On Wachspress pentagonal patches

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Wachspress quadrilateral patches have been recently studied from the point of view of applications to surface modelling in CAGD [1], [3], [4]. Some more applications for defining barycentric coordinates for arbitrary polygons have also been presented in [5] [9]. The purpose of the present paper is to introduce non-negative Wachspress rational basis functions for surface modelling on pentagonal partitions. Interpolation formula for function values and directional derivatives at the vertices of pentagon has been presented. Conditions for $C^1$ continuity of the composite surface have also been studied in the paper.

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1 Introduction

Wachspress finite elements were introduced with the purpose of providing nice and computationally convenient basis functions for approximating the solutions of BVPs over a polygonal mesh [6]. Recent studies include applications of linear Wachspress basis function for finite element solution to problems of solid mechanics and biomechanics. [2], [6], [7]. An interesting application of these rational polygonal elements in boundary color interpolation problems of computer graphics has been studied in [2]. Although linear Wachspress elements as introduced in [8] remain non negative on their support element, higher order basis functions assume negative values also. One of the main requirements for implementations of basis functions in surface modelling and FEM etc. is that basis functions should be non negative and should provide partition of unity. Keeping this in view, Wachspress basis functions over quadrilateral elements have been introduced in [1] where nice iterative algorithms have been presented to generate higher degree basis functions which remain nonnegative on the element. Composite Wachspress surfaces with $C^1$- continuity and formula for subdivision have been studied in [3], [4]. Condition for $C^k$ continuity have also been presented in [3]. In the present paper we have defined a set of quadratic Wachspress basis functions over convex pentagonal elements. These patches are potentially good candidates for boundary color interpolation and other surface modelling applications in CAGD.

2 Formulation

In this section we first introduce some definitions and notations which will be frequently used in subsequent discussions.

Let $P$ be a convex pentagon with vertices $\{v^i\}_{i \in Z_5}$ in $R^2$ labelled in counter clockwise direction. We say that $P$ is nondegenerate if no two sides of $P$ are parallel. Any two sides of $P$ which are not adjacent are called opposite sides. Throughout we shall assume that $P$ is nondegenerate.

Let $l_i$ denote the linear form which vanishes on the edge of $P$ joining $v^i$ and $v^{i+1}$. Consider the five points of intersection $\{\nu^i\}_{i \in Z}$ obtained by extending the opposite sides of $P$. These five points are known as exterior intersection points (EIP) of $P$. We now denote by $q$ the quadratic form which represents the conic passing through these five EIPs. We are now in a position to define linear Wachspress basis function $W_{1,i}$ for interpolation of data specified at the vertices of pentagon $P$ as follows (cf [8]).

$$W_{1,i}(u) = \frac{K_i l_{i+1}(u) l_{i+2}(u) l_{i+3}(u)}{q(u)},$$

for $u \in P$, $i \in Z_5$, where $K_i$ is chosen so that $W_{1,i}(v^i) = 1$. These basis functions are non-negative and form partition of unity. Further, they can also be viewed as barycentric coordinates for the pentagon $P$. These pentagonal linear Wachspress elements have found interesting applications in web enabled color interpolation schemes useful in instrumentation imaging and internet graphic [2]. This motivates us to define higher degree basis functions over a convex pentagonal element so that their possible application would further strengthen the results obtained in [2],[6],[7].

We are now in a position to introduce quadratic Wachspress blending functions as follows

$$W_{2,i}(u) = W_{1,i}^2(u); \quad W_{2,i,j}(u) = 2W_{1,i}(u)W_{1,j}(u), \quad i \neq j.$$
It is immediate that $W_{2,i}$ and $W_{2,i,j}$ are nonnegative on $P$ and reduce to quadratic polynomials on the perimeter of $P$. It can be shown that out of fifteen blending functions, the set $E = \{W_{2,i}, W_{2,i,i+1}, i \in Z_5\}$ is linearly independent. Further any four functions from the set $U = \{W_{2,i}, i \in Z_5\}$ can be added to form a linearly independent set of fourteen elements. However, the complete set of fifteen elements $\{W_{2,i}, W_{2,i,i+1}, W_{2,i,i+2}, i \in Z_5\}$ is linearly dependent. For the purpose of boundary data interpolation we employ only the set $E$ with ten basis functions since none of the functions from the set $U$ contribute a non zero value on the perimeter of the pentagon.

### 3 Interpolation and $C^1$—Continuity Conditions

Let $f$ be a real valued function defined over $P \in R^2$ whose first partial derivatives exist and are continuous. Consider the function $W(u)$ defined by

$$W(u) = \sum_{i \in Z_5} C_i W_{2,i} + \sum_{i \in Z_5} C_{i,i+1} W_{2,i,i+1}$$

(3)

which satisfies the following interpolatory conditions given on the vertices of $P \in R^2$.

$$(f - W)(v') = 0; \quad D(f - W)(v')(v'^{+1} - v') = 0, \quad i \in Z_5$$

(4)

where $Df(v)(q-p)$ is the directional derivative of $f$ at $v$ in the direction of $(q-p)$. One immediately gets $C_i = f(v_i) = f_i$. Further,

$$C_{j,j+1} = \left[ \prod_{k=1}^{4} M_{j+k} A_{j+1} + \prod_{k=1}^{4} L_{j+k} A_{j+1} - M_{j+1} L_{j+2} L_{j+3} L_{j+4} A_{j+2} \right] / \gamma, \quad \text{with}$$

(5)

$$M_j = \left( \prod_{i \in Z_5} L_i + \prod_{i \in Z_5} M_i \right)$$

(6)

$$\gamma = \left( K_{j-1} l_{i+1} l_{i+2} D_{l_{i}} + \right) (v^j)(v^{j+1} - v^j), \quad L_{j-1} = \left( K_{j-1} l_{i+1} l_{i+2} D_{l_{i}} \right) (v^j)(v^{j+1} - v^j)$$

(7)

To demonstrate the conditions for global $C^1$—smoothness of the composite surface, we consider specific pentagon $P$ defined by the vertices $v^0 = (0,0)$, $v^1 = (h,0)$, $v^2 = \left( \frac{2h+k}{3}, \frac{2(k-h)}{3} \right)$, $v^3 = \left( \frac{2(k-h)}{3}, \frac{2h}{3} \right)$, $v^4 = (0,h)$, with $h, k > 0$ and $h < k$. Pentagon $P$ is the mirror image of $P$ about the $x$—axis. The patch $W$ defined on $P$ can be defined by equation (4), where as expression of $\hat{W}$ defined on $P$ can be obtained by replacing $W_{2,i}$, $W_{2,i,i+1}$, $C_i, C_{i,i+1}$ by $\hat{W}_{2,i}$, $\hat{W}_{2,i,i+1}$, $\hat{C}_i$, $\hat{C}_{i,i+1}$ with appropriate modifications in the definitions of $W_{2,i}$ and $W_{2,i,i+1}$. Continuity of composite patch imposes the conditions $C_0 = \hat{C}_0$, $C_1 = \hat{C}_1$, $C_{0,1} = \hat{C}_{0,1}$. The $C^1$—continuity constraints are as given below.

$$d(C_0 + \hat{C}_0) + 20kh(k-h)(C_{4,0} + C^\gamma_{4,0}) = 0$$

(9)

$$d(C_1 + \hat{C}_1) + 3h(2k^2 - 11hk + 24h^2)(C_{4,0} + C^\gamma_{4,0}) = 0$$

(10)

$$d(C_{0,1} + \hat{C}_{0,1}) + 12h(k-h)(k-2h)(C_{4,0} + C^\gamma_{4,0}) = 0$$

(11)

$$d(C_{1,2} + \hat{C}_{1,2}) + 20h(k-h)k(C_{4,0} + C^\gamma_{4,0}) = 0$$

(12)

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### References