Performance Analysis of Coded Cooperation Diversity

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Abstract—In a multi-user environment, coded cooperation creates transmit diversity for small mobiles (e.g. handsets) that cannot support more than one antenna. Coded cooperation allows these mobiles to share their antennas via a simple and effective coding method. In this work we present an analytical methodology for evaluating the performance of coded cooperation. We develop tight bounds for bit and block error probabilities, showing in the process that coded cooperation achieves maximal diversity. We demonstrate the validity of these bounds via simulations.

I. INTRODUCTION

Uplink transmit diversity normally would require multiple antennas at the mobile, a requirement that may be impractical for a variety of reasons. Cooperation diversity is a method to allow single-antenna mobiles to share their antennas and thus achieve transmit diversity. Cooperation diversity was introduced in a CDMA framework in [1], and extended and asymptotically analyzed in a TDMA framework in [2]. Recognizing that these methods are equivalent to repetition codes, [3], [4] introduced coded cooperation which is simple, applies to various multiple access schemes, removes error propagation, and provides coding gain at no extra cost.

In this work we present an analytical methodology for evaluating the performance of coded cooperation. In developing these results, we use tools and techniques from Craig [5], Simon and Alouini [6], and Malkamäki and Leib [7]. The results we obtain provide accurate estimates for the bit error rate (BER) and block error rate (BLER) of coded cooperation. In addition, these results show that the coded cooperation framework does achieve full diversity. We demonstrate the validity of these bounds via simulations.

II. CODED COOPERATION

Our system uses BPSK modulation, with all users having the same transmit power. The channels between users (interuser) and each user and the base station (uplink) are mutually independent with flat Rayleigh fading and zero-mean additive white Gaussian noise (AWGN). We assume that all receivers have channel state information and employ coherent detection, so that in the subsequent analysis we need only consider the fading coefficient magnitudes. In addition, we assume that interuser channels are reciprocal, as in [1]. That is, two cooperating users see identical fading coefficients between them.

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The users encode blocks of $K$ source bits using a cyclic redundancy check (CRC) code concatenated with an error correcting code, so that, for an overall rate $R$ code, we have $N = K/R$ total code bits per source block. Two users cooperate by dividing the transmission of their $N$-bit code words into two successive time segments, or frames. In the first frame, each user transmits a rate $R_1 > R$ code word with $N_1 = K/R_1$ bits. This can be viewed as a subset of the total $N$ allocated code bits that contains all the original information. Each user also receives and decodes the partner’s transmission. If the user successfully decodes the partner’s rate $R_1$ code word, determined by checking the CRC bits, the user computes and transmits $N_2$ additional parity bits for the partner’s data in the second frame, where $N_1 + N_2 = N$. These additional parity bits are selected such that they can be combined with the first frame code word to produce a more powerful rate $R$ code word. If the user does not successfully decode the partner, $N_2$ additional parity bits for the user’s own data are transmitted. Each user always transmits a total of $N$ bits per source block over the two frames, and we define the level of cooperation as $N_2/N$, the ratio of the partner’s bits transmitted. Note also that the users need only transmit in their own multiple access channels. Figure 1 illustrates the operation of the scheme.

In general, various FEC coding methods can be used within this coded cooperation framework. For example, one can use block or convolutional codes, and use either puncturing or product codes for cooperation. In addition to the general framework, [3] and [4] also propose a simple but very effective implementation using rate-compatible punctured convolutional (RCPC) codes [8]. In this implementation, the overall rate $R$ code is selected from a given RCPC code family (for example,
the mother code). The code word for the first frame is then obtained by applying the puncturing matrix corresponding to rate $R_1$, and the additional parity bits for the second frame are those bits that are punctured in the first frame.

Since the users act independently in the second frame, not knowing whether their partner successfully decoded their data, there are four possible scenarios for the transmission of the second frame. First, if both users successfully decode each other, they both send additional parity bits for each other, resulting in the scenario depicted in Figure 1. Second, if neither user is successfully decoded, the system reverts to the non-cooperative case for that block. Finally, if one user decodes the partner but not vice versa, then both users will transmit one user’s additional parity bits in the second frame. These two copies will be optimally combined at the base station prior to decoding.

III. PAIRWISE ERROR PROBABILITY

The pairwise error probability (PEP) for a coded system is defined as deciding in favor of code word $e = (e_1, e_2, \ldots, e_N)$ when code word $c = (c_1, c_2, \ldots, c_N)$ was transmitted. In general, for a binary code with BPSK modulation, coherent detection, and maximum-likelihood (ML) decoding, the PEP conditioned on the set of fading coefficient magnitudes $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ can be written as [6, (12.13)]

$$P(c \rightarrow e | \alpha) = Q \left( \sqrt{2 \sum_{n \in n} \gamma_n} \right)$$

where $Q(x)$ denotes the Gaussian $Q$-function, and

$$\gamma_n = \frac{\alpha_n^2 E_n}{N_0}$$

is the instantaneous signal-to-noise ratio (SNR) of received bit $n$. The set $\eta$ is the set of all $n$ for which $e_n \neq c_n$, and the cardinality of $\eta$ is equal to the Hamming distance $d$ between code words $c$ and $e$. For linear codes, the PEP depends only on $d$ and not the particular code words $c$ and $e$, so that the conditional PEP is typically denoted by $P(d|\alpha)$ or $P(d|\gamma)$.

A. Cooperation with Slow Fading

For the case of slow fading, the fading coefficients for each user uplink channel are constant over the code word. As a result, we have $\alpha_n^{(i)} = \alpha^{(i)}$ and $\gamma_n^{(i)} = \gamma^{(i)}$ constant for all $n$, where the superscript $(i)$ denotes User $i$’s uplink channel. For two-user cooperation, when both users successfully decode each other, each user’s coded bits are divided between the two user channels. We can thus write (1) as

$$P(d|\gamma^{(1)}, \gamma^{(2)}) = Q \left( \sqrt{2d_1\gamma^{(1)} + 2d_2\gamma^{(2)}} \right).$$

where $d_1$ and $d_2$ are the numbers of bits in the Hamming weight $d$ that are transmitted through User 1’s channel and User 2’s channel respectively, such that $d_1 + d_2 = d$.

To obtain the unconditional PEP we must average (3) over the fading distributions, as

$$P(d) = \int_{0}^{\infty} \int_{0}^{\infty} P(d|\gamma^{(1)}, \gamma^{(2)}) p(\gamma^{(1)}) p(\gamma^{(2)}) d\gamma^{(1)} d\gamma^{(2)}$$

where $p(x)$ is the probability density function (PDF) of $x$. We can obtain an exact solution to (4) using the techniques of Simon and Alouini [6]. The first step is to use the following alternative representation for the Gaussian $Q$-function, originally derived by Craig [5], and then applied to performance analysis in fading channels in [6]

$$Q(x) = \frac{1}{\pi} \int_{0}^{\pi/2} \exp \left( -\frac{x^2}{2\sin^2 \theta} \right) d\theta, \quad x \geq 0.$$  

Using (5) in (3) and (4) gives

$$P(d) = \frac{1}{\pi} \int_{0}^{\pi/2} \left[ \int_{0}^{\infty} \exp \left( -\frac{d_1\gamma^{(1)}}{\sin^2 \theta} \right) p(\gamma^{(1)}) d\gamma^{(1)} \right] \times \left[ \int_{0}^{\infty} \exp \left( -\frac{d_2\gamma^{(2)}}{\sin^2 \theta} \right) p(\gamma^{(2)}) d\gamma^{(2)} \right] d\theta.$$  

Using the results given in [6], we can evaluate the two inner integrals in (6). For the case of Rayleigh fading, we obtain

$$P(d) = \frac{1}{\pi} \int_{0}^{\pi/2} \left[ 1 + \frac{d_1\gamma^{(1)}}{\sin^2 \theta} \right]^{-1} \left[ 1 + \frac{d_2\gamma^{(2)}}{\sin^2 \theta} \right]^{-1} d\theta,$$

where $\gamma^{(i)}$ is the average SNR for User $i$’s uplink channel. Equation (7) is an exact expression for the unconditional PEP and is easily evaluated with numerical integration techniques (we can obtain a closed form expression using [6, (5A.58)-(5A.60)], but it provides no additional insight to the coded cooperation scheme).

We can obtain the following upper bound for (7) by noting that the integrand is maximized for $\sin^2 \theta = 1$, so that

$$P(d) \leq \frac{1}{2} \left( \frac{1}{1 + d_1\gamma^{(1)}} \right) \left( \frac{1}{1 + d_2\gamma^{(2)}} \right).$$

In (8) we see that, for large SNR, the PEP is inversely proportional to the product of the average SNR for the user uplink channels. Thus, provided that $d_1$ or $d_2$ is not zero, full diversity order of two is achieved when both partners successfully receive each other and cooperate. This is a significant improvement over no cooperation (see (17)).

For the case in which User 1 does not successfully decode User 2, but User 2 successfully decodes User 1, both users send the same additional parity bits for User 1 in the second frame, which are then optimally combined at the base station. In this case, the conditional PEP for User 1 becomes

$$P(d|\gamma^{(1)}, \gamma^{(2)}) = Q \left( \sqrt{2d_1\gamma^{(1)} + 2d_2(\gamma^{(1)} + \gamma^{(2)})} \right).$$

$$= Q \left( \sqrt{2d_2\gamma^{(2)} + 2d_2\gamma^{(2)}} \right)$$

where $d_1$ and $d_2$ are the numbers of bits in the Hamming weight $d$ that are transmitted through User 1’s channel and User 2’s channel respectively, such that $d_1 + d_2 = d$. 


and unconditional PEP becomes
\[ P(d) = \frac{1}{\pi} \int_{0}^{\pi/2} \left( 1 + \frac{d \gamma_1(1)}{\sin^2 \theta} \right)^{-1} \left( 1 + \frac{d \gamma_2(2)}{\sin^2 \theta} \right)^{-1} d\theta \]
\[ \leq \frac{1}{2} \left( \frac{1}{1 + d \gamma_1(1)} \right) \left( \frac{1}{1 + d \gamma_2(2)} \right). \quad (10) \]

**B. Cooperation with Fast Fading**

In fast fading when both users successfully decode each other in the first frame, each user’s coded bits are again split between the two uplink channels. However, the fading coefficients for each channel are no longer constant over the code word, but are i.i.d. for each code bit. Thus we can rewrite (1) as

\[ P(d|\gamma^{(1)},\gamma^{(2)}) = Q \left( \sqrt{2 \sum_{n \in \eta^{(1)}} \gamma_n^{(1)} + 2 \sum_{n \in \eta^{(2)}} \gamma_n^{(2)}} \right) \quad (11) \]

where the set \( \eta^{(i)} \) is the set of all \( n \) for which \( c_n \neq e_n \) corresponding to bits transmitted through User \( i \)'s channel. The cardinalities of \( \eta^{(1)} \) and \( \eta^{(2)} \) are \( d_1 \) and \( d_2 \) respectively.

Averaging over the fading to obtain the unconditional PEP now involves a \( d \)-fold integration. Again however, the techniques of \[6\] provide a tractable solution. Applying (5) gives the following integral expression for unconditional PEP

\[ P(d) = \frac{1}{\pi} \int_{0}^{\pi/2} \left( 1 + \frac{\gamma_1(1)}{\sin^2 \theta} \right)^{-d_1} \left( 1 + \frac{\gamma_2(2)}{\sin^2 \theta} \right)^{-d_2} d\theta \]
\[ \leq \frac{1}{2} \left( \frac{1}{1 + \gamma_1(1)} \right)^{d_1} \left( \frac{1}{1 + \gamma_2(2)} \right)^{d_2}. \quad (13) \]

Each of the inner integrals in (12) has the same form as those in (6), so that for Rayleigh fading we easily obtain

\[ P(d) = \frac{1}{\pi} \int_{0}^{\pi/2} \left( 1 + \frac{\gamma_1(1)}{\sin^2 \theta} \right)^{-d_1} \left( 1 + \frac{\gamma_2(2)}{\sin^2 \theta} \right)^{-d_2} d\theta \]
\[ \leq \frac{1}{2} \left( \frac{1}{1 + \gamma_1(1)} \right)^{d_1} \left( \frac{1}{1 + \gamma_2(2)} \right)^{d_2}, \quad (13) \]

where we assume that \( \gamma_i^{(i)} \) is constant over \( n \). (Again we can obtain a closed form expression for the integral in (13) from \[6, 5A.58)-(5A.60)], but this gives no further insight). The bound in (13) indicates that the diversity order for fast fading is equal to the Hamming weight. Since this is also the case for no cooperation (see (18)), this result suggests that the improvement over no cooperation in fast fading is limited. This is consistent with the experimental results presented in [4].

Similar to the slow fading case, when User 1 does not successfully decode User 2, but User 2 successfully decodes User 1, \( d \) becomes \( d \) in (11) and (13), giving for User 1

\[ P(d|\gamma^{(1)},\gamma^{(2)}) = Q \left( \sqrt{2 \sum_{n \in \eta^{(1)}} \gamma_n^{(1)} + 2 \sum_{n \in \eta^{(2)}} \gamma_n^{(2)}} \right) \quad (14) \]

\[ P(d) = \frac{1}{\pi} \int_{0}^{\pi/2} \left( 1 + \frac{\gamma_1(1)}{\sin^2 \theta} \right)^{-d_1} \left( 1 + \frac{\gamma_2(2)}{\sin^2 \theta} \right)^{-d_2} d\theta \]
\[ \leq \frac{1}{2} \left( \frac{1}{1 + \gamma_1(1)} \right)^{d_1} \left( \frac{1}{1 + \gamma_2(2)} \right)^{d_2}. \quad (15) \]

**C. No Cooperation**

For the non-cooperative case in slow fading, we can obtain the conditional and unconditional PEP by simply setting \( d_1 = d \) and \( d_2 = 0 \) in (3) and (7), resulting in

\[ P(d|\gamma) = Q \left( \sqrt{2d\gamma} \right) \quad (16) \]

\[ P(d) = \frac{1}{\pi} \int_{0}^{\pi/2} \left( 1 + \frac{\gamma(1)}{\sin^2 \theta} \right)^{-d} d\theta \leq \frac{1}{2} \left( \frac{1}{1 + \gamma} \right)^d. \quad (17) \]

Similarly for fast fading, we see that the conditional PEP is given by (1), and the unconditional PEP based on (13) is

\[ P(d) = \frac{1}{\pi} \int_{0}^{\pi/2} \left( 1 + \frac{\gamma(1)}{\sin^2 \theta} \right)^{-d} d\theta \leq \frac{1}{2} \left( \frac{1}{1 + \gamma} \right)^d. \quad (18) \]

These are not new results (see for example \[8, 9\]) but are included here because they are used to compute the end-to-end performance of the coded cooperation scheme in Section IV-B.

**IV. BIT AND BLOCK ERROR RATE ANALYSIS**

The probabilities calculated so far depend on the mode of operation (cooperative versus non-cooperative). In this section, we calculate the probability of operating in the cooperative mode, which leads us to the computation of end-to-end error probabilities.

**A. Cooperation Analysis**

Cooperation depends on the instantaneous quality of the interuser channel — a user operates in the cooperative mode in the second frame whenever the partner’s first frame is correctly decoded. As mentioned before, the two users each with two possibilities give rise to four cases. We parameterize these four cases with \( \Theta = \{1, 2, 3, 4\} \).

We intend to calculate the conditional interuser block error probabilities in terms of error event probabilities. We use the approach of \[7, 10\] and bound the BLER via:

\[ P_{\text{block}}(\gamma) \leq 1 - (1 - P_E(\gamma))^B \quad (19) \]
\[ \leq B \cdot P_E(\gamma) \quad (20) \]

where \( B \) is the number of trellis branches in the code word, and \( P_E(\gamma) \) is the error event probability as a function of \( \gamma \), the vector state of the channel. Note that for a rate \( 1/n \) convolutional code (or one obtained by puncturing a rate \( 1/n \) code), \( B \) equals the uncoded block length \( K \). \( P_E(\gamma) \) is upper bounded by \[11, (4.4.5)]

\[ P_E(\gamma) \leq \sum_{d=d_f}^{\infty} a(d) P(d|\gamma) \quad (21) \]
where \( d_f \) is the free distance of the code and \( a(d) \) is the number of error events with Hamming weight \( d \). Now we can express the probabilities for individual cases:

- **Case \((\Theta = 1)\) - both users decode each other correctly.**
  Both users act in the cooperative mode.
  \[
  P(\Theta = 1|\gamma) = (1 - P_{\text{block}}^{(1)}(\gamma))(1 - P_{\text{block}}^{(2)}(\gamma)) \\
  \leq (1 - P_E^{(1)}(\gamma))^B (1 - P_E^{(2)}(\gamma))^B \\
  \leq (1 - B P_E^{(1)}(\gamma))(1 - B P_E^{(2)}(\gamma))
  \]

- **Case \((\Theta = 2)\) - neither user decodes the other correctly.**
  The system falls back into non-cooperative mode.
  \[
  P(\Theta = 2|\gamma) = P_{\text{block}}^{(1)}(\gamma) \cdot P_{\text{block}}^{(2)}(\gamma) \\
  \leq [1 - (1 - P_E^{(1)}(\gamma))^B][1 - (1 - P_E^{(2)}(\gamma))^B] \\
  \leq B^P P_E^{(1)}(\gamma) \cdot P_E^{(2)}(\gamma)
  \]

- **Case \((\Theta = 3)\) - User 2 decodes User 1, but not vice versa.**
  User 2 acts in cooperative mode, but not User 1.
  \[
  P(\Theta = 3|\gamma) = (1 - P_{\text{block}}^{(1)}(\gamma)) \cdot P_{\text{block}}^{(2)}(\gamma) \\
  \leq 1 - P_E^{(1)}(\gamma)^B [1 - (1 - P_E^{(2)}(\gamma))^B] \\
  \leq B \cdot (1 - B P_E^{(1)}(\gamma)) \cdot P_E^{(2)}(\gamma)
  \]

- **Case \((\Theta = 4)\) - User 1 decodes User 2, but not vice versa.**
  User 1 acts in cooperative mode, but not User 2.
  \[
  P(\Theta = 4|\gamma) = P_{\text{block}}^{(1)}(\gamma) \cdot (1 - P_{\text{block}}^{(2)}(\gamma)) \\
  \leq [1 - (1 - P_E^{(1)}(\gamma))^B](1 - P_E^{(2)}(\gamma))^B \\
  \leq B \cdot P_E^{(1)}(\gamma) \cdot (1 - B P_E^{(2)}(\gamma))
  \]

To calculate end-to-end error probabilities, we need the unconditional \( P(\Theta) \), which is calculated by:

\[
P(\Theta) = \int \gamma P(\Theta|\gamma) \cdot p(\gamma) \, d\gamma
\]

For each of conditional probabilities above we found two expressions. As noted in [7], simple expressions of the type (20) do not yield tight bounds in the case of slow fading. For our slow fading calculations we use approximations of the type (19) with the limit-before-average technique given in [7]. For example, for case \( \Theta = 1 \) a tight upper bound is obtained by:

\[
P(\Theta = 1) \leq \int_\gamma \left(1 - \min \left[1, \sum_i a_1(d)P(d|\gamma)\right]\right)^B \\
\times \left(1 - \min \left[1, \sum_i a_2(d)P(d|\gamma)\right]\right)^B p(\gamma) \, d\gamma
\]

Other cases can be calculated similarly.

**B. End-to-End Error Analysis**

The overall end-to-end unconditional BER is equal to the average of the unconditional BER over the four possible transmission scenarios discussed in Section IV-A as

\[
P_b = \sum_{i=1}^4 P_b(\Theta)P(\Theta = i)
\]

where \( P_b \) denotes the BER. The end-to-end BLER has an identical expression.

The conditional BLER is given by (19)–(21), and the unconditional BER is bounded by [11, (4.8)]

\[
P_b(\gamma, \Theta) \leq \frac{1}{k_e} \sum_{d=d_f}^\infty c(d)P(d|\gamma, \Theta)
\]

where \( c(d) \) is the number of information bit errors corresponding to code words or error events with Hamming weight \( d \), and \( k_e \) is the number of input bits per each branch of the code trellis.

As mentioned previously, for sufficiently tight bounds for slow fading, we must use the limit-before-average technique from [7]. The unconditional BER and BLER are given by

\[
P_b(\Theta) \leq \int_\gamma \min \left[\frac{1}{2}, P_b(\gamma, \Theta)\right] p(\gamma) \, d\gamma
\]

\[
P_{\text{block}}(\Theta) \leq \int_\gamma \min \left[1, P_{\text{block}}(\gamma, \Theta)\right] p(\gamma) \, d\gamma.
\]

For fast fading, it is sufficient to use the unconditional (on fading) \( P(d|\Theta) \) directly in the summations (21) and (27).

The following lists the relevant PEP expressions corresponding to each of the four transmission scenarios already noted. These expressions are used as described above to compute the end-to-end BER and BLER for User 1:

- **Case \((\Theta = 1)\) - (3), (13).**
- **Case \((\Theta = 2)\) - (16), (18).**
- **Case \((\Theta = 3)\) - (9), (15).**
- **Case \((\Theta = 4)\) - (16), (18)**
  (with \( a(d) \) and \( c(d) \) for rate \( R_1 \) code).

**V. Numerical Results**

In this section, we compare the BER and BLER bounds with simulation results for the RCPC code implementation of coded cooperation. In all the examples we use the family of RCPC codes with memory \( M = 4 \) and puncturing period \( P = 8 \) given by Hagenauer [8], and we use a source block size of \( K = 128 \) bits. For slow fading, we use overall code rate \( R = 1/4 \), and for fast fading we use \( R = 2/5 \).

Figure 2 shows BER and BLER results for slow Rayleigh fading with an interuser channel of 10dB average SNR, and both user uplink channels having equal average SNR. Furthermore, we consider both 50% and 25% cooperation. Figure 3 shows BER and BLER results for fast Rayleigh fading. The interuser channel average SNR is again 10dB, but now the users have different quality uplink channels. User 1’s uplink channel is fixed at 10dB, while User 2’s channel varies from 0dB to 10dB. For this case, we consider 30% cooperation.

In order to compute the BER and BLER as described in Sections III and IV, we need the code distance spectra \( a(d) \) and \( c(d) \), as well as the partitioning of the Hamming weight \( d \) into \( d_1 \) and \( d_2 \) for all error events. For the purposes of this work,
we compute the analytical BER and BLER with the assumption that $d_1/d_2 = N_1/N_2$. In other words, for each error event, $d$ is partitioned into $d_1$ and $d_2$ according to the cooperation percentage. The results presented here indicate that this is a reasonable assumption. In addition, for convenience we truncate the summations (21) and (27) to the first six terms of $a(d)$ and $c(d)$, which are those reported in [8]. With this truncation, the analytical results shown here are only approximations that approach the bounds from below, which is why they appear to be below the simulation results in Figures 2 and 3 for some cases and in some SNR ranges. Nevertheless, in each of the examples presented we see that the analytical approximations agree very well with the simulations. This validates the methodology used to derive the bounds and also illustrates a computationally-efficient analytical means to obtain accurate BER and BLER estimates for coded cooperation.

REFERENCES


