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Necessary and sufficient condition for stability robustness under known additive perturbation

A. I. G. VARDULAKIS†

A necessary and sufficient condition is derived for the stability robustness of a unity feedback closed-loop system involving a strictly proper plant \( P_0 \) and a stabilizing compensator \( C_o \) under the assumption that the plant \( P_0 \) is perturbed to \( P_0 + \Delta P_0 \) where \( \Delta P_0 \) is a known strictly proper rational matrix.

Notation

- \( \mathbb{R} \) the field of reals
- \( \mathbb{C} \) the field of complex numbers
- \( \mathbb{C}_0^- = \{ s \in \mathbb{C}, \text{Re}(s) < 0 \} \)
- \( \mathbb{C}_0^+ = \{ s \in \mathbb{C}, \text{Re}(s) \geq 0 \} \)
- \( \mathbb{C}_0^+ = \mathbb{C}_0^+ \cup \{ \infty \} \)
- \( \mathbb{R}(s) \) the field of rational functions with coefficients in \( \mathbb{R} \)
- \( \mathbb{P}(s) \) the ring of proper rational functions
- \( \Omega \) any subset of \( \mathbb{C} \) which is symmetrically located with respect to \( \mathbb{R} \) and which excludes at least one point \( -\alpha \in \mathbb{R} \) (\( \alpha > 0 \))
- \( \Omega := \Omega \cup \{ \infty \} \)
- \( \mathbb{R}(s) \) the field of rational functions \( t(s) \in \mathbb{R}(s) \) which have no poles in \( \Omega \), i.e. of 'proper and \( \Omega \)-stable' rational functions.
- \( k^p \times m \) the set of \( p \times m \) matrices with elements in \( k \) if \( k \) is a set.

1. Introduction

Let \( \Sigma(P_0) \) be a linear multivariable system which is free of unstable hidden modes and whose input–output behaviour is described by a \( p \times m \) strictly proper rational transfer function matrix \( P_0 \) (the 'plant'). Consider now the closed-loop unity feedback system \( \Sigma_{e_1}(P_0, C_o) \) of Fig. 1 which involves a 'stabilizing compensator' \( C_o \) such that \( \Sigma_{e_1}(P_0, C_o) \) is internally stable and the closed-loop transfer function matrix

\[
H_{e_1}(P_0, C_o) : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]

has arbitrary desired poles in \( \mathbb{C}_0^- \). Now let the nominal plant \( P_0 \) be perturbed to \( P_0 + \Delta P_0 =: P \) where \( \Delta P_0 \) is a known \( p \times m \) proper rational matrix. In this paper we give a simple necessary and sufficient condition that has to be satisfied by \( \Delta P_0 \) so that if the additively perturbed system \( \Sigma(P_0 + \Delta P_0) \) is also free of unstable hidden modes, the closed-loop system \( \Sigma_{e_1}(P_0 + \Delta P_0, C_o) \) of Fig. 2 is also internally stable.

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Set of plants stabilizable by a compensator which stabilizes a nominal plant $P_0$

Let $\Sigma(P_0)$ be a linear, time-invariant multivariable system which is free of unstable hidden modes and whose (nominal) transfer function matrix is $P_0 \in \mathbb{P}_{pr \times m}(s)$ and let $P_0 = A_1^{-1} B_1 = B_2 A_2^{-1}$ be left and right coprime in $\bar{\Omega}$ fractional representations of $P_0$, i.e. let the matrices $[A_1, B_1] \in S^{p \times (p+m)}$ and $[A_2, B_2] \in S^{m+p \times m}$ be respectively $S$-right and left invertible.

It is then known (Desoer et al. 1980, Vidyasagar 1985, Vidyasagar et al. 1982, Saeks and Murray 1981, Youla et al. 1976, Callier and Desoer 1982) that another system $\Sigma(C_0)$ (which is also free of unstable hidden modes) with transfer function matrix $C_0 \in \mathbb{P}_{pr \times p}(s)$ is a 'stabilizing compensator' for $\Sigma(P_0)$ such that the unity feedback closed-loop system denoted by $\Sigma_{uf}(P_0, C_0)$ of Fig. 1 is internally stable (i.e. all it modes, including the uncontrollable and unobservable ones lie in $C_0$) iff

$$\|P_0(\infty) C_0(\infty)\| \neq 0$$

and the transfer function matrix $H_m(P_0, C_0) : [u_1, u_2]^T \rightarrow [y_1, y_2]^T$ is an element of $S^{(m+p) \times (m+p)}$ where $\tilde{\Omega} \equiv \tilde{\mathbb{C}}^+$. It is also well known that if $C_0$ is a stabilizing compensator for $\Sigma(P_0)$ then $C_0$ has a unique left and right coprime in $\bar{\Omega}$ fractional representations

$$C_0 = D_1^{-1} N_1 = N_2 D_2^{-1}$$

satisfying the Bezout identity

$$\begin{bmatrix} D_1 & N_1 \\ -B_1 & A_1 \end{bmatrix} \begin{bmatrix} A_2 & -N_2 \\ B_2 & D_2 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}$$

Multiplying (3) on the left and right by the $S$-unimodular matrices

$$\begin{bmatrix} I_m & W \\ 0 & I_p \end{bmatrix}$$

and
Stability robustness under additive perturbation

\[
\begin{bmatrix}
I_m & -W \\
0 & I_p
\end{bmatrix},
\]
respectively, where \( W \in \mathbb{S}^{m \times r} \) such that \([D_1 - WB_1](\infty) \neq 0 \) and \([D_2 - B_2 W](\infty) \neq 0 \), we obtain the identity

\[
\begin{bmatrix}
D_1 - WB_1 & N_1 + WA_1 \\
-B_1 & A_1
\end{bmatrix}
\begin{bmatrix}
A_2 & -(N_2 + A_2 W) \\
B_2 & D_2 - B_2 W
\end{bmatrix}
= \begin{bmatrix}
I_m & 0 \\
0 & I_p
\end{bmatrix}
\]
which, due to the above results, shows clearly that the set \( \phi(p_0) \) of all stabilizing compensators \( C_w \) of \( \Sigma(p_0) \) is parametrized by \( C_0 \) and \( W \) and is given by

\[
\phi(p_0) = \{ C_w = (D_1 - WB_1)^{-1}(N_1 + WA_1) = (N_2 + A_2 W)(D_2 - B_2 W)^{-1} W \in \mathbb{S}^{m \times r} \\
\text{such that } [D_1 - WB_1](\infty) \neq 0, [D_2 - B_2 W](\infty) \neq 0 \}
\]
(5)

(See the references cited above.)

Consider now the nominal plant \( p_0 \) a (nominal) stabilizing compensator \( C_0 \in \phi(p_0) \) and denote by \( \psi(C_w) \) the set of all 'plants' stabilizable by a compensator \( C_w \in \phi(p_0) \). By duality to the above characterization of \( \phi(p_0) \) we can parametrize \( \psi(C_w) \). Then we have the following result.

**Proposition 1**

Let \( p_0 = A_1^{-1} B_1 = B_2 A_2^{-1} \in \mathbb{R}^{p \times m}(s) \), \( C_0 = D_1^{-1} N_1 = N_2 D_2^{-1} \in \mathbb{R}^{m \times p}(s) \), a nominal stabilizing compensator of \( \Sigma(p_0) \), and \( C_w \in \phi(p_0) \). Then the set \( \psi(C_w) \) of all 'plants' stabilizable by \( C_w \) which stabilizes \( p_0 \) is given by:

\[
\psi(C_w) = \psi(p_0, C_0) = \{ p_{Q,w} = [A_1 + Q(N_1 + WA_1)]^{-1} [B_1 - Q(D_1 - WB_1)] \\
= [B_2 - (D_2 - B_2 W)Q][A_2 + (N_2 + A_2 W)Q]^{-1}, \\
Q \in \mathbb{S}^{p \times m}, \ W \in \mathbb{S}^{m \times r} \}
\]
(6)

\([A_1 + Q(N_1 + WA_1)](\infty) \neq 0, \ [A_2 + (N_2 + A_2 W)Q](\infty) \neq 0 \)
(7)

**Proof**

Multiplying (4) on the left and right respectively by the \( S \)-unimodular matrices

\[
\begin{bmatrix}
I_m & 0 \\
Q & I_p
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
I_m & 0 \\
-Q & I_p
\end{bmatrix},
\]
where \( Q \in \mathbb{S}^{p \times m} \) and is such that conditions (7) are satisfied, we obtain the identity

\[
\begin{bmatrix}
D_1 - WB_1 & N_1 + WA_1 \\
-[B_1 - Q(D_1 - WB_1)] & A_1 + Q(N_1 + WA_1)
\end{bmatrix}
\begin{bmatrix}
A_2 + (N_2 + A_2 W)Q & -(N_2 + A_2 W) \\
B_2 - (D_2 - B_2 W)Q & D_2 - B_2 W
\end{bmatrix}
= \begin{bmatrix}
I_m & 0 \\
0 & I_p
\end{bmatrix}
\]
(8)

which, owing to the above results clearly shows that the set \( \psi(C_w) \) is given by (6).

\[\square\]
Corollary 1
The set \( \psi(C_0) \) of all plants stabilizable by \( C_0 \in \phi(P_0) \) is given by
\[
\psi(C_0) = \{ P_Q = (A_1 + QN_1)^{-1}(B_1 - QD_1) = (B_2 - D_2Q)(A_2 + N_2Q)^{-1}, \quad Q \in S^{p \times m} \}
\]
(6)
\[
|[A_1 + QN_1](\infty)| \neq 0, \quad |[A_2 + N_2Q](\infty)| \neq 0
\]
(7)

Proof
Just put \( W = 0 \) in (6).

We investigate now conditions under which the elements of the set \( \psi(C_0) \) are proper rational matrices. To this end the next well-known lemma is in order.

Lemma 1 (Vardulakis and Karcanias 1983)
Let \( T \in \mathbb{R}^{p \times m}(s) \) and let \( D_L \in S^{p \times p}, \quad N_L \in S^{p \times m}, \quad D_R \in S^{m \times m} \) right coprime in \( \Omega \), \( N_R \in S^{p \times m} \), \( D_R \in S^{m \times m} \) right coprime in \( \Omega \) and such that \( T = D_L^{-1}N_L = N_RD_R^{-1} \). Then

(i) \( T \in \mathbb{R}^{p \times m}(s) \) iff \( D_L, D_R \) are biproper or equivalently iff \( |D_L(\infty)| \neq 0, |D_R(\infty)| \neq 0 \).

Moreover if (i) holds true then \( T \) is strictly proper iff \( N_L, N_R \) are strictly proper or equivalently iff \( N_L(\infty) = 0, N_R(\infty) = 0 \).

If we assume now that \( P_0 \) is strictly proper, then from the above lemma:
\[
|A_1(\infty)| \neq 0, \quad |A_2(\infty)| \neq 0, \quad B_1(\infty) = 0, \quad B_2(\infty) = 0
\]
and hence from (3) we obtain
\[
D_1(\infty)A_2(\infty) = I_m, \quad i.e. \quad |D_1(\infty)| = |A_2(\infty)|^{-1} \neq 0 \quad \text{which again by virtue of Lemma 1 implies that} \quad C_0 \in \mathbb{R}^{p \times m}(s).
\]
Moreover
\[
(D_1 - WB_1)(\infty) = D_1(\infty) - W(\infty)B_1(\infty) = D_1(\infty)
\]
(9)
so that for every \( W \in S^{m \times p} \), every \( C_w \in \phi(P_0) \) will be proper. On the other hand, if \( P_0 \) is proper but not strictly proper then it is known that \( W \in S^{m \times p} \) can be chosen so that \( C_w \) is proper or even strictly proper (Vidyasagar 1985). Of course a strictly proper compensator \( C_w \) also always exists in the case of a strictly proper plant \( P_0 \).

We look now more closely into the set \( \psi(C_0) \). According to Lemma 1, if \( P_0 \) is proper then the matrices \( A_1 \) and \( A_2 \) are biproper and if either \( Q \in S^{p \times m} \) and/or \( QN_1 \in S^{p \times p} \) is not strictly proper then the matrices \( A_1 + QN_1, \quad A_2 + N_2Q \) appearing in (6) might turn out to be non-biproper. Thus in general, and even if \( P_0 \) is strictly proper, the set \( \psi(C_0) \) will also contain non-proper 'plants' \( P_0 \). On the other hand, if \( Q \in S^{p \times m} \) is chosen to be strictly proper then \( P_Q \in \mathbb{R}^{p \times m}(s) \). If \( P_0 \) is strictly proper, then the strictly proper elements \( P_Q \) of \( \psi(C_0) \) can be characterized by \( Q \). We have the following result.

Proposition 2
Let \( P_0 \in \mathbb{R}^{p \times m}(s) \) be strictly proper with \( P_0 = A_1^{-1}B_1 = B_2A_2^{-1} \) and let \( C_0 \in \mathbb{R}^{p \times m}(s) \) with \( C_0 = D_L^{-1}N_L = N_RD_R^{-1} \) such that the Bezout identity (3) is satisfied, i.e. let \( C_0 \in \phi(P_0) \). Then

(i) for every \( Q \in S^{p \times m} \) which is strictly proper
\[
P_Q := (A_1 + QN_1)^{-1}(B_1 - QD_1) = (B_2 - D_2Q)(A_2 + N_2Q)^{-1} \in \psi(C_0)
\]
(10)
and it is strictly proper.
(ii) Every strictly proper plant \( P_Q \in \psi(C_0) \) can be expressed as in (10) for some strictly proper \( Q \in S^{p \times m} \).

**Proof**

(i) From Corollary 1 it follows that for every strictly proper \( Q, P_Q \in \psi(C_0) \).

Moreover, we have

\[
(A_1 + QN_1)(\infty) = A_1(\infty) + Q(\infty)N_1(\infty) = A_1(\infty)
\]

\[
(B_1 + QD_1)(\infty) = B_1(\infty) + Q(\infty)D_1(\infty) = 0
\] (11) (12)

By Lemma 1 the above equations imply that \( P_Q \) is strictly proper.

(ii) Let \( P_Q \in R_0^{p \times m}(s) \) and be strictly proper such that \( P_Q \in \psi(C_0) \). Then, according to Corollary 1, \( P_Q \) can be written as in (10) for some \( Q \in S^{p \times m} \). Now as seen in (9) our hypothesis that \( P_\delta \) is strictly proper implies that \( D_1 \) is biproper, and since also by hypothesis \( P_Q \) is strictly proper, from Lemma 1 and (10) it follows that we must have

\[
(B_1 - QD_1)(\infty) = 0
\] (13)

which holds true iff \( Q \) is strictly proper.

In order to determine \( Q \) we proceed as follows. Let \( P_Q = B_QA_Q^{-1} \) with \( B_Q \in S^{p \times m}, A_Q \in S^{m \times m} \) right coprime in \( \Omega \). Now \( P_Q \in \psi(C_0) \) implies that

\[
D_1A_Q + N_1B_Q = U \in S^{m \times m}
\] (14)

is an \( S \)-unimodular matrix. Also, according to the above we must have that

\[
P_Q = B_QA_Q^{-1} = (A_1 + QN_1)^{-1}(B_1 - QD_1)
\] (15)

for some \( Q \in S^{p \times m} \). Solving (18) with respect to \( Q \) we obtain the expression for \( Q \):

\[
Q = (B_1A_Q - A_1B_Q)U^{-1}
\] (16)

which is clearly strictly proper since (16) is a right coprime in \( \Omega \) fractional representation and by assumption both \( P_\delta \) and \( P_Q \), or equivalently \( B_1 \) and \( B_Q \), are all strictly proper (see Lemma 1).

3. **Necessary and sufficient condition for robust stability of an additively perturbed plant**

Let us now consider the closed-loop unity feedback system \( \Sigma_{c_1}(P_\delta, C_0) \) of Fig. 1 where \( P_\delta \in R_0^{p \times m}(s) \) is strictly proper and \( C_0 \in \phi(P_\delta) \), and let us make the following assumptions.

(i) The nominal plant \( P_\delta \) is additively perturbed to \( P_\delta + \Delta P \in R_\Gamma^{p \times m}(s) \) where \( \Delta P \in R_\Gamma^{p \times m}(s) \) strictly proper and known.

(ii) The additively perturbed system \( \Sigma(P) \) is also free of unstable hidden modes and \( \left| \mathcal{H}(\Gamma) + \left[ P_\delta(s) + \Delta P(s) \right] C_0(s) \right| \neq 0 \) as a rational function, i.e. that the perturbed closed-loop system is 'well posed'.

If we now consider the closed-loop unity feedback system \( \Sigma_{c_1}(P_\delta + \Delta P_\delta, C_0) \) of Fig. 2 then in view of the parametrization in (6') of the set of 'plants' stabilizable by \( C_0 \in \phi(P_\delta) \) we can state the following.
Proposition 3

Under the above assumptions and notation the following statements are equivalent:

(i) $\Sigma_21(P_0 + \Delta P_0, C_0)$ is internally stable and $H_{\text{in}}(P_0 + \Delta P_0, C_0) \in S^{(p+m)\times(p+m)}$

(ii) The additively perturbed plant $P := P_0 + \Delta P_0$ is stabilizable by $C_0 \in \phi(P_0)$

(iii) $P := P_0 + \Delta P_0 \in \psi(C_0)$

(iv) There exists a strictly proper $Q \in S^{p \times m}$ such that

$$P := P_0 + \Delta P_0 = (A_1 + QN_1)^{-1}(B_1 - QD_1) = (B_2 - D_2Q)(A_2 + N_2Q)^{-1}$$

Solving (17) with respect to $Q$ we obtain the following necessary and sufficient condition for the robust stability of an additively perturbed plant.

Theorem 1

Let $P_0 = A^{-1}B_1 = B_2A^{-1} \in \mathbb{R}^{p \times m}(s)$ and strictly proper, $C_0 = D_1^{-1}N_1 = N_2D_2^{-1} \in \mathbb{R}^{m \times r}(s)$ a pair of nominal plant and nominal stabilizing compensator so that (3) is satisfied. Let $P_0$ be additively perturbed to $P_0 + \Delta P_0 := P \in \mathbb{R}^{p \times m}(s)$. Then $\Sigma_21(P_0 + \Delta P_0, C_0)$ is internally stable iff

$$Q := -D_2^{-1}[(P_0 + \Delta P_0)C_0 + I_r]^{-1}\Delta P_0 A_2$$

$$= -A_1\Delta P_0[C_0(P_0 + \Delta P_0) + I_m]^{-1}D_1^{-1} \in S^{p \times m}$$

and is strictly proper.

Example

Consider an SISO system with a 'high frequency' model $P$ where $P = \frac{100}{(s - 1)\times(s + 100)}$ and a 'low-frequency' model of the plant is $P_0 = \frac{1}{(s - 1)}$. Obviously pure gain feedback can stabilize $P_0$, e.g. $C_0 = 2$ is a stabilizing compensator for $P_0$. Now

$$\Delta P_0 = P - P_0 = \frac{-s}{(s - 1)(s + 100)}$$

Also $P_0$ can be written as

$$P_0 = \frac{1}{s + a}\left[\frac{s - 1}{s + a}\right]^{-1} = BA^{-1}, a > 0$$

and $(A, B)$ are coprime in $\mathbb{C}$ so that from criterion (18)

$$Q = -\left[\frac{100}{(s - 1)(s + 100)}\right]^{2 + 1} = \frac{-s - 1}{s - 1} \frac{s(s - 1)}{(s - 1)(s + 100)(s + a)} \in S$$

and is strictly proper. Thus $C_0 = 2$ also 'stabilizes' the 'high-frequency' model $P$.

Starting from (17) we can also give a parametrization of the set $\Delta \Pi$ of all allowable perturbations $\Delta P_0$ of a strictly proper plant $P_0$ such that if $C_0$ stabilizes $P_0$ then $C_0$ also stabilizes $P_0 + \Delta P_0$. A similar result to the one below has also appeared in Huang and Lin (1987).
Proposition 4

Let \( P_0 \in \mathbb{P}_{pr}^{r \times m}(s) \) be strictly proper with \( P_0 = A_1^{-1}B_1 = B_2A_2^{-1} \) and \( C_0 = \phi(P_0) \) with \( C_0 = D_1N_1 = N_2D_2^{-1} \) so that (3) is satisfied. Then the set of all additive perturbations \( \Delta P_0 \) such that \( C_0 \) also stabilizes \( P_0 + \Delta P_0 \) is given by

\[
\Delta \Pi = \{ \Delta P_0 | \quad Q = -(A_1 + QN_1)^{-1}QA_2^{-1}C \in S^{r \times m} \text{ and strictly proper} \}
\]

(19)

Proof

Solving (17) with respect to \( \Delta P_0 \) we have

\[
\Delta P_0 = (A_1 + QN_1)^{-1}(B_1 - QD_1) - B_2A_2^{-1} = (A_1 + QN_1)^{-1}
\]

\[
\times [(B_1 - QD_1)A_2 - (A_1 + QN_1)B_2]A_2^{-1}
\]

\[
= -(A_1 + QN_1)^{-1}Q(D_1A_2 + N_1B_2)A_2^{-1} = -(A_1 + QN_1)^{-1}QA_2^{-1}
\]

(20)

where we have made use of the facts that \( B_1A_1 - A_1B_2 = 0 \) and \( D_1A_1 + A_1B_2 = I_r \).

Notice that every \( \Delta P_0 \in \Delta \Pi \) is strictly proper.

\( \square \)

4. Robustness of stability for stable plants

As before, let \( P_0 \in \mathbb{P}_{pr}^{r \times m}(s) \) be strictly proper, \( C_0 = \phi(P_0) \) with \( P_0 = A_1^{-1}B_1 = B_2A_2^{-1} \), \( C_0 = D_1N_1 = N_2D_2^{-1} \) so that the Bezout identity (3) is satisfied and consider the internally stable closed-loop system \( \Sigma_{c1}(P_0, C_w) \) where \( C_w \in \phi(P_0) \) as given by (5) and the non-singularity constraints are automatically satisfied. Now let \( P_0 \) be perturbed to \( P_0 + \Delta P_0 \) where \( \Delta P_0 \) is strictly proper.

Assuming that \( \Sigma(P_0 + \Delta P_0) \) is also free of unstable hidden modes, for \( C_w \) to maintain the internal stability of \( \Sigma_{c1}(P_0 + \Delta P_0, C_w) \) we must have that (i) \( \|I_p + [P_0(s) + \Delta P_0(s)]C_w(s)\| \neq 0 \) and (ii) \( P_0 + \Delta P_0 \in \psi(C_w) \) where \( \psi(C_w) \) is given by (6).

Equivalently \( \Sigma_{c1}(P_0 + \Delta P_0, C_w) \) is internally stable iff

\[
Q := -(D_2 - B_3W)^{-1}[(P_0 + \Delta P_0)C_w(I_p + I_m)]^{-1} \Delta P_0 A_2
\]

\[
= -A_1 \Delta P_0 [C_w(P_0 + \Delta P_0) + I_m]^{-1}(D_1 - WB_1) \in S^{r \times m}
\]

(21)

If \( \Sigma(P_0) \) is (open-loop) internally stable then \( P_0 \in S^{r \times m}(\text{with } \Omega \equiv C^+) \) and we can take \( B_1 = B_2 = P_0, A_1 = I_p, A_2 = I_m, D_1 = I_m, D_2 = I_p, N_1 = Q_{m,p}, N_2 = O_{p,m} \) so that \( \phi(P_0) \) is given by (Desoer et al. 1980, Vidyasagar 1985)

\[
\phi(P_0) = \{ C_w = (I_m - WP_0)^{-1}W = W(I_p - P_0 W)^{-1}W \in S^{m \times m} \}
\]

(22)

Noticing that the expressions in (22) constitute fractional representations of \( C_w \) left and right coprime in \( \Omega \) and substituting these for \( C_w \) in condition (21) we obtain

\[
Q = -(I_p - P_0 W)^{-1}[(P_0 + \Delta P_0)W(I_p - P_0 W)^{-1} + I_p]^{-1} \Delta P_0
\]

\[
= -[(\Delta P_0 W + I_p)^{-1} \Delta P_0
\]

(23)

and thus we can state the following result.

Proposition 5

Let \( P_0 \in S^{r \times m} \) and be strictly proper. Let \( C_w = \phi(P_0) \) and let \( P_0 \) be additively perturbed to \( P_0 + \Delta P_0 \) where \( \Delta P_0 \) is strictly proper and known. Then under condition (i) \( \Sigma_{c1}(P_0 + \Delta P_0, C_w) \) is internally stable iff

\[
Q := -(\Delta P_0 W + I_p)^{-1} \Delta P_0 = -\Delta P_0 (W \Delta P_0 + I_m)^{-1} \in S^{r \times m}
\]

(24)
Now from the fact that $Q$ in (24) gives the closed-loop transfer function matrix of the configuration in Fig. 3, we can rephrase Proposition 5 as follows.

**Figure 3.**

![Block diagram](image)

**Proposition 6**

Let $P_0 \in S^{p \times m}$ and $P \in P^{p \times m}_s(s)$ both be strictly proper. Then there exists a compensator $C_w$ stabilizing both, i.e., $P_0, P$ are simultaneously stabilizable iff their difference $\Delta P_0 := P - P_0$ is stabilizable by a proper and $\Omega$-stable compensator $W$ placed on the feedback loop.

Finally, solving (24) with respect to $\Delta P_0$ we obtain the following.

**Corollary 2**

Let $P_0, C_w$ be as in Proposition 5. Then any $\Delta P_0$ such that $\Sigma_{c_1}(P_0 + \Delta P_0, C_w)$ is internally stable must have fractional representations that are left and right coprime in $\Omega$ and given by

$$\Delta P_0 = -Q(I_m + WQ)^{-1} = -(I_p + QW)^{-1}Q$$

for some strictly proper $Q \in S^{p \times m}$.

**REFERENCES**


