Opportunistic Relay Selection with a Direct Link

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Abstract—The performance of relay selection in the presence of a direct source-destination link has been an open problem, essentially because a direct link introduces cross-dependencies in the relay selection that significantly complicate its analysis. Thus, the previous studies of relay selection by and large have been forced to assume that the network geometry is such that the direct link is weak enough to be ignored. This paper addresses and solves this open problem. Several relaying protocols are analyzed with relay selection in the high-SNR regime using the diversity-multiplexing tradeoff, including AF, DF, NAF, DDF, and CF. In several cases, simplified selection criteria are developed.

I. INTRODUCTION

Relay selection is an area of practical importance that has already produced a rich literature. An early example appeared in [1]. Bletsas et al. [2] investigated amplify-and-forward (AF) relay selection, followed by many other works. Decode-and-forward (DF) relay selection has also received attention, e.g. [3] and many others. The diversity-multiplexing tradeoff (DMT) for relay selection has been investigated in [4] for addressing the multiplexing loss of DF relaying.

The relay selection literature has so far addressed only the cases without a direct link between the source and the destination. This is due to statistical dependencies and the resulting complicated effect of the selection function on the performance criterion. In this work we consider a parallel multi-relay network with a direct link, where the system performance is maximized by selecting one relay. Under a variety of orthogonal and non-orthogonal protocols, we calculate the exact diversity-multiplexing tradeoff.

A summary of our results is as follows: we show that opportunistic orthogonal AF and DF relay selection achieve a DMT similar to opportunistic NAF. The opportunistic DDF achieves a DMT that matches the upper bound for low multiplexing gains and does not benefit from relay selection at multiplexing gains higher than 1/2. The CF opportunistic relay selection outperforms all relaying protocols at high multiplexing gains.

II. SYSTEM MODEL

The system consists of one source, one destination and n relays, all half-duplex single-antenna nodes. There is a viable link between the source and destination. Wireless links experience flat, quasi-static block fading. A block fading model is used where each message is transmitted within one coherence interval, which is sufficiently long to support codewords that allow standard coding arguments to be used.

\[ h_{si}, h_{id}, \text{and} h_{sd} \] represent the channel coefficients for the source-relay and relay-destination for relay \( i \), and the source-destination link, respectively. They are identically distributed circularly symmetric complex Gaussian random variables. \( n_i \sim \mathcal{C}N(0,1) \) and \( n_d \sim \mathcal{C}N(0,1) \) are additive white Gaussian noise (AWGN) at relay \( i \) and the receiver, respectively. The relays and the destination have receive-side channel state information (CSI).

III. OPPORTUNISTIC PARALLEL RELAY SELECTION

A. Orthogonal Relaying

The selection criterion is as follows. We first check the direct link. If the source-destination direct link does not support the transmission rate, the best relay is allowed to access the channel as follows. The source transmits for half the transmission interval and the selected relay transmits for the second half of the transmission interval. A relay is selected such that the outage probability is minimized (i.e., the end-to-end mutual information is maximized). We use a lemma from [5] to facilitate the analysis throughout this paper.

Lemma 1: Consider a system that opportunistically switches between \( n \) access modes where an access mode is defined as a set of active transmitters, receivers, and respective links during a given transmission interval. The following DMT can be achieved by the opportunistic system

\[ d(r) = d_1(r) + d_2(r) + \ldots + d_n(r), \]

where \( d_i(r) \) is the individual conditional DMT for access mode \( i \) and is defined as

\[ d_i(r) = - \lim_{\rho \to \infty} \frac{\log \mathbb{P}(e_i|e_{i-1}, \ldots, e_1)}{\log \rho}, \]

and \( \mathbb{P}(e_i|e_{i-1}, \ldots, e_1) \) is the probability of error in access mode \( i \) given that all the previous access modes are in error. This DMT can be achieved with a selection criterion.
that guarantees that the system is in outage only when all subsystems are in outage, and by using codebooks that achieve each mode’s conditional DMT.

We define \( n + 1 \) access modes. The first access mode, the non-relayed access mode, is such that no relay is allowed to transmit and the source transmits during the whole transmission interval. The rest of the access modes, i.e., the relay assisted access modes, each selects a relay for the usual two-interval relaying orthogonal framework.

1) Upper bound: Consider a genie that provides the relays with perfect knowledge of the source message. Each of the \( n \) relay-assisted access modes is transformed into a MISO channel with 2 transmit antennas and one receive antenna. All the relay-assisted modes share one link: the source destination link. The non-relayed mode is not affected and is a point to point link between the source and the destination.

**Theorem 1:** An upper bound for the opportunistic parallel relay channel with \( n \) orthogonal relays is

\[
d(r) = (n + 1)(1 - r).
\]

**Proof:** Using Lemma 1, the genie upper bound DMT is

\[
d(r) = d_0(r) + d_1(r) + \ldots + d_n(r),
\]

where \( d_0(r) \) is the DMT of the non-relay mode (Mode 0). This mode consists of a simple direct link whose DMT is \( d_0(r) = (1 - r)^+ \) and \( d_i(r), i = 1, \ldots, n \) are the conditional DMTs of the relay-assisted modes that are given by Equation (2).

Although the DMT expressions are in terms of error events, in the remainder of this paper the distributions are expressed in terms of outage events. Using sufficiently long codewords guarantees that the errors are dominated by outage.

For Mode 1, the DMT is given by

\[
d_1(r) = -\lim_{\rho \to \infty} \log \frac{\log \mathbb{P}(O_1|O_0)}{\log \rho}
\]

where \( O_i \) represents the outage event for Mode \( i \). The conditional outage is

\[
\mathbb{P}(O_1|O_0) = \mathbb{P}\left(\{[h_{sd}]^2 + [h_{id}]^2 < \rho^r - \frac{1}{\rho}\}\right) \left| \left\{[h_{sd}]^2 < \rho^r - 1\right\}\right.
\]

Using results from Appendix 1 and defining \( g_i(r, \rho) \triangleq \frac{\rho^r - 1}{\rho} \), the conditional outage probability can be calculated as follows

\[
\mathbb{P}(O_1|O_0) = \int_0^{g_1(r, \rho)} \frac{ze^{-z}}{1 - e^{-g_1(r, \rho)}} dz = \frac{1 - e^{-\rho^r - 1 - \rho^{-1}}}{1 - e^{-\rho^r - 1}} = \rho^{r-1}.
\]

Hence, \( d_1(r) = (1 - r)^+ \). For Modes \( i > 1 \) one can show that

\[
\mathbb{P}(O_1, \ldots, O_{i-1}) = \mathbb{P}\left([h_{sd}]^2 + [h_{id}]^2 < \rho^r - 1\left| O_0, \ldots, O_{i-1}\right.\right) \leq \mathbb{P}\left([h_{id}]^2 < \rho^{-1}\left| O_0, \ldots, O_{i-1}\right.\right) = \mathbb{P}\left([h_{id}]^2 < \rho^{r-1}\right)
\]

Hence,

\[
\mathbb{P}\left(\left|h_{id}\right|^2 < \rho^{r-1}\right) \geq \mathbb{P}(O_1|O_0, \ldots, O_{i-1}) \geq \mathbb{P}(O_1|O_0) \rho^{-1} \geq \mathbb{P}(O_1|O_0, \ldots, O_{i-1}) \geq \rho^{r-1},
\]

where \( \mathbb{P}(O_1|O_0) \) is calculated using the same technique used for calculating \( \mathbb{P}(O_1|O_0) \). Hence, \( d_i(r) = (1 - r)^+ \) for \( 1 < i \leq n \). Substituting conditional DMTs into Equation (4) completes the proof.

Notice that the selection order does not affect the result but can make the calculation easier. Selecting the mode \( i^* \) where \( i^* = \arg \max_{i \in [0, \ldots, n]} I_i \) gives the same result as above.

2) Amplify and Forward Orthogonal Relaying: If the direct link is in outage, the selected relay amplifies its received signal during the first transmission interval and forwards it to the destination in the second half. The instantaneous mutual information of the non-relay mode is given by \( I_0 = \log(1 + |h_{sd}|^2/\rho) \). The instantaneous mutual information for the relay-assisted modes under orthogonal AF is given by [6, 7]

\[
I_i = \frac{1}{2} \log \left(1 + |h_{sd}|^2 + \rho f_h|h_{sd}|^2, i \in \{1, \ldots, n\},
\]

where \( f(x, y) = \frac{xy}{x + y + 1} \). At high SNR, the selected relay \( i^* \) is such that \( i^* = \max_{i \in [0, \ldots, n]} \frac{|h_{sd}|^2}{|h_{id}|^2 + |h_{sd}|^2} \).

**Theorem 2:** The DMT of the orthogonal opportunistic AF parallel relay channel is given by

\[
d(r) = (1 - r)^+ + n(1 - 2r)^+.
\]

**Proof:** The total DMT is given by Equation (4). To calculate its components, we start with the non-relay mode (Mode 0) whose DMT is \( d_0(r) = (1 - r)^+ \). At high SNR, the conditional outage of Mode 1 is

\[
\mathbb{P}(O_1|O_0) = \mathbb{P}\left([h_{sd}]^2 + \frac{|h_{id}|^2}{|h_{id}|^2 + \rho^2} < \frac{\rho^{2r - 1}}{\rho - 1}\right)
\]

In order to calculate the conditional outage probability distribution, we first calculate the conditional density function of \( Z = |h_{sd}|^2 + V \) where \( V = |h_{id}|^2 |h_{sd}|^2 \). The term \( |h_{sd}|^2 |h_{id}|^2 \) represents the harmonic mean of two independent exponential random variables. Using the result of [8], the harmonic mean of two exponential random variables with exponential parameters \( \lambda \) can be approximated by an exponential random variable with exponential parameter \( \lambda + \lambda = 2\lambda \).

Using Appendix 1 and defining \( g_1(r, \rho) = \frac{e^{r-1}}{\rho} \) and \( g_2(r, \rho) = \frac{e^{r-1}}{\rho-1} \), the conditional outage probability is

\[
\mathbb{P}(O_1|O_0) = 2 \int_0^{g_1(r, \rho)} \frac{e^{-2z}(e^z - 1)}{1 - e^{-g_1(r, \rho)}} dz + 2 \int_{g_2(r, \rho)}^{g_1(r, \rho)} \frac{e^{-2z}(e^{g_1(r, \rho)} - 1)}{1 - e^{-g_1(r, \rho)}} dz
\]

\[
= -2e^{2r-1} - e^{r-1} + e^{2r-2} + 1 + e^{-r-1} \geq \rho^{2r-1}.
\]
From Equation (11) and (5), \( d_1(r) = (1 - 2r)^+ \). For Mode 2, the conditional outage can be shown to be
\[
P(O_2|O_1, O_0) \geq P(O_2|O_0) \cong \rho^{2r-1},
\]
where (12) follows the same proof as \( P(O_1|O_0) \). Also one can show using the same technique used in Equation (7) that
\[
P(O_2|O_1, O_0) \leq P\left( f(|h_{sd}|^2, |h_{id}|^2) < \rho^{2r-1} \right) \cong \rho^{2r-1}.
\]
Hence, \( P(O_2|O_1, O_0) \cong \rho^{2r-1} \) and \( d_2(r) = (1 - 2r)^+ \) for \( i = 3, \ldots, n \). Substituting conditional DMTs into Equation (4) completes the proof. ■

3) Decode and Forward Orthogonal Relaying: The transmission interval is divided into two halves. If the direct link is in outage, a relay is selected to transmit in the second transmission interval from the group of relays that decoded the message in the first transmission interval. The selection process is according to the maximum end-to-end mutual information.

**Theorem 3:** The DMT of the orthogonal opportunistic DF parallel relay channel is given by
\[
d(r) = (1 - r)^+ + n(1 - 2r)^+.
\]
**Proof:** The total DMT is given by Equation (4) and \( d_0(r) = (1 - r)^+ \). The conditional outage probability of the first relay-assisted mode is given by
\[
P(O_1|O_0) = P\left\{ \frac{1}{2} \log(1 + U) < r \log \rho \right\} \bigg\} \left\{ \log(1 + |h_{sd}|^2 \rho) < r \log \rho \right\} \bigg\}
\]
where the random variable \( U \) is defined as
\[
U = \left\{ \frac{2|h_{sd}|^2}{|h_{sd}|^2 + |h_{id}|^2} \bigg| \frac{|h_{sd}|^2}{|h_{sd}|^2 + |h_{id}|^2} \geq \frac{\rho^{2r-1}}{\rho} \right\}.
\]
\[
P(O_1|O_0) = \frac{1}{\rho^{2r-1}} \left[ \frac{\rho^{2r-1}}{\rho} - \frac{1}{\rho} \right] \left( \frac{\rho^{2r-1}}{\rho} - \frac{1}{\rho} \right) + P\left\{ |h_{sd}|^2 + |h_{id}|^2 < \frac{\rho^{2r-1}}{\rho} \right\} \left\{ |h_{sd}|^2 < \frac{\rho^{2r-1}}{\rho} \right\}
\]
One can show that \( \frac{1}{2} \rho^{2r-1} \geq \frac{\rho^{2r-1}}{\rho} \), therefore
\[
P\left\{ |h_{sd}|^2 < \frac{\rho^{2r-1}}{\rho} \bigg| \left\{ |h_{sd}|^2 < \frac{\rho^{2r-1}}{\rho} \right\} \right\} \cong \frac{1}{2}
\]
Using results from Appendix I and defining \( g_1(r, \rho) \triangleq \frac{\rho^{2r-1}}{\rho} \) and \( g_2(r, \rho) \triangleq \frac{\rho^{2r-1}}{\rho} \)
\[
P\left\{ \bigg| |h_{sd}|^2 + |h_{id}|^2 < g_2(r, \rho) \bigg| \left\{ |h_{sd}|^2 < g_1(r, \rho) \right\} \right\} = \int_0^{g_1(r, \rho)} \frac{z e^{-z}}{1 - e^{-g_1(r, \rho)}} dz + \int_{g_1(r, \rho)}^{g_2(r, \rho)} \frac{g_1(r, \rho) e^{-z}}{1 - e^{-g_1(r, \rho)}} dz
\]
\[
\triangleq \frac{1 - e^{-\rho^{2r-1}} - \rho^{r-1} e^{-\rho^{2r-1}}}{1 - e^{-\rho^{2r-1}}} \leq \rho^{2r-1}.
\]
Substituting (17) and (18) into (14), the conditional probability of outage is given by
\[
P(O_1|O_0) \cong \rho^{2r-1} + \rho^{2r-1}(1 - \rho^{2r-1}) \cong \rho^{2r-1}.
\]
Hence, the conditional DMT for the first relay-assisted access mode is \( d_1(r) = (1 - 2r)^+ \). Using the same argument as the orthogonal AF, it follows that \( P(O_1|O_0, \ldots, O_{i-1}) \cong \rho^{2r-1} \) and hence \( d_i(r) = (1 - 2r)^+ \) for \( i = 2, \ldots, n \). Substituting conditional DMTs into Equation (4) completes the proof. ■

B. Non-Orthogonal Relaying

The source transmits throughout the transmission interval. The relay listens for a portion of the transmission interval and transmits the remaining part. The destination receives the source and the relay messages superimposed over each other. In this case, the channel has \( n \) access modes as opposed to \( n + 1 \) in the case of orthogonal relaying. We do not consider the direct link alone as an access mode, because it can be shown that in non-orthogonal modes, the end-to-end mutual information always improved with a relay. The selected relay is the one that maximizes the end-to-end mutual information.

1) Upper Bound: A genie is assumed to provide the relays with an error-free version of the source message. We also assume full cooperation between the source and the relays. The relay that maximizes the instantaneous end-to-end mutual information is selected to transmit simultaneously with the source. Each of the \( n \) relays has an independent link to the destination and they all share the same source-destination link.

**Theorem 4:** An upper bound for the opportunistic parallel relay channel with \( n \) non-orthogonal relays is
\[
d(r) = (n + 1)(1 - r)^+.
\]
**Proof:** The genie-aided opportunistic parallel relay channel is equivalent to a MISO system with \( n + 1 \) transmit antennas and one receive antenna. The performance of the opportunistic genie-aided relay selection is therefore upper bounded by a \( (n + 1) \times 1 \) MISO system with antenna selection that chooses for each codeword transmission two transmit antennas. The \( (n + 1) \times 1 \) antenna selection allows configurations that do not have a counterpart in the opportunistic modes in our channel, therefore due to the extra flexibility the MISO system with antenna selection upper bounds the performance of the genie-aided opportunistic parallel relay channel.

The DMT of a \( M \times N \) MIMO link with \( L_r < M \) selected transmit antennas and \( L_r < N \) selected receive antennas is upper bounded by a piecewise linear function obtained by connecting the following \( K + 2 \) points [9]
\[
\left\{ \frac{(n, (M_r - n)(M_r - n))}{\min(L_r, L_r)} \right\}_n^{K+2},
\]
where \( L_r < M \) and \( L_r < N \).
The random variables $v_1, u_1^{(i)}$, and $v_2^{(i)}$ represent the exponential order of $1/|h_{sd}|^2$, $1/|h_{xi}|^2$, and $1/|h_{id}|^2$, respectively. Each of these random variables has a probability density function $p(x)$ that is asymptotically equal to $\rho^{-x}$ for $x \geq 0$ and 0 otherwise [11]. The set $O$ represents the outage event for the opportunistic network. The opportunistic system is considered in outage when no access mode is viable, i.e., $O = O_1^+ \cap \ldots \cap O_n^+$, where $O_i$ represents the outage region for Mode $i$ and is defined by [11]

$$O_i^+ = \left\{(v_1, v_2^{(i)}, u_1^{(i)}) \in R^3 \mid (l-2m_i^2)(1-v_1)^+ + m_i^2 \max\{2(1-v_1), 1-(v_2^{(i)}+u_1^{(i)})\}^+ < rl\right\}. \quad (27)$$

$m_i^2$ is rank of the amplification matrix and $l$ is the block length. The solution to Equations (26) and (27) is facilitated by the knowledge that $d_o(r)$ is maximized when $m_i^2$ is maximum, i.e., $m_i^2 = l/2$, leading to result.

3) Dynamic Decode and Forward: The relay listens to the selected source until it has enough information to decode the message. The relay uses the rest of the transmission interval to send the decoded information to the destination. The selection criterion is, once again, according to the maximum end-to-end mutual information.

**Theorem 6:** The maximum DMT for the $n$ parallel relay channel with opportunistic DDF relay selection is

$$d_{DDF}(r) = \begin{cases} 
(n+1) (1-r) + n(1-2r)^+ & \text{for} \quad \frac{r}{n+1} \geq r \geq 0 \\
(n+1) - n \frac{r}{n+1} + \frac{2}{1-r} & \text{for} \quad \frac{r}{n+1} > \frac{2}{1-r} \\
\frac{r}{n+1} & \text{for} \quad r \geq \frac{2}{1-r} 
\end{cases} \quad (28)$$

**Proof:** Following Equations (24), (25), and (26), for the relay channel characterized by Relay $i$, the outage region under DDF can be shown to be [11]

$$O_i^+ = \left\{(v_1, v_2^{(i)}, u_1^{(i)}) \in R^3 \mid t_i(1-v_1)^+ + (1-t_i)^+(1-min(v_1, v_2^{(i)})^+ < r \right\}, \quad (29)$$

where $t_i$ is the listening-time ratio of the relay $i$, with $r \leq t_i \leq 1$. In the following we outline the solution of Equations (26) and (29) for a two-relay channel. The generalization to $n$ users follows the same lines.

Our strategy for solving the optimization problem is to partition the optimization space into eight regions, solve the optimization problem over each region as a function of $t_1$ and $t_2$, maximize over $t_1$ and $t_2$ and then find the minimum of the eight solutions. The eight regions correspond to the Cartesian product of whether each of the three positive variables $v_1^{(i)}, v_1^{(i)}, v_2^{(i)}$ is greater than or less than 1. Following the calculations, which are lengthy and therefore omitted, the DMT for DDF is established.

4) Compress and Forward: Following [12], the relay uses Wyner-Ziv compression and the destination uses the received signal from the source as side information to decode the signal. The compression rate ensures that the compressed signal reaches the destination error-free. Yuksel and Erkip [12] show that the optimal DMT, $d(r) = 2(1-r)^+$, is achieved.
The relay \( i^* = \arg \max_i I_i \) is selected. At high-SNR, using results from \([12]\), the selected user \( i^* \) can be proven to be

\[
i^* = \arg \max_i \frac{(|h_{si}|^2 + |h_{sd}|^2)(|h_{ai}|^2 + |h_{ad}|^2)}{|h_{si}|^2 + |h_{sd}|^2} + |h_{ai}|^2 + |h_{ad}|^2.
\]

Following \([12]\), the DMT for the \( n \) parallel relay channel with opportunistic CF relay selection can be shown to be

\[
d(r) = \max_t \min(d_{MAC}(r,t), d_{BC}(r,t)),
\]

\[
d_{MAC} = -\lim_{\rho \to \infty} \log \min_{p(x_s,x_i)} \frac{P(I_{MAC} < r \log \rho)}{\log \rho},
\]

\[
d_{BC} = -\lim_{\rho \to \infty} \log \min_{p(x_s,x_i)} \frac{P(I_{BC} < r \log \rho)}{\log \rho},
\]

where \( t \) is the the time ratios vector, \( t = [t^{(1)}, \ldots, t^{(n)}] \), \( q \) represents the state of the relay (listening vs. transmitting), \( p(x_s,x_i|q) \) is the probability density of the codebooks generated for the source and the relay, and \( I_{BC} \) and \( I_{MAC} \) represent the total mutual information across the source and the destination cutsets, respectively. It can be shown that

\[
I_{MAC} \leq (1 - t^{(i)}) \log(1 + |h_{sd}|^2) + (1 - t^{(i)}) \log(1 + |h_{di}|^2).
\]

We can show that \( P(I_{MAC} < r \log \rho) \approx \rho^{-d_{MAC}(r)} \) where

\[
d_{MAC}(r) = \inf_{v_1 + \sum_{i=1}^{n} v_i^{(i)}}\left\{(1 - t^{(i)})(1 - v_1^{(i)}) \leq r\right\}.
\]

and the outage event \( O^+ \) is defined as

\[
O^+ = \left\{(v_1, v_2^{(i)}) \in R^2+ \left| (1 - t^{(i)})(1 - v_1^{(i)}) + t^{(i)}(1 - \min(v_1, v_2^{(i)})) \leq r\right.\right\}.
\]

We provide the solution for the optimization problem for the case of 2 relays, the generalization to the \( n \)-relay case follows the same lines. The DMT for the source is

\[
d_{MAC}(r) = \begin{cases} 
3(1 - r) & r \leq 1 \\
3 - r & r \geq 1 \\
\frac{3 - r}{1 - t_C} & r \leq 1 - t_C \\
\frac{1 - r}{1 - t_C} & r \geq 1 - t_C \\
\frac{1 - r}{1 - t_V} & r \leq 1 - t_V \\
\frac{1 - r}{1 - t_V} & r \geq 1 - t_V
\end{cases}
\]

where \( t_V = \min\{t^{(1)}, t^{(2)}\} \) and \( t_C = \max\{t^{(1)}, t^{(2)}\} \). The above equation gives the destination cut DMT for various values of \( t^{(1)} \) and \( t^{(2)} \). Similarly, the source cutset DMT can be obtained by switching \( t^{(1)} \) by \( 1 - t^{(1)} \) and \( t^{(2)} \) by \( 1 - t^{(2)} \). Maximizing over values of \( t^{(1)} \) and \( t^{(2)} \), the DMT is

\[
d_{CF}(r) = \begin{cases} 
3 - 4r & r < \frac{3}{4} \\
\frac{3}{4} & \frac{3}{4} \leq r \leq 1
\end{cases}
\]

IV. CONCLUSION

This paper calculates the DMT for relay selection under the existence of a direct link, thus solving an important open problem. Many prevalent relaying protocols are analyzed under this condition, and the results are reported.

APPENDIX I

SUM OF AN EXPONENTIAL RANDOM VARIABLE AND A CLIPPED EXPONENTIAL RANDOM VARIABLE

Define \( Z = X + Y \), where \( X \) and \( Y \) are exponential random variables with exponential parameters \( \lambda_1 \) and \( \lambda_2 \), respectively. Conditioned on the event \( B = \{Y < \alpha\} \), the conditional pdf of \( Y \) is given by

\[
f_{Y|B}(y) = \begin{cases} 
\lambda_2 \alpha^{-\lambda_2} & y \leq \alpha \\
0 & y > \alpha
\end{cases}
\]

Hence, if \( \lambda_1 \neq \lambda_2 \), the conditional pdf of \( Z = X + Y \) is

\[
f_{Z|B}(z) = \begin{cases} 
\lambda_1 \alpha \lambda_2^{\alpha-1} z^{-\lambda_2}(1-\alpha^{-\lambda_2}) & z \leq \alpha \\
\lambda_1 \lambda_2(\lambda_1 - \lambda_2)(\lambda_2 - \alpha^{-\lambda_2}) & z > \alpha
\end{cases}
\]

If \( \lambda_1 = \lambda_2 = \lambda \), the conditional pdf of \( Z \) is

\[
f_{Z|B}(z) = \begin{cases} 
\lambda_1 \alpha \lambda_2^{\alpha-1} z^{-\lambda} & z \leq \alpha \\
0 & z > \alpha
\end{cases}
\]

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