + t(t + 1)/2)r - (t - 1) \log_2 r + 1 \leq f^*(2, rt) \leq a_t r + 1, \]

where

\[ a_t = \frac{1 + \sqrt{2}}{2} (2 + \sqrt{2})^{t-1} - \frac{\sqrt{2} - 1}{2} (2 - \sqrt{2})^{t-1}. \]

Thus, for fixed \( t \), \( f^*(2, rt) \) and hence \( f(2, rt) \) are linear in \( r \). On the other hand, it follows easily from the pigeon hole principle that

Proposition 2.

\[ f(2, rt) \leq (r(t - 1) + 1)(r + 1). \]

Thus for fixed \( r \) the function \( f(2, rt) \) is linear in \( t \).

3. Zero-sum sets

The theorem of Erdős, Ginzburg and Ziv can be considered as a generalization of the pigeon hole principle for two boxes: if we restrict the coloring \( \mathcal{A} \) in Theorem 1 to take only the values of 0 and 1, then the theorem becomes equivalent to the pigeon hole principle for the distribution of \( 2m - 1 \) objects into two boxes. Before considering the generalization for any number of boxes we shall state a supplement to Theorem 1 proved by Bialostocki and Dierker, which will be used in the following.
Theorem 8 (Bialostok and Dierker [3]). Let $S$ be a finite set with $|S|=2m-2$. If $\Delta : S \rightarrow Z_m$ is a coloring and $S$ does not contain a zero-sum $m$-element subset $T$, then $\Delta(S) = \{a, b\}$ and $|\Delta^{-1}(a)| = |\Delta^{-1}(b)| = m-1$.

Denote by $Z_m^{(k)}$ the disjoint union of $k$ copies of $Z_m$ ($Z_m = Z_m^{(1)}$) and let $\infty$ denote a color with $\infty \notin Z_m^{(k)}$. The following generalization of the pigeon hole principle for any number of boxes is an immediate consequence of Theorem 1.

Corollary 1. Let $S$ be a finite set with $|S|=(2m-2)k+1$ ($|S|=(2m-2)k+m$). If $\Delta : S \rightarrow Z_m^{(k)}$ ($\Delta : S \rightarrow \{\infty\} \cup Z_m^{(k)}$) is a coloring, then $S$ contains an $m$-element subset $T$ which is zero-sum in one of the copies of $Z_m$ (which either is zero-sum in one of the copies of $Z_m$ or is $\infty$-monochromatic).

Definition 2. Let $m$ and $r$ be positive integers with $m, r \geq 2$.

1. Then $f(m,r)$ is the minimum integer $n$ such that for every coloring $\Delta : [1, n] \rightarrow [1, r]$ there exist two $m$-element subsets of $[1, n]$, $B_1$ and $B_2$, satisfying:
   (i) $B_1 \subsetneq B_2$, and
   (ii) $\text{diam}(B_1) \leq \text{diam}(B_2)$, and
   (iii) $B_i$ is monochromatic for $i = 1, 2$.

2. Then $f(m, Z_m^{(k)})$ ($f(m, \{\infty\} \cup Z_m^{(k)})$) is the minimum integer $n$ such that for every coloring $\Delta : [1, n] \rightarrow Z_m^{(k)}$ ($\Delta : [1, n] \rightarrow \{\infty\} \cup Z_m^{(k)}$) there exist two $m$-element subsets of $[1, n]$, $B_1$ and $B_2$, satisfying:
   (i) $B_1 \subsetneq B_2$, and
   (ii) $\text{diam}(B_1) \leq \text{diam}(B_2)$, and
   (iii) $B_i$ is zero-sum in one of the $Z_m$'s for $i = 1, 2$, ($B_i$ either is zero-sum in one of the $Z_m$'s or is $\infty$-monochromatic for $i = 1, 2$).

The following inequalities are obvious:

$$f(m, 2k) \leq f(m, Z_m^{(k)}) \leq f(m, km),$$

$$f(m, 2k + 1) \leq f(m, \{\infty\} \cup Z_m^{(k)}) \leq f(m, km + 1).$$

In Section 2 we determined the values of $f(m, 2)$ and $f(m, 3)$. In the following section we will prove that $f(m, 2) = f(m, Z_m) = 5m - 3$ and $f(m, 3) = f(m, \{\infty\} \cup Z_m) = 9m - 7$. This suggests that the following might be true:

$$f(m, 2k) = f(m, Z_m^{(k)}) \quad \text{and} \quad f(m, 2k + 1) = f(m, \{\infty\} \cup Z_m^{(k)}).$$

4. Zero-sum generalizations

Lemma 6. Let $d, m$ be positive integers, where $m \geq 2$, and let $\Delta : [1, 2m + d] \rightarrow Z_m$ be a coloring. Then the following holds:
(i) either there exists a zero-sum m-element subset \( B \subseteq [1, 2m+d] \) with \( \text{diam}(B) \geq m+d \),

(ii) or there exist two zero-sum m-element subsets \( B_1, B_2 \subseteq [1, 2m+d] \) with \( B_1 \prec_p B_2 \) and \( \text{diam}(B_1) = \text{diam}(B_2) = m-1 \).

Proof. Let \( [1, 2m+d] \) be the disjoint union of \( P, Q \) and \( R \) where \( P = [1, m] \), \( Q = [m+1, m+d] \) and \( R = [m+d+1, 2m+d] \). Consider the sets \( P \cup (R \setminus \{m+d+1\}) \) and \((P \setminus \{m\}) \cup R\). Since \(|P \cup (R \setminus \{m+d+1\})| = |(P \setminus \{m\}) \cup R| = 2m-1\), by Theorem 1 each of the two sets above contains a zero-sum m-element subset \( B_1 \) and \( B_2 \), respectively. If \( B_1 \cap P \neq \emptyset \) and \( B_1 \cap (R \setminus \{m+d+1\}) \neq \emptyset \), then (i) follows. Similarly, if \( B_2 \cap (P \setminus \{m\}) \neq \emptyset \) and \( B_2 \cap R \neq \emptyset \), then (i) follows. Otherwise, we deduce that \( B_1 = P \) and \( B_2 = R \), yielding (ii). \( \square \)

Theorem 9. For every positive integer \( m \),

\[ f(m, Z_m) = 5m - 3. \]

Proof. The string \( 10^{m-1}1^{m-1}0^{m-1}1^{2m-2} \), which corresponds to a coloring \( \Delta : [1, 5m-4] \rightarrow \{0, 1\} \), implies the lower bound \( f(m, Z_m) \geq 5m - 3 \). Next, we show the upper bound \( f(m, Z_m) \leq 5m - 3 \). Let \( \Delta : [1, 5m-3] \rightarrow Z_m \) be a coloring. By Theorem 1 there exists a zero-sum m-element subset \( B_1 \subseteq [1, 2m-1] \) with \( \text{diam}(B_1) \leq 2m-2 \). Thus, it is sufficient to show that the set \([2m, 5m-3] \) either contains an m-element subset \( B_2 \) with \( \text{diam}(B_2) \geq 2m-2 \) or it contains two zero-sum m-element subsets \( B_1 \) and \( B_2 \) satisfying \( B_1 \prec_p B_2 \) with \( \text{diam}(B_1) \leq \text{diam}(B_2) \). Applying Lemma 6 to the set \([2m, 5m-3]\) with \( d = m-2 \) yields the required conclusion. \( \square \)

Theorem 10. For every positive integer \( m \),

\[ f(m, \{\infty\} \cup Z_m) = 9m - 7. \]

Proof. The string \( \infty 0^{m-1}1^{m-1}0^{m-1}1^{m-1}0^{m-1}1^{m-1}0^{m-1}1^{2m-2} \), which shows that \( f(m, Z_m) \geq 9m - 8 \). Next, we will prove that \( f(m, \{\infty\} \cup Z_m) \leq 9m - 7 \). Let \( \Delta : [1, 9m-7] \rightarrow \{\infty\} \cup Z_m \) be a coloring and consider the set \([1, 3m-2]\). Either it contains an \( \infty \)-monochromatic m-element subset or a \((2m-1)\)-element subset, say \( A \), satisfying \( \Delta(A) \subseteq Z_m \). In view of Theorem 1 we get an m-element subset of \([1, 3m-2]\), say \( B_1 \), with \( \text{diam}(B_1) \leq 3m-3 \) where \( B_1 \) is either \( \infty \)-monochromatic or zero-sum. We continue the proof by considering the set \([3m-1, 9m-7]\). The proof of Theorem 10 will be completed once we prove the following lemma. \( \square \)

Lemma 7. Let \( \Delta : [1, 6m-5] \rightarrow \{\infty\} \cup Z_m \) be a coloring. Then one of the following holds:

(i) either there exists an m-element subset \( B \) of \([1, 6m-5]\) satisfying

(a) \( \text{diam}(B) \geq 3m - 3 \), and

(b) \( B \) either is \( \infty \)-monochromatic or is zero-sum,
(ii) or there exist two \( m \)-element subsets \( B_1 \) and \( B_2 \) of \([1, 6m-5]\), satisfying

(a) \( B_1 \prec B_2 \), and
(b) \( \text{diam}(B_1) \leq \text{diam}(B_2) \), and
(c) \( B_i \) either is \( \infty \)-monochromatic or is zero-sum for \( i = 1, 2 \).

**Proof.** We will distinguish three cases according to the cardinalities of the sets \( \Delta^{-1}(Z_m) \) and \( \Delta^{-1}(\infty) \).

*Case a:* \( |\Delta^{-1}(Z_m)| \leq 2m - 2 \). It follows in this case that \( |\Delta^{-1}(\infty)| \geq 4m - 3 \). Hence there exists an \( \infty \)-monochromatic \( m \)-element subset \( B \) of \( \Delta^{-1}(\infty) \) with \( \text{diam}(B) \geq 4m - 4 \geq 3m - 3 \), and (i) follows.

*Case b:* \( |\Delta^{-1}(\infty)| \leq m - 1 \). Then \( |\Delta^{-1}(Z_m)| \geq 5m - 4 \). Let \( P \) and \( R \) be the sets of the first \( m \) elements and the last \( 2m - 1 \) elements of \( \Delta^{-1}(Z_m) \), respectively. Let \( a \) be the last element of \( P \) and let \( b \) be the first element of \( R \) and consider the sets \( S_1 = \{P\} \cup R \) and \( S_2 = P \cup \{R \setminus \{b\} \} \). Notice the following.

**Observation 1.** (1) If there is a zero-sum \( m \)-element subset \( B \) of \( S_1 \) satisfying \( B \cap \{P\} \neq \emptyset \) and \( B \cap R \neq \emptyset \), then \( \text{diam}(B) \geq 3m - 3 \) and (i) follows.

(2) If there is a zero-sum \( m \)-element subset \( B \) of \( S_2 \) satisfying \( B \cap P \neq \emptyset \) and \( B \cap \{R \setminus \{b\}\} \neq \emptyset \), then \( \text{diam}(B) \geq 3m - 3 \) and (i) follows.

Therefore, we assume now that in Observation 1 neither (1) nor (2) holds. In the following we will use a variant of Lemma 6.

**Lemma 8.** Let \( d, m \) be positive integers with \( m \geq 2 \). Let \( A \) be a subset of the positive integers with \( |A| = 2m + d \), \( A = \{a_1, a_2, \ldots, a_{2m+d}\} \) with \( a_1 < a_2 < \cdots < a_{2m+d} \). Then for every coloring \( \Delta : A \to \mathbb{Z}_m \) the following holds:

(i) either there exists a zero-sum \( m \)-element subset \( B \subseteq A \) with \( B \cap \{a_1, a_2, \ldots, a_m\} \neq \emptyset \) and \( B \cap \{a_{m+d+2}, a_{m+d+3}, \ldots, a_{2m+d}\} \neq \emptyset \) or with \( B \cap \{a_1, a_2, \ldots, a_{m-1}\} \neq \emptyset \) and \( B \cap \{a_{m+d+1}, a_{m+d+2}, \ldots, a_{2m+d}\} \neq \emptyset \), i.e. \( \text{diam}(B) \geq m + d \),

(ii) or the two sets \( B_1 = \{a_1, a_2, \ldots, a_m\} \) and \( B_2 = \{a_{m+d+1}, a_{m+d+2}, \ldots, a_{2m+d}\} \) are each zero-sum.

The proof of Lemma 8 is similar to the proof of Lemma 6 and therefore we omit it.

We shall apply Lemma 8 with \( d = m - 2 \) to \( S_1 \) and \( S_2 \). Applying Lemma 8 to \( S_1 \), in view of Observation 1(1), we get a zero-sum \( m \)-element subset \( B_2 \subseteq R \) with \( \text{diam}(B_2) \geq 2m - 2 \), as \( |P \setminus \{a\}| = m - 1 \). Let \( R^* \) be an arbitrary \((m - 1)\)-element subset of \( R \setminus \{b\} \). By Theorem 1, the set \( P \cup R^* \) contains a zero-sum \( m \)-element subset \( B_1 \). In view of Observation 1(2), the set \( B_1 \) is zero-sum with \( \text{diam}(B_1) = m + |P \setminus \Delta^{-1}(\infty)| - 1 \leq 2m - 2 \). Thus \( B_1 \) and \( B_2 \) satisfy (ii) of Lemma 7.

*Case c:* \( |\Delta^{-1}(Z_m)| \geq 2m - 1 \) and \( |\Delta^{-1}(\infty)| \geq m \).

**Subcase c1:** \( d(1) = \infty \). If \( \text{last}(\infty) \geq 3m - 2 \), then (i) follows. Hence, we can assume that \([3m-2, 6m-5] \subseteq \Delta^{-1}(Z_m) \). Applying Lemma 6 with \( d = m - 2 \) to
we may suppose that there exists a zero-sum \( m \)-element subset \( B_2 \) of \( [3m-2, 6m-5] \) with \( \text{diam}(B_2) \geq 2m-2 \). Hence it follows that \([1,2m-1] \cap \Delta^{-1}(\infty) \leq m-1\), yielding \([1,2m-1] \cap \Delta^{-1}(Z_m) \geq m\). Let \( P \) and \( R \) be the set of the first \( m \) elements and the last \( 2m-2 \) elements of \( \Delta^{-1}(Z_m) \), respectively. Clearly \( P \subseteq [1, 2m-1] \) and \( R = [4m-2, 6m-5] \). Moreover, since \( \Delta(1) = \infty \), we get \( \text{diam}(P) \leq 2m-3 \).

Notice that if there would exist a zero-sum \( m \)-element subset \( B \) of \( P \cup R \) satisfying \( B \cap P \neq \emptyset \) and \( B \cap R \neq \emptyset \), then \( \text{diam}(B) \geq 3m-3 \) and (i) follows. Now we shall apply Lemma 8 with \( d = m-2 \) to \( P \cup R \). From the above it follows that we can assume that (ii) of Lemma 8 is implied, consequently \( P \) is a zero-sum \( m \)-element set. Let \( B_1 = P \) and let \( B_2 \) be as above and (ii) of Lemma 7 is satisfied.

**Subcase c2:** \( \Delta(1) \in Z_m \). Let \( Q = [\text{first}(\infty), \text{last}(\infty)] \). If \( |Q| \geq 3m-2 \), then (i) follows. Hence we can assume that \( |Q| \leq 3m-3 \) and consequently \( |[1, 6m-5]| \geq 3m-2 \).

First, we shall show that \( \Delta(6m-5) \in Z_m \). If \( \Delta(6m-5) = \infty \), then first \((\infty) \geq 3m-1 \) and we get a zero-sum \( m \)-element subset \( B \subseteq [1, 2m-1] \) with \( \text{diam}(B) \leq 2m-2 \). This implies that \( |Q| \leq 2m-2 \) and hence \([1, 4m-3] \subseteq \Delta^{-1}(Z_m) \). Now we can apply Lemma 6 with \( d = 2m-3 \) to \([1, 4m-3] \) and the lemma follows. Thus we can suppose that \( \Delta(6m-5) \in Z_m \).

By Lemma 3, we can assume in the following that

\[
|\Delta(\Delta^{-1}(Z_m))| \geq 3. \tag{2}
\]

Set \( P = [1, 2m-1] \) and \( R = [4m-2, 6m-5] \). We will distinguish three cases.

First suppose that \( |P \cap \Delta^{-1}(Z_m)| \geq m-1 \) and that \( |R \cap \Delta^{-1}(Z_m)| \geq m-1 \). Let \( P' \subseteq P \cap \Delta^{-1}(Z_m) \) and \( R' \subseteq R \cap \Delta^{-1}(Z_m) \) with \( 1 \in P' \), and \( 6m-5 \in R' \) and with \( |P'| = |R'| = m-1 \). If \( |\Delta(P' \cup R')| \geq 3 \), then by Theorem 2 there exists a zero-sum \( m \)-element subset \( B \subseteq P' \cup R' \) with \( B \cap P' \neq \emptyset \), \( B \cap R' \neq \emptyset \). Hence, \( \text{diam}(B) \geq m + 2m-2 = 3m-3 \) and (i) follows. Otherwise, if \( P' \cap R' \) contains no zero-sum \( m \)-element subset, the set \( P' \cup R' \) is colored by only two colors, say, \( a \) and \( b \), where \( \Delta^{-1}(a) \cap (P' \cup R') = |\Delta^{-1}(b) \cap (P' \cup R')| = m-1 \). By (2), there exists \( x \in [1, 6m-5] \) \( \cap (P' \cup R') \) with \( \Delta(x) \neq \{a, b\} \). Clearly, \( P' \cup R' \cup \{x\} \) contains a zero-sum \( m \)-element subset \( B \). As \( \Delta(x) \notin \{a, b\} \), we infer that \( \Delta(B) = \{a, b, \Delta(x)\} \). But then, w.l.o.g. \( 1 \in B \) or \( 6m-5 \in B \), and clearly \( B \cap P' \neq \emptyset \), \( B \cap R' \neq \emptyset \). Therefore, \( \text{diam}(B) \geq 4m-4 \geq 3m-3 \) and (i) follows.

Next assume that \( |P \cap \Delta^{-1}(Z_m)| \leq m-2 \). Then \( |P \cap \Delta^{-1}(\infty)| \geq m+1 \), i.e. \( \text{first}(\infty) = d \leq m-1 \). We can assume that \( \text{last}(\infty) \leq 3m-4 + d \), as otherwise (i) follows. But then the set \( A = [3m-3+d, 6m-5] \) is colored by elements of \( Z_m \) only, and, hence, by Lemma 8, w.l.o.g. case (i), as \( |A| = 3m-1-d \), \( A \) contains a zero-sum \( m \)-element subset \( B_2 \) with \( \text{diam}(B_2) \geq 2m-d-1 \). Moreover, \( P \) contains an \( \infty \)-monochromatic \( m \)-element subset \( B_1 \) with \( \text{diam}(B_1) \leq 2m-d-1 \) and (i) follows.

Finally, assume that \( |R \cap \Delta^{-1}(Z_m)| \leq m-2 \). Then \( \text{last}(\infty) = d \geq 5m-3 \), and hence we may suppose that \( \text{first}(\infty) \geq d-3m+4 \geq 2m+1 \). Thus, \( P \) contains a zero-sum \( m \)-element subset \( B_1 \) with \( \text{diam}(B_1) \leq 2m-2 \). Therefore, we can assume that \( \text{first}(\infty) \geq d-2m+3 \). Then, the set \( A = [1,d-2m+2] \cup [d+1, 6m-5] \) is
colored by elements from $Z_m$ only. As $|A| = 4m - 3$, by Lemma 8 in case (i) $A$ contains a zero-sum $m$-element set $B$ with $\text{diam}(B) \geq 3m - 3$ or, in case (ii), the set $B_1 = \{1, 2, \ldots, m\}$ and the set $B_2$ consisting of the last $m$ elements of $A$ are each zero-sum, hence (ii) follows, which finishes the proof of Lemma 7. □

5. Zero-sum sets for $m$ prime

In this section we shall state and prove stronger versions of Theorem 1 in the case where $m$ is a prime. We start with the well-known theorem of Cauchy–Davenport.

**Theorem 11 (Davenport [8]).** If $A$ and $B$ are subsets of $Z_m$ where $m$ is a prime, then $|A + B| = |\{a + b | a \in A \text{ and } b \in B\}| \geq \min\{m, |A| + |B| - 1\}$.

An important particular case of the above theorem is the case where $|B| = 2$, which gives $|A + B| \geq \min\{m, |A| + 1\}$. Using this and induction we obtain the following.

**Corollary 2.** If $A_1, A_2, \ldots, A_{m-1}$ are subsets of $Z_m$, where $m$ is a prime and $|A_i| = 2$ for $i = 1, 2, \ldots, m-1$, then $A_1 + A_2 + \cdots + A_{m-1} = Z_m$.

**Lemma 9.** Let $m$ be a positive integer and let $S$ be a set with $|S| = 2m - 1$. Then for every coloring $\Delta : S \rightarrow C$, where $C$ is an arbitrary set of colors, the following holds:

(i) either there exists a monochromatic subset $A$ of $S$ with $|A| = m$,

(ii) or for each $s \in S$ the elements of $S \setminus \{s\}$ can be partitioned into $\{x_i, y_i\}_{i=1}^{m-1}$ such that $\Delta(x_i) \neq \Delta(y_i)$ for $i = 1, 2, \ldots, m-1$.

**Proof.** Suppose (i) is not satisfied. Fix $x \in S$ and order the elements of $S \setminus \{x\}$ such that the first $v_1$ are monochromatic, the next $v_2$ are monochromatic and so on up to the last $v_k$ that are monochromatic, where w.l.o.g. $v_1 \geq v_2 \geq \cdots \geq v_k$. Let the ordered set $S$ be written accordingly as the sequence $s_1, s_2, \ldots, s_{2m-1}$ with $s_{2m-1} = s$. Define $x_i = s_i$ and $y_i = s_i + m - 1$ for $i = 1, 2, \ldots, m-1$. Since $v_1 \leq m-1$ we get $\Delta(x_i) \neq \Delta(y_i)$ for $i = 1, 2, \ldots, m-1$ and hence (ii) follows. □

By combining Corollary 2 with Lemma 9 we obtain the following generalized version of Theorem 1.

**Theorem 12.** Let $S$ be a set of cardinality $2m - 1$, where $m$ is a prime, and let $\Delta : S \rightarrow Z_m$ be a coloring. Then the following holds:

(i) either there is a monochromatic subset $T$ of $S$ cardinality $m$,

(ii) or for each fixed $s \in S$,

(a) the elements of $S \setminus \{s\}$ can be partitioned into $\{x_i, y_i\}_{i=1}^{m-1}$ such that $\Delta(x_i) \neq \Delta(y_i)$ for $i = 1, 2, \ldots, m-1$, and

(b) $\{\sum_{z \in T} \Delta(z) | T \in \mathcal{F}\} = Z_m$, where $\mathcal{F}$ is the set of all transversals through the $m-1$ sets $\{x_i, y_i\}_{i=1}^{m-1}$. 


Lemma 10. Let \( m \geq 3 \) be a prime and let \( \Delta : [1, 3m-2] \to \mathbb{Z}_m \) be a coloring. If \( \mathcal{B} = \{ B | B \subseteq [1, 3m-2], |B| = m \text{ and } \text{diam}(B) \leq 2m-2 \} \), then the following holds:

(i) either there exists a monochromatic set \( B \in \mathcal{B} \),
(ii) or \( \{ \sum_{x \in B} \Delta(x) | B \in \mathcal{B} \} = \mathbb{Z}_m \),
(iii) or there exist two monochromatic \( m \)-element subsets of \( [1, 3m-2] \) with \( B_1 \subsetneq B_2 \) and \( \text{diam}(B_1) = \text{diam}(B_2) = m-1 \).

Proof. Let \( P \) and \( R \) denote the sets of the first \( m \) elements and the last \( m \) elements of \( [1, 3m-2] \), respectively. We consider two cases.

Case a: \( R \) is monochromatic. Suppose \( \Delta(R) = a \). If there is an \( x \in P \) with \( \Delta(x) = a \), then (i) follows. Thus we can assume that \( \Delta(x) \neq a \) for every \( x \in P \). If \( P \) is monochromatic, then (iii) follows. Finally, we assume that there exist \( u, w \in P \) with \( \Delta(u) \neq \Delta(w) \). Consider the set \( S = P' \cup R' \) where \( P' = P \setminus \{ u, w \} \) and \( R' = [2m, 3m-2] \). The set \( S \) can be partitioned into \( \{ x_i, y_i \}_{i=1}^{m-2} \) and \( \{ 3m-2 \} \), where \( x_i \in P' \) and \( y_i \in R' \). Let \( T \) be a transversal through \( \{ u, w \} \) and the \( m-2 \) sets above. Since \( 3m-2 \in T \) and since \( T \cap R \neq \emptyset \), we have \( \text{diam}(T) \geq 2m-2 \) and hence (ii) follows by Corollary 2.

Case b: \( R \) is not monochromatic. Let \( u, w \in R \) with \( \Delta(u) \neq \Delta(w) \), and let \( S \) be a monochromatic subset of \( [1, 3m-2] \) of maximum cardinality. If \( |S| \geq 2m-1 \), then (i) follows. Thus we can assume that \( |S| < 2m-2 \). If either \( |S| < 2m-2 \) or \( 1 \in S \), then we can find \( m-2 \) pairwise disjoint subsets of \( [1, 3m-2] \setminus \{ u, w, 1 \} \), say \( \{ x_i, y_i \}_{i=1}^{m-2} \) satisfying \( \Delta(x_i) \neq \Delta(y_i) \) for \( i = 1, 2, \ldots, m-2 \). Let \( T \) be a transversal through the sets \( \{ u, w \}, \{ x_i, y_i \}_{i=1}^{m-2} \) and \( \{ 1 \} \). Since \( 1 \in T \) and since \( T \cap R \neq \emptyset \) we get \( \text{diam}(T) \geq 2m-2 \) and (ii) follows by Corollary 2. So, we assume that \( |S| = 2m-2 \) and \( 1 \notin S \). Observe that \( S \) is a segment of \( [1, 3m-2] \), as otherwise (i) follows. Let \( M = [1, 3m-2] \setminus S \). If \( M \) is monochromatic, then since \( 1 \in M \) and \( M \cap R \neq \emptyset \) we get \( \text{diam}(M) \geq 2m-2 \) and (i) follows. Thus we can assume that \( M \) is not monochromatic and \( u \in M \) or \( w \in M \). Let \( \{ x_i, y_i \}_{i=1}^{m-2} \) be \( m-2 \) pairwise disjoint sets such that \( x_i \in S \) and \( y_i \in [1, 3m-2] \setminus (S \cup \{ u, w \}) \) and let \( x \in S \setminus (\bigcup_{i=1}^{m-2} \{ x_i, y_i \}) \cup \{ u, w \} \). Consider a transversal \( T \) through the sets \( \{ u, w \}, \{ x_i, y_i \}_{i=1}^{m-2} \) and \( \{ x \} \). Clearly \( x \in T \). If there is an \( x' \in T \) where \( x' \neq x \) and \( x' \neq x \), then since \( S \) is a segment of \( 2m-2 \) elements, there is a set \( B \) satisfying \( \sum_{b \in B} \Delta(b) = \sum_{x \in T} \Delta(x) \) and \( \text{diam}(B) > \text{diam}(T) > \text{diam}(S) \geq 2m-3 \). Otherwise \( 1 \in T \) and since \( T \cap R \neq \emptyset \) we get \( \text{diam}(T) \geq 2m-2 \). Consequently in either case (ii) follows. \( \square \)

Theorem 13. Let \( m \) be a prime and let \( \Delta : [1, 5m-3] \to \mathbb{Z}_m \) be a coloring. Let

\[ \mathcal{B} = \{ (B_1, B_2) | B_1, B_2 \subseteq [1, 5m-3], |B_1| = |B_2| = m, B_1 \subsetneq B_2, \text{ and } \text{diam}(B_1) \leq \text{diam}(B_2) \} \].

Then the following holds:

(i) either there exists a pair \((B_1, B_2) \in \mathcal{B}\) such that \( B_1 \) and \( B_2 \) are each monochromatic,

(ii) or for every \( a \in \mathbb{Z}_m \) there exists a pair \((B_1, B_2) \in \mathcal{B}\) such that \( B_1 \) is monochromatic and \( \sum_{x \in B_2} \Delta(x) = a \).
(iii) or for every $a \in \mathbb{Z}_m$ there exists a pair $(B_1, B_2) \in \mathcal{B}$ such that $\sum_{x \in B_1} \Delta(x) \equiv a \mod m$ and $B_2$ is monochromatic.

(iv) or for every pair $(a, b) \in \mathbb{Z}_m \times \mathbb{Z}_m$ there exists a pair $(B_1, B_2) \in \mathcal{B}$ such that $\sum_{x \in B_1} \Delta(x) \equiv a \mod m$ and $\sum_{x \in B_2} \Delta(x) \equiv b \mod m$.

Proof. Combining the results of Lemma 9 for the set $[1, 2m-1]$ and the results of Lemma 10 for the set $[2m, 5m-3]$ the proof follows. \qed

References