SYNCHRONIZED ABANDONMENTS IN A SINGLE SERVER UNRELIABLE QUEUE

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Abstract

We consider a single server unreliable queue represented by a 2-dimensional continuous time Markov chain. At failure times, all present customers leave the system. Moreover, customers become impatient and perform synchronized abandonments, as long as the server is down. We analyze this model and derive the main performance measures using results from the basic $q$-hypergeometric series.

Keywords: queueing system; unreliable server; catastrophes; synchronized abandonments; basic $q$-hypergeometric series; Markov chain; stationary distribution; sojourn time; busy period

1 Introduction

In the queueing literature, there exists a significant number of papers dealing with queueing systems with abandonments. In the majority of the papers, the source of the impatience has been taken to be either a long wait already experienced at a queue, or a long wait anticipated by the customers. Recently, Altman and Yechiali (2006, 2008), Perel and Yechiali (2009) and Yechiali (2007) considered systems with server(s) alternating between on and off periods, where customers’ impatience is due to the absence of the server(s). Such systems can model satisfactorily the reneging behavior of waiting customers in real systems with servers that are temporarily unavailable due to either scheduled vacations or failures.

Models with servers alternating between on and off periods and customers’ abandonments are generally hard to analyze. In a Markovian framework, the state of such a system is typically represented by a vector $(n, i)$, where $n$ records the number of customers and $i$ the state of the server(s). However, the abandonments that the customers perform independently lead to transitions out of a state $(n, 0)$ to a state $(n - 1, 0)$ with rates proportional to $n$. In this sense, the independent abandonments of the customers give rise to ‘spatially inhomogeneous’ continuous time Markov chains. This is also the distinctive feature in queueing models with an infinite number of servers, retrial queueing models and population growth models in Mathematical Biology, since every individual is associated with births and deaths. In most cases these models are mathematically untractable and it is not possible to conclude with closed form results.
There are only a few research works trying to extend matrix analytic methods or other analytic tools for models with this type of spatial inhomogeneity. In general, the authors use truncation or generalized truncation ideas to study the systems (see e.g. Artalejo and Pozo (2002) and the references therein) or they apply generating function methods. However, we have to note that in this case the partial generating functions of the number of individuals in system satisfy a system of linear differential equations in contrast with the ‘spatially homogeneous’ case where they satisfy linear algebraic equations. Such systems are usually untractable or can be solved in terms of hypergeometric series (see e.g. Altman and Yechiali (2006, 2008), Artalejo and Gomez-Corral (1997), Baykal-Gursoy and Xiao (2004), Keilson and Servi (1993), Krishnamoorthy et al. (2005), Perel and Yechiali (2009) and Yechiali (2007)).

The aim of the present paper is to extend the study of queues with disasters and impatient customers when the system is down that was introduced by Yechiali (2007). Yechiali (2007) considered Markovian queues subject to disasters that remove all the customers from the system and turn the server down. As long as the server is down, he assumed that the arrivals continue to come, but the customers become impatient and perform independent abandonments, that is, every customer sets on his own exponential patience clock and leaves the system when his patience time expires. In the present paper we assume that the customers are impatient but they perform synchronized abandonments. Such a model is motivated by remote systems where customers have to wait for a certain transport facility to abandon the system. Then, whenever the facility visits the system, the present customers can decide whether to leave the system or not. Therefore, we have synchronized departures for some of the customers. More specifically, we assume that the abandonment opportunities occur according to a Poisson process at rate $\xi$, whenever the server is down (so we can think that the transportation facility’s arrivals occur according to this process). Then, at an abandonment opportunity epoch, every customer decides to abandon the system with probability $p$ or remains in the system waiting for service with probability $q = 1 - p$, independently of the others. This implies that there exist binomial transition rates of the form $\xi \binom{n}{n'} p^{n-n'} q^{n'}$, from a state $(n,0)$ to states $(n',0)$, for $0 \leq n' \leq n$. Similar Markov chains occur in Mathematical Biology in the study of population processes subject to binomial catastrophes (see e.g. Artalejo et al. (2007), Brockwell et al. (1982), Economou (2004) and Economou and Fakinos (2008)). Moreover, Neuts (1994) studied a 1-dimensional discrete-time model with similar dynamics. In the same framework, Economou and Kapodistria (2009) studied a model with synchronized services, while Adan et al. (2009) studied a model with server’s vacations and synchronized abandonments.

We show that the framework of basic $q$-hypergeometric series enables us to express in closed form the main performance measures of this system. In general, the theory of $q$-hypergeometric series can facilitate the computations regarding systems with this kind of binomial transitions arising from synchronization (see also Economou and Kapodistria (2009) and Adan et al. (2009)).

The paper is organized as follows. In section 2 we describe the model and introduce the appropriate notation. In section 3 we carry out the equilibrium analysis of the system state and derive several exact formulas and iterative algorithmic schemes. Some limiting regimes with a particular interest are discussed in more detail. In section 4 we study the sojourn time of a customer, while in section 5 we treat the system busy period distribution of the model.
2 Model description and notation

We consider an $M/M/1$ queueing system in which customers arrive according to a Poisson process at rate $\lambda$. The service is provided by a single server, who serves the customers on a FCFS basis. The successive service times are independent exponentially distributed random variables with rate $\mu$. The system is subject to failures that occur when the server is at a functioning state, according to a Poisson process at rate $\eta$. At a failure epoch, the server is turned off and all present customers are forced to leave the system. Then, the repair process starts immediately. The repair times are exponentially distributed random variables with rate $\gamma$. While the server is off, the stream of new arrivals continues. However, since the server is down, these new customers become impatient and perform synchronized abandonments in the following way: A transportation facility is set on and it arrives at the system according to a Poisson process at rate $\xi$. Every arrival epoch of the transportation facility constitutes an abandonment opportunity for the present customers. We suppose that each one of them decides to abandon the system with probability $p$ or remains in the system with probability $q$, independently of the others.

The system is represented by a continuous-time Markov chain $\{(N(t),I(t)) : t \geq 0\}$, where $N(t)$ is the number of customers in the system at time $t$ and $I(t)$ denotes the state of the server at time $t$ ($0=\text{off (under repair)}$ and $1=\text{on (functioning)}$), $t \geq 0$. The corresponding transition rate diagram is given in figure 1.

![Figure 1: Transition rate diagram of $\{(N(t),I(t))\}$](image)

Let $(\pi(n,i) : n \geq 0 \text{ and } i = 0,1)$ denote the equilibrium distribution of $\{(N(t),I(t))\}$. We also define the partial probability generating functions $\Pi_0(z)$ and $\Pi_1(z)$ by

$$
\Pi_0(z) = \sum_{n=0}^{\infty} \pi(n,0)z^n \text{ and } \Pi_1(z) = \sum_{n=0}^{\infty} \pi(n,1)z^n .
$$

In section 3 we will determine $\Pi_0(z)$ and $\Pi_1(z)$ in terms of $q$-hypergeometric series (also known as basic hypergeometric series). Moreover, in sections 4 and 5 we will also see that the study of the sojourn times and the busy period distribution of this system is also facilitated using the theory of $q$-hypergeometric series.

There exists a rich theory for the class of $q$-hypergeometric series and their $q$-calculus which enables fast calculations and simplifications. In the queueing theory literature there exist only
few papers where this theory has been applied (see e.g. Ismail (1985), Kemp and Newton (1990) and Kemp (2005)). For this reason and for the sake of self-completeness we will briefly summarize the basic definitions and results of this theory. The interested reader can find more details on the definitions and the results below (with proofs and extensions) in Gasper and Rahman (2004), Chapters 1-3 and Appendices I-III.

The $q$-hypergeometric series are series of the form $\sum_{n=0}^{\infty} c_n$ where $c_0 = 1$ and $\frac{c_{n+1}}{c_n}$ is a rational function of $q^n$ for a deformation parameter $q$, which is taken to satisfy $|q| < 1$. They were initially introduced by Heine, in 1846, who developed the basic theory, following Gauss’ fundamental paper on hypergeometric series. Observing that the ratio $\frac{c_{n+1}}{c_n}$, being rational in $q^n$, can be written in the form

$$\frac{c_{n+1}}{c_n} = \frac{(1-a_1 q^n)(1-a_2 q^n) \cdots (1-a_r q^n)}{(1-q^{n+1})(1-b_1 q^n) \cdots (1-b_s q^n)} (-q^n)^{1+s-r} z,$$

we have that every such series assumes the canonical form

$$r \phi_s \left( \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array} ; q, z \right) = r \phi_s (a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} [(-1)^n q^{(n)}]^{1+s-r} z^n,$$

(2.2)

where $(a; q)_0 = 1$ and $(a; q)_n = (1-a)(1-aq)(1-aq^2) \cdots (1-aq^{n-1})$, $n \geq 1$. In the definition of a $q$-series through (2.2), it is assumed that $b_i \neq q^{-m}$ for $m = 0, 1, \ldots$ and $i = 1, 2, \ldots, s$. This is the standard $r \phi_s$ notation for $q$-series that is currently used instead of the old one without the factor $[(-1)^n q^{(n)}]^{1+s-r}$. For $|q| < 1$, the $r \phi_s$ series converges absolutely for all $z$ when $r \leq s$ and for $|z| < 1$ when $r = s + 1$. We use the abbreviation $(a_1, a_2, \ldots, a_r; q)_n$ to denote the product $(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n$. The quantity $(a; q)_n$ is referred to as the $q$-shifted factorial. We also define $(a; q)_\infty = \prod_{k=0}^{\infty} (1-aq^k)$ and use the abbreviation $(a_1, a_2, \ldots, a_r; q)_\infty$ to denote the product $(a_1; q)_\infty (a_2; q)_\infty \cdots (a_r; q)_\infty$. A $q$-calculus has been developed that parallels the theory of hypergeometric functions. The most important summation formula for the $q$-hypergeometric series is given by the $q$-binomial theorem (see e.g. Gasper and Rahman (2004) Section 1.3)

$$1 \phi_0 (a; -; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1.$$

(2.3)

Of particular importance are the two $q$-analogues of the exponential function $e^z$. They are the functions $e_q(z) = \frac{1}{(z; q)_\infty} = 1 \phi_0 (0; -; q, z)$, $|z| < 1$ and $E_q(z) = (-z; q)_\infty = 0 \phi_0 (-; -; q, -z)$. The $q$-binomial theorem enables to express the $q$-exponential functions in the form of $q$-series. The definite $q$-integral on an interval $[0, a]$ is defined by

$$\int_0^a f(t) dq t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n.$$

As $q \to 1^-$ the $q$-analogues reduce to their standard counterparts. In particular we have the relationships:

$$\lim_{q \to 1^-} e_q(z(1-q)) = e^z, \quad \lim_{q \to 1^-} \frac{(q^a z; q)_\infty}{(z; q)_\infty} = (1-z)^{-a}, \quad \lim_{q \to 1^-} \int_0^a f(t) dq t = \int_0^a f(t) dt.$$

(2.4)
As \( q \to 0^+ \) we can easily see that
\[
\lim_{q \to 0^+} e_q(z) = \frac{1}{1 - z}, \quad \lim_{q \to 0^+} r_{q+1} \phi_r \left( a_1, a_2, \ldots, a_{r+1}; b_1, b_2, \ldots, b_r ; q, q \right) = 1. \tag{2.5}
\]

### 3 The equilibrium state distribution

The balance equations of the model are given as follows:

\[
(\lambda + \gamma + \xi)\pi(0, 0) = \eta \sum_{j=0}^{\infty} \pi(j, 1) + \xi \sum_{j=0}^{\infty} p^j \pi(j, 0) \tag{3.1}
\]

\[
(\lambda + \gamma + \xi)\pi(n, 0) = \lambda \pi(n - 1, 0) + \xi \sum_{j=n}^{\infty} \binom{j}{n} p^{j-n} q^n \pi(j, 0), \quad n \geq 1 \tag{3.2}
\]

\[
(\lambda + \eta)\pi(0, 1) = \mu \pi(1, 1) + \gamma \pi(0, 0) \tag{3.3}
\]

\[
(\lambda + \mu + \eta)\pi(n, 1) = \lambda \pi(n - 1, 1) + \mu \pi(n + 1, 1) + \gamma \pi(n, 0), \quad n \geq 1. \tag{3.4}
\]

These equations can be solved efficiently by employing generating function methods and we obtain the following.

**Theorem 3.1** The partial probability generating functions \( \Pi_0(z) \) and \( \Pi_1(z) \) are given by

\[
\Pi_0(z) = \frac{\eta \gamma}{(\gamma + \eta)(\lambda(1-z) + \gamma + \xi)} \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} \binom{j}{n} p^{j-n} q^n \pi(j, 0) z^n, \quad |z| < 1 + \frac{\gamma + \xi}{\lambda} \tag{3.5}
\]

\[
\Pi_1(z) = \frac{\gamma(1-z_0) z \Pi_0(z) - \gamma(1-z_0) \pi_0(z)}{(\lambda + \mu + \eta) z - \lambda z^2 - \mu(1-z_0)}, \quad |z| < \min \left( 1 + \frac{\gamma + \xi}{\lambda}, z_1 \right), \tag{3.6}
\]

where

\[
z_0 = \frac{\lambda + \mu + \eta - \sqrt{(\lambda + \mu + \eta)^2 + 4 \lambda \mu}}{2 \lambda} \in (0, 1), \tag{3.7}
\]

\[
z_1 = \frac{\lambda + \mu + \eta + \sqrt{(\lambda + \mu + \eta)^2 + 4 \lambda \mu}}{2 \lambda} \in (1, \infty). \tag{3.8}
\]

The convergence of the series is absolute in the corresponding open disks and uniform in every compact subset of them.

**Proof.** Summing equations (3.1) and (3.2) over all \( n = 0, 1, \ldots \) we obtain that

\[
\gamma \Pi_0(1) = \eta \Pi_1(1). \tag{3.9}
\]

Equation (3.9) together with the normalization equation \( \Pi_0(1) + \Pi_1(1) = 1 \) yields that

\[
\Pi_0(1) = \frac{\eta}{\gamma + \eta}, \quad \Pi_1(1) = \frac{\gamma}{\gamma + \eta}. \tag{3.10}
\]

Multiplying equations (3.1) and (3.2) by \( z^0 \) and \( z^n \) respectively and summing for all \( n = 0, 1, \ldots \), we obtain that

\[
(\lambda + \gamma + \xi) \Pi_0(z) = \lambda z \Pi_0(z) + \xi \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} \binom{j}{n} p^{j-n} q^n \pi(j, 0) z^n + \eta \sum_{j=0}^{\infty} \pi(j, 1)
\]

\[
= \lambda z \Pi_0(z) + \xi \sum_{j=0}^{\infty} z \pi(j, 0) (p + qz)^j + \eta \Pi_1(1)
\]

\[
= \lambda z \Pi_0(z) + \xi \Pi_0(p + qz) + \eta \Pi_1(1).
\]
Hence
\[ \Pi_0(z) = \frac{\xi}{\lambda(1 - z) + \gamma + \xi} \Pi_0(1 - q + qz) + \frac{\eta \Pi_1(1)}{\lambda(1 - z) + \gamma + \xi}. \]  
(3.11)

Iterating equation (3.11) yields
\[
\Pi_0(z) = \Pi_0(1 - q^{n+1} + q^{n+1}z) \prod_{i=0}^{n} \frac{1}{\lambda q^i(1 - z) + \gamma + \xi} 
+ \eta \Pi_1(1) \sum_{k=0}^{n} \frac{1}{\lambda q^k(1 - z) + \gamma + \xi} \prod_{i=0}^{k-1} \frac{\xi}{\lambda q^i(1 - z) + \gamma + \xi},
\]  
(3.12)

where we assume that for \( k = 0 \) the empty product \( \prod_{i=0}^{k-1} \) is by definition equal to 1. Taking the limit as \( n \to \infty \) in (3.12) we obtain
\[
\Pi_0(z) = \eta \Pi_1(1) \sum_{k=0}^{\infty} \frac{1}{\lambda q^k(1 - z) + \gamma + \xi} \prod_{i=0}^{k-1} \frac{\xi}{\lambda q^i(1 - z) + \gamma + \xi}.
\]  
(3.13)

Expressing equation (3.13) in terms of the canonical form of \( q \)-series we obtain that
\[
\Pi_0(z) = \frac{\eta \Pi_1(1)}{\lambda(1 - z) + \gamma + \xi} (0, q; -\frac{\lambda q(1-z)}{\gamma + \xi}, \frac{\xi}{\gamma + \xi}).
\]  
(3.14)

Then by substituting \( \Pi_1(1) \) from (3.10) yields equation (3.5). The absolute convergence of the series (3.5) is guaranteed for \( z \in \{ z \in \mathbb{C} : |z| < 1 + \frac{\gamma + \xi}{\lambda} \} \). Indeed, we first observe that the
\( \phi_1 \left( 0, q; -\frac{\lambda q(1-z)}{\gamma + \xi}, \frac{\xi}{\gamma + \xi} \right) \) series converges absolutely (see the comments after the definition (2.2)) for all \( z \) such that \(-\frac{\lambda q(1-z)}{\gamma + \xi} \neq q^{-m} \), for \( m = 0, 1, 2, \ldots \), consequently it converges for \( z \) with \( |z| < 1 + \frac{\gamma + \xi}{\lambda} \). Moreover, the denominator in (3.5) does not vanish for \( z \neq 1 + \frac{\gamma + \xi}{\lambda} \). Hence the partial probability generating function \( \Pi_0(z) \) as given in equation (3.5) converges for \( |z| < 1 + \frac{\gamma + \xi}{\lambda} \).

Multiplying equations (3.3) and (3.4) by \( z^0 \) and \( z^n \) respectively and summing for all \( n = 0, 1, \ldots \) we obtain
\[
(\lambda + \mu + \eta)\Pi_1(z) - \mu \pi(0, 1) = \lambda z \Pi_1(z) + \frac{\mu}{z} (\Pi_1(z) - \pi(0, 1)) + \gamma \Pi_0(z),
\]
or equivalently
\[
((\lambda + \mu + \eta)z - \lambda z^2 - \mu)\Pi_1(z) = -\mu(1 - z)\pi(0, 1) + \gamma z \Pi_0(z).
\]  
(3.15)

Equation (3.15) is identical to Yechiali (2007) equation (2.6). We observe that the quadratic polynomial
\[
f(z) = (\lambda + \mu + \eta)z - \lambda z^2 - \mu
\]  
(3.16)

has two real roots \( z_0 \) and \( z_1 \) given by (3.7) and (3.8). Setting \( z = z_0 \) in equation (3.15) (note that \( \Pi_1(z_0) \) converges since \( z_0 < 1 \)) yields
\[
\pi(0, 1) = \frac{\gamma z_0 \Pi_0(z_0)}{\mu(1 - z_0)}.
\]  
(3.17)
Substituting (3.17) into (3.15) and solving for $\Pi_1(z)$ yields (3.6). Having established the radius of convergence for $\Pi_0(z)$, it can be easily checked that the partial probability generating function $\Pi_1(z)$ given by (3.6) converges for $z \in \{z \in \mathbb{C} : |z| < \min(1 + \frac{2 + \xi}{\lambda}, z_1)\}$.

We can now use (3.5)-(3.6) to derive explicit expressions for some important performance measures of the system. More specifically, we will derive the factorial moments of the number of customers in the system. Note that the moments of all orders exist since the partial probability generating function $\Pi_1(z)$ has coefficients and constant term that depend on $n$ in product closed form for any desired $n$. Nevertheless, a closed form for $\Pi_1^{(n)}(1)$ can be obtained by applying results from the theory of $q$-series, in particular using Heine’s transformation formulas (3.18)-(3.20).

Differentiating (3.11) $n$ times and setting $z = 1$ yields

$$\Pi_0^{(n)}(1) = \frac{n\lambda}{\gamma + \xi(1 - q^n)}\Pi_0^{(n-1)}(1), \quad n \geq 1. \tag{3.22}$$

Differentiating (3.15) $n$ times and setting $z = 1$ yields

$$\begin{align*}
\eta\Pi_1^{(1)}(1) + (\mu + \eta - \lambda)\Pi_0^{(0)}(1) &= \gamma\Pi_0^{(1)}(1) + \gamma\Pi_0^{(0)}(1) + \mu\pi(0,1) \\
\eta\Pi_1^{(n)}(1) + n(\mu + \eta - \lambda)\Pi_1^{(n-1)}(1) - n(n-1)\lambda\Pi_1^{(n-2)}(1) &= \gamma\Pi_0^{(n)}(1) + n\gamma\Pi_0^{(n-1)}(1), \quad n \geq 2. \tag{3.23}
\end{align*}$$

Equations (3.22) and (3.23) form an iterative scheme with initial conditions for $\Pi_0^{(0)}(1) = \Pi_0(1)$ and $\Pi_1^{(0)}(1) = \Pi_1(1)$ given by (3.10). The first order scheme given by (3.22) gives easily $\Pi_1^{(n)}(1)$ in product closed form for any desired $n$. However, the second order scheme given by (3.23) has coefficients and constant term that depend on $n$, so its direct solution seems impossible. Nevertheless, a closed form for $\Pi_1^{(n)}(1)$ can be obtained by applying results from the theory of $q$-series, in particular using Heine’s transformation formulas (3.18)-(3.20).
Moreover from equation (3.6) we obtain that the equilibrium number of customers in the system are given by

\[ m^{(n)} = \frac{\gamma}{\gamma + \eta} \frac{n!}{z_0} \sum_{k=0}^{n} \frac{\lambda^n}{\eta^k \prod_{i=1}^{n} \left( \gamma + \xi(1 - q^i) \right)} \left[ (-1)^k z_0 \left( z_1 - 1 \right)^{k+1} + z_1 \left( 1 - z_0 \right)^{k+1} \right] \]

+ \gamma z_0 \Pi_0(z_0) \prod_{i=1}^{n} \left( \frac{\lambda/\eta}{\eta(z_1 - z_0)} \left[ (-1)^{n-1} (z_1 - 1)^n + (1 - z_0)^n \right] \right)

+ \frac{\gamma}{\gamma + \eta} \frac{\lambda^n}{\prod_{i=1}^{n} \left( \gamma + \xi(1 - q^i) \right)}, \ n \geq 1, \tag{3.24} \]

where \( z_0 \) and \( z_1 \) are given in equations (3.7) and (3.8) respectively. In particular

\[ E[N] = \frac{\gamma}{\gamma + \eta} \frac{z_0 \Pi_0(z_0)}{1 - z_0} + \frac{\lambda}{\lambda - \mu} + \frac{\lambda}{\gamma + \xi(1 - q)} . \tag{3.25} \]

**Proof.** The factorial moments exponential generating function \( P(z) \) is given by

\[ P(z) = \sum_{n=0}^{\infty} m^{(n)} \frac{z^n}{n!} = E \left[ \sum_{n=0}^{\infty} \left( \frac{N}{n} \right) z^n \right] = E[(1 + z)^N] = \Pi_0(1 + z) + \Pi_1(1 + z). \tag{3.26} \]

We have already shown in theorem 3.1 that \( \Pi_0(z) \) and \( \Pi_1(z) \) converge in a neighborhood of 1, hence \( P(z) \) is well defined in a neighborhood of 0. Using Heine’s transformation formula (3.19), equation (3.5) assumes the form

\[ \Pi_0(z) = \frac{\eta}{\gamma + \eta} \left( 0, q; \frac{\xi q}{\gamma + \xi}, q, \frac{\lambda(1 - z)}{\gamma + \xi} \right) = \frac{\eta}{\gamma + \eta} \sum_{n=0}^{\infty} \lambda^n \prod_{i=1}^{n} \left( \gamma + \xi(1 - q^i) \right) (z - 1)^n, \]

that gives

\[ \Pi_0(1 + z) = \frac{\eta}{\gamma + \eta} \sum_{n=0}^{\infty} \lambda^n \prod_{i=1}^{n} \left( \gamma + \xi(1 - q^i) \right) z^n. \tag{3.27} \]

Moreover from equation (3.6) we obtain that

\[ \Pi_1(1 + z) = \frac{\gamma}{(1 + z)} \Pi_0(1 + z) + \frac{z_0 \Pi_0(z_0)}{1 - z_0} \frac{z}{f(1 + z)}, \tag{3.28} \]

where \( f(z) = -\lambda(z_0 - z)(z_1 - z) \) is given in equation (3.16). By partial fraction expansion and elementary algebra we obtain that

\[ \frac{1}{f(1 + z)} = \frac{1}{\lambda(z_1 - z_0)(1 - z_0)} \frac{1}{1 + \frac{z}{1 - z_0}} + \frac{1}{\lambda(z_1 - z_0)(z_1 - 1)} \frac{1}{1 - \frac{z}{z_1 - 1}} \]

\[ = \sum_{n=0}^{\infty} \frac{1}{\lambda(z_1 - z_0)} \left[ (-1)^n \frac{1}{(1 - z_0)^{n+1}} + \frac{1}{(z_1 - 1)^{n+1}} \right] z^n \]

\[ = \sum_{n=0}^{\infty} \frac{(\lambda/\eta)^n}{\eta(z_1 - z_0)} \left[ (-1)^n (z_1 - 1)^{n+1} + (1 - z_0)^{n+1} \right] z^n, \tag{3.29} \]
and then easily
\[
\frac{1 + z}{f(1+z)} = \sum_{n=0}^{\infty} \frac{(\lambda/\eta)^n}{\eta(z_1 - z_0)} \left[(-1)^n z_0(z_1 - 1)^{n+1} + z_1(1-z_0)^{n+1}\right] z^n. \tag{3.30}
\]

Now (3.27) and (3.30) imply easily that
\[
\frac{(1 + z)\Pi_0(1+z)}{f(1+z)} = \frac{1}{(\gamma + \eta)(z_1 - z_0)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\lambda^n}{\eta^k \prod_{i=1}^{n-k}(\gamma + \xi(1-q_i))} \times \left[(-1)^k z_0(z_1 - 1)^{k+1} + z_1(1-z_0)^{k+1}\right] z^n, \tag{3.31}
\]
where \(z_0\) and \(z_1\) are given in equations (3.7) and (3.8), respectively. Plugging (3.29) and (3.31) into (3.28) and using (3.27), we can expand (3.26) in powers of \(z\). After some straightforward algebra we obtain (3.24) and in particular, for \(n = 1\), we deduce (3.25). \[\blacksquare\]

In the model we have two types of lost customers, those that abandon the system due to impatience during the repair phase of the server and those that are forced to leave the system at the failure epochs of the server. We will now calculate the proportion of customers who leave the system at the failure epochs, the proportion of customers who abandon the system because of impatience and the proportion of served customers. We begin by computing the corresponding rates.

When the system is in state \((n,1), n \geq 0\), the failure rate of the server is \(\eta\) and when a failure occurs all present customers leave the system, therefore the rate of lost customers due to failures, \(R_{\text{failures}}\), is given by
\[
R_{\text{failures}} = \sum_{n=0}^{\infty} \eta n \pi(n,1) = \eta \Pi_1'(1). \tag{3.32}
\]

Similarly, the rate of served customers, \(R_{\text{served}}\), is given by
\[
R_{\text{served}} = \sum_{n=1}^{\infty} \mu n \pi(n,1) = \mu (\Pi_1(1) - \pi(0,1)). \tag{3.33}
\]

Finally, the rate of abandonments due to impatience when the server is down, \(R_{\text{abandonments}}\), is given by
\[
R_{\text{abandonments}} = \lambda - R_{\text{failures}} - R_{\text{served}} = \xi(1-q)\Pi_0'(1). \tag{3.34}
\]

Clearly, \(\Pi_0'(1)\) can be directly obtained from equation (3.22), i.e. \(\Pi_0'(1) = \frac{\eta}{\gamma+\eta+\xi(1-q)}\), while \(\Pi_1'(1)\) can be obtained as \(\Pi_1'(1) = E[N] - \Pi_0'(1) = \frac{E[N]}{\eta(1-z_0)} + \frac{\lambda-\mu}{\gamma+\eta} + \frac{\gamma}{\gamma+\eta+\xi(1-q)}\), since the mean number of customers in the system, \(E[N]\), is given in equation (3.25). The percentages of lost customers due to failures, served customers and lost customers due to abandonments are readily computed by dividing (3.32)-(3.34) by the overall rate \(\lambda\).

We now turn our attention to the behavior of the model under certain limiting regimes. To emphasize the dependence on the parameters of the model in the rest of this section, we will denote \(\pi(n,i)\), \(\Pi_0(z)\) and \(\Pi_1(z)\) by \(\pi(n,i; \lambda, \mu, \xi, p, \gamma, \eta)\), \(\Pi_0(z; \lambda, \mu, \xi, p, \gamma, \eta)\) and \(\Pi_1(z; \lambda, \mu, \xi, p, \gamma, \eta)\)
respectively. Note that $\xi p$ can be thought of as the effective abandonment rate per customer. Indeed the overall abandonment time of a customer is a geometric sum of exponentially distributed random variables with rate $\xi$ and so we can easily see that it is also exponentially distributed with parameter $\xi p$. Under this perspective, if we have two models with the same parameters $\lambda, \mu, \gamma$ and $\eta$ that differ only in $\xi$ and $p$, but with $\xi p = \xi^*$ fixed, we can think that the models have identical arrival rates $\lambda$, service rates $\mu$, effective abandonment rates per customer $\xi^*$, setup rates $\gamma$ and catastrophe rates $\eta$ and differ only in the ‘level of synchronization’ $p$. Indeed, the case $p \to 0^+$ corresponds to no synchronization since the customers depart almost singly at the abandonment epochs. On the contrary, the case $p \to 1^-$ corresponds to full synchronization since almost all present customers depart simultaneously from the system when an abandonment opportunity occurs.

We are interested in studying the equilibrium behavior of the system for the case where $\lambda, \mu, \xi^*, \gamma$ and $\eta$ are kept fixed in the two limiting cases $p \to 0^+ (q \to 1^-)$ and $p \to 1^- (q \to 0^+)$. The case $p \to 0^+$ corresponds exactly to the model studied by Yechiali (2007) where the customers perform independent abandonments. In theorem 3.3 we derive the equilibrium state probability generating functions while in theorem 3.4 we obtain the factorial moments of the state distribution.

**Theorem 3.3** For a system with arrival rate $\lambda$, service rate $\mu$, effective abandonment rate per customer $\xi^*$, setup rate $\gamma$ and catastrophe rate $\eta$, the generating functions

$$\Pi^{(1)}_i(z) = \lim_{q \to 1^-} \Pi_i(n, i; \lambda, \mu, \frac{\xi^*}{1-q}, 1-q, \gamma, \eta), \quad i = 0, 1,$$

in the limiting case of no synchronization are given by

$$\Pi^{(1)}_0(z) = \frac{\eta \gamma}{\eta + \gamma} \frac{1}{1-z} \int_0^1 \left(1-t\right)^{\xi^*-1} e^{-\frac{\lambda}{\eta + \gamma}(t-z)} dt,$$

$$\Pi^{(1)}_1(z) = \frac{\gamma(1-z_0)z_0 \Pi^{(1)}_0(z) - \gamma(1-z)z_0 \Pi^{(1)}_0(z_0)}{((\lambda + \mu + \eta)z - \lambda z^2 - \mu)(1-z_0)},$$

where $z_0 = \frac{\lambda + \mu + \eta - \sqrt{\gamma^2 - 4\eta \mu}}{2\lambda}$.

**Proof.** Using Heine’s transformation formula (3.18), equation (3.5) assumes the following form

$$\Pi_0(z) = \frac{\eta}{\gamma + \eta} \frac{\gamma}{\gamma + \xi} \frac{(q; q)_\infty}{(q; q)_{\xi + \xi}} \frac{2\phi_1 \left( -\frac{\lambda(1-z)}{\gamma + \xi}, \frac{\xi}{\gamma + \xi}; 0; q, q \right)}{\gamma + \eta \gamma + \xi}.$$

We use (3.21) and we obtain that

$$\frac{(1-q) \phi_1 \left( -\frac{\lambda(1-z)}{\gamma + \xi}, \frac{\xi}{\gamma + \xi}; 0; q, q \right)}{(q; q)_\infty} = \frac{(q; q)_\infty}{(q; q)_\infty} \int_0^1 \frac{(gt; q)_\infty}{(-\frac{\lambda(1-z)q}{\gamma + \xi}, \frac{\xi}{\gamma + \xi}; q)_\infty} d_q t.$$

Plugging (3.38) into (3.37) and simplifying several terms we have that (3.37) assumes the form

$$\Pi_0(z) = \frac{\eta}{\gamma + \eta \gamma(1-q) + \xi^*} \int_0^1 \frac{(gt; q)_\infty}{(-\frac{\lambda(1-z)q}{\gamma + \xi}, \frac{\xi}{\gamma + \xi}; q)_\infty} d_q t$$

$$= \frac{\eta}{\gamma + \eta \gamma(1-q) + \xi^*} \int_0^1 \frac{(qt; q)_\infty}{(-\frac{\lambda(1-z)q}{\gamma + \xi}, \frac{\xi}{\gamma + \xi}; q)_\infty} e_q \left( -\frac{\lambda(1-z)q}{\gamma + \xi} \right) d_q t.$$
Replacing $\xi$ by $\frac{\xi^*}{1-q}$ and using (2.4) we have

$$\lim_{q \to 1^-} e_q \left( -\frac{\lambda(1-z)t}{\gamma + \xi} \right) = \lim_{q \to 1^-} e_q \left( -\frac{\lambda(1-z)t}{\gamma - (1-q) + \xi^*(1-q)} \right) = e^{-\frac{\lambda^*}{\gamma}(1-z)t}. \quad (3.40)$$

$$\lim_{q \to 1^-} \frac{(qt; q)_{\infty}}{q^t} = \lim_{q \to 1^-} \frac{(q(1 + \gamma(1-q))e^{\xi^*})_{\infty}}{(\frac{q(1 + \gamma(1-q))e^{\xi^*}}{\gamma(1-q)+\xi^*}; q)_{\infty}} = (1-t)^{\xi^*}. \quad (3.41)$$

Taking the limit as $q \to 1^-$ in (3.39), taking into account equations (3.40) and (3.41) and setting $t = \frac{\xi^*}{1-q}$ yields (3.35), which is Yechiali (2007) formula (2.7). Equation (3.36) results immediately from (3.6).

**Theorem 3.4** The factorial moments $m_{(n)}^{(1)} = E[N(N-1)(N-2)\cdots(N-n+1)]$ of the equilibrium number of customers in the system are given by

$$m_{(n)}^{(1)} = \frac{\gamma}{\gamma + \eta} \frac{n!}{z_1 - z_0} \sum_{k=0}^{n} \frac{\lambda^n}{\eta \prod_{i=1}^{k}(\gamma + \xi^*)} \left[ (-1)^k z_0(1 - 1)^{k+1} + z_1(1 - z_0)^{k+1} \right]$$

$$+ \gamma \frac{z_0 \Pi_{(1)}^{(0)}(z_0)}{1 - z_0} \frac{n!(\lambda/\eta)^{n-1}}{\eta(z_1 - z_0)} \left[ ((-1)^n - 1)^n(1 - 1)^{n} + (1 - z_0)^{n} \right]$$

$$+ \frac{\eta}{\gamma + \eta} \frac{n!\lambda^n}{\prod_{i=1}^{n}(\gamma + \xi^*)}, \quad n \geq 1, \quad (3.42)$$

where $z_0$ and $z_1$ are given in equations (3.7) and (3.8) respectively and $\Pi_{(1)}^{(0)}(z_0)$ is given by (3.35). In particular

$$E^{(1)}[N] = \frac{\gamma z_0 \Pi_{(1)}^{(0)}(z_0)}{\eta(1 - z_0)} + \frac{\gamma}{\gamma + \eta} \frac{\lambda - \mu}{\lambda} + \frac{\lambda}{\gamma + \xi^*}. \quad (3.43)$$

**Proof.** We replace $\xi$ by $\xi^*/(1-q)$ in (3.24) and (3.25) and we take the limit as $q \to 1^-$. We obtain (3.42) and (3.43) respectively.

The expression (3.43) gives the mean number of customers in Yechiali (2007) model. Indeed, Yechiali (2007) equations (2.10) and (2.13) can be used to derive independently our equation (3.43).

We now derive the corresponding results for the other extreme case of full synchronization, i.e. when $p \to 1^-$. In theorem 3.5 we derive the equilibrium state probability generating functions while in theorem 3.6 we obtain the factorial moments of the state distribution.

**Theorem 3.5** For a system with arrival rate $\lambda$, service rate $\mu$, effective abandonment rate per customer $\xi^*$, setup rate $\gamma$ and catastrophe rate $\eta$, the equilibrium state distribution $\pi_{(n)}^{(2)}(n, i) = \lim_{q \to 0^+} \pi(n, i; \lambda, \mu, \xi^*/1-q, 1-q, \gamma, \eta)$ in the limiting case of full synchronization is given by

$$\pi_{(n, 0)}^{(2)} = c_1 \left( \frac{\lambda}{\gamma + \xi^* + \lambda} \right)^n, \quad n \geq 0 \quad (3.44)$$

$$\pi_{(n, 1)}^{(2)} = \begin{cases} c_2 [n(1 - z_0) + 1] \left( \frac{\lambda}{\lambda + \gamma + \xi^*} \right)^n, & n \geq 0, \quad \text{when } z_1 = \frac{\lambda + \gamma + \xi^*}{\lambda}, \\ c_2 \left[ c_3 \left( \frac{1}{z_1} \right)^n + c_4 \left( \frac{\lambda}{\lambda + \gamma + \xi^*} \right)^n \right], & n \geq 0, \quad \text{when } z_1 \neq \frac{\lambda + \gamma + \xi^*}{\lambda}. \end{cases} \quad (3.45)$$
where
\[
c_1 = \frac{\eta \gamma + \xi^*}{\gamma + \eta \gamma + \xi^* + \lambda}, \\
c_2 = \frac{\eta \xi^*}{\gamma + \eta \mu(1 - z_0) \lambda(1 - z_0) + \gamma + \xi^*}, \\
c_3 = \frac{\lambda + \gamma + \xi^* - \mu}{\lambda + \gamma + \xi^* - \lambda z_1}, \\
c_4 = \frac{\mu(1 - z_0)}{\mu - z_0(\lambda + \gamma + \xi^*)},
\]
and \(z_0, z_1\) are given in equations (3.7) and (3.8), respectively.

**Proof.** We take the limit as \(q \to 0^+\) in (3.37), using (2.5). This yields
\[
\Pi_0^{(2)}(z) = \lim_{q \to 0^+} \frac{\eta \gamma}{\gamma + \eta \gamma + \xi} \left( -\frac{\lambda(1 - z)}{\gamma + \xi^*} \right) + 2 \phi \left( \frac{\xi}{\gamma + \xi} ; 0, q, q \right)
\]
\[
= \frac{\eta}{\gamma + \eta \gamma + \xi^*} (1 + \frac{\lambda(1 - z)}{\gamma + \xi^*} - \frac{\lambda}{\lambda + \gamma + \xi^*} z)
\]
(3.46)

Expanding \((\gamma + \xi^* + \lambda - \lambda z)^{-1}\) in power series and equating the coefficients of \(z^n\) yields (3.44). Taking the limit as \(q \to 0^-\) in equation (3.6) and plugging equation (3.46) yields after some calculations
\[
\Pi_1^{(2)}(z) = \frac{\eta \gamma}{\gamma + \eta \lambda + \gamma + \xi^*} \left( -\frac{\lambda(1 - z)}{\gamma + \xi^*} \right) + 2 \phi \left( \frac{\xi}{\gamma + \xi} ; 0, q, q \right)
\]
(3.47)

Using partial fraction expansion, we express \(\Pi_1^{(2)}(z)\) in power series and equating the coefficients of \(z^n\) for the two cases \(z_1 = \frac{\lambda + \gamma + \xi^*}{\lambda}\) and \(z_1 \neq \frac{\lambda + \gamma + \xi^*}{\lambda}\) we obtain the two branches of (3.45). ■

**Theorem 3.6** The factorial moments \(m^{(2)}(n) = E[N(N - 1)(N - 2) \cdots (N - n + 1)]\) of the equilibrium number of customers in the system are given by
\[
m^{(2)}(n) = \frac{\lambda^n}{\gamma + \eta z_1 - z_0} \sum_{k=0}^{n} \frac{n!}{\gamma \eta^k (\gamma + \xi^*)^{n-k}} \left[ (-1)^k z_0 (z_1 - 1)^{k+1} + z_1 (1 - z_0)^{k+1} \right]
\]
\[
+ \frac{\lambda z_0 \Pi_0^{(2)}(z_0) n! (\lambda/\eta)^{n-1}}{1 - z_0} \left[ (-1)^{n-1} (z_1 - 1) + (1 - z_0)^{n} \right]
\]
\[
+ \frac{\eta n! \lambda^n}{\gamma + \eta (\gamma + \xi^*)} \text{, } n \geq 1,
\]
(3.48)

where \(z_0\) and \(z_1\) are given in equations (3.7) and (3.8) respectively and \(\Pi_0^{(2)}(z)\) is given by (3.46). In particular
\[
E^{(2)}[N] = \frac{\gamma}{\eta} \frac{z_0 \Pi_0^{(2)}(z_0)}{1 - z_0} + \frac{\gamma}{\gamma + \eta} (\frac{\lambda - \mu}{\gamma} + \frac{\lambda}{\gamma + \xi^*}).
\]
(3.49)

**Proof.** We replace \(\xi\) by \(\xi^*/(1 - q)\) in (3.24) and (3.25) and we take the limit as \(q \to 0^+\). We obtain (3.48) and (3.49) respectively. ■
4 Sojourn times

Let $S$ denote the unconditional total sojourn time of an arbitrary customer in the system, regardless of whether he completes service or not. Moreover, let $S_{(n,i)}$ denote the conditional total sojourn time of a tagged customer in the system, given that upon arrival he finds the system in state $(n,i)$.

We employ first-step analysis excluding arrivals, because future arrivals do not influence the tagged customer. Indeed, by conditioning on whether the next transition is a service completion or a failure when the system is up we obtain the equations

$$E[S_{(0,1)}] = \frac{1}{\mu + \eta}, \quad (4.1)$$

$$E[S_{(n,1)}] = \frac{1}{\mu + \eta} + \frac{\mu}{\mu + \eta} E[S_{(n-1,1)}], \quad n \geq 1. \quad (4.2)$$

Similarly, by conditioning on whether the next transition corresponds to a repair completion or to an abandonment opportunity, we obtain the equations

$$E[S_{(n,0)}] = \frac{1}{\gamma + \xi} + \frac{\gamma}{\gamma + \xi} E[S_{(n,1)}] + \frac{\xi q}{\gamma + \xi} \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^i E[S_{(i,0)}], \quad n \geq 0. \quad (4.3)$$

The system of recursive relations (4.1)-(4.3) can be solved explicitly employing a generating function approach and using the theory of $q$-hypergeometric series. The solution is summarized in the following theorem.

**Theorem 4.1** The conditional expected total sojourn time of a tagged customer in the system $S_{(n,i)}$, given that upon his arrival he finds the system in state $(n,i)$, is given as follows

$$E[S_{(n,0)}] = \frac{1}{\gamma + \xi} + \frac{\gamma}{\gamma + \xi} E[S_{(n,1)}] + \frac{\gamma \mu}{\eta(\gamma + \xi)} \sum_{k=0}^{n} \frac{1}{\eta(1-q^{k+1})} \left(-\frac{\eta}{\eta + \mu}\right)^{k} \binom{n}{k}, \quad (4.4)$$

$$E[S_{(n,1)}] = \frac{1}{\eta} \left(1 - \left(\frac{\mu}{\mu + \eta}\right)^{n+1}\right), \quad n \geq 0. \quad (4.5)$$

**Proof.** Iterating equation (4.2), using (4.1), yields immediately (4.5). We now define the generating functions of the mean conditional expected total sojourn times $S_i(z), \ i = 0, 1$, given as

$$S_i(z) = \sum_{n=0}^{\infty} E[S_{(n,i)}] z^n, \ |z| < 1, \ i = 0, 1. \quad (4.6)$$

The convergence of these series in the open unit disk will be proved below, as they appear in the calculations. Indeed, multiplying equation (4.5) with $z^n$ and adding for all $n \geq 0$ results to

$$S_1(z) = \frac{1}{\eta} \left(\frac{1}{1-z} - \frac{\mu}{\mu + \eta}\right), \quad (4.7)$$

which is readily seen to converge in the open unit disk. Regarding the series $S_0(z)$, multiplying equations (4.3) with $(\gamma + \xi)z^n$ and adding for all $n \geq 0$ yields

$$(\gamma + \xi)S_0(z) = \frac{1}{1-z} + \gamma S_1(z) + \xi q \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^i E[S_{(i,0)}] z^n \quad (4.8)$$

$$= \frac{1}{1-z} + \gamma S_1(z) + \xi q \sum_{i=0}^{\infty} E[S_{(i,0)}] \left(\frac{q}{p}\right)^i \sum_{n=i}^{\infty} \binom{n}{i} (pz)^n.$$
Using now the ‘upper’ binomial coefficients generating function
\[
\sum_{n=i}^{\infty} \binom{n}{i} x^n = \frac{x^i}{(1-x)^{i+1}}, \, |x| < 1,
\] (4.9)
we have that (4.8) assumes the form
\[
(\gamma + \xi)S_0(z) = \frac{1}{1-z} + \gamma S_1(z) + \frac{\xi q}{1-(1-q)z} S_0\left(\frac{qz}{1-pz}\right).
\] (4.10)
We observe that equation (4.10) can be put in the form
\[
S_0(z) = \frac{H(z)}{G(z)} S_0(T(z)) + \frac{K(z)}{G(z)},
\] (4.11)
where
\[
T(z) = \frac{qz}{1-(1-q)z}
\] (4.12)
and
\[
G(z) = \gamma + \xi, \quad H(z) = \xi \frac{T(z)}{z}, \quad K(z) = \frac{1}{1-z} + \gamma S_1(z).
\] (4.13)
The solution of (4.11) can be done by iteration. To this end it seems convenient to introduce here an operator notation: The transformation \(T(z)\) defined by (4.12) is a linear fractional transformation and therefore its \(k\)-th compositions defined by \(T_0(z) = z\) and \(T_k(z) = T(T_{k-1}(z))\), \(k \geq 1\), can be computed in closed form. Indeed, it can be proved inductively that
\[
T_k(z) = \frac{q^k z}{1-(1-q^k)z}, \, k \geq 0.
\] (4.14)
By iterating (4.11) \(n\) times we obtain
\[
S_0(z) = \sum_{k=0}^{n} \frac{K(T_k(z))}{H(T_k(z))} \prod_{i=0}^{k} \frac{H(T_i(z))}{G(T_i(z))} + S_0(T_{n+1}(z)) \prod_{i=0}^{n} \frac{H(T_i(z))}{G(T_i(z))}.
\] (4.15)
However, note that
\[
\frac{H(T_i(z))}{G(T_i(z))} = \frac{\xi T_{i+1}(z)}{\gamma + \xi T_i(z)}
\]
and
\[
\frac{K(T_k(z))}{H(T_k(z))} = \left(\frac{1}{1-T_k(z)} + \gamma S_1(T_k(z))\right) \frac{T_k(z)}{\xi T_{k+1}(z)}
\]
so (4.15) assumes the form
\[
S_0(z) = \sum_{k=0}^{n} \left[ \frac{1}{1-T_k(z)} + \gamma S_1(T_k(z)) \right] \frac{T_k(z)}{\xi T_{k+1}(z)} \left(\frac{\xi}{\gamma + \xi}\right)^{k+1} \frac{T_{k+1}(z)}{z}
\]
\[+ S_0(T_{n+1}(z)) \left(\frac{\xi}{\gamma + \xi}\right)^{n+1} \frac{T_{n+1}(z)}{z}, \quad (4.16)\]
and by taking the limit as \(n \to \infty\) we obtain
\[
S_0(z) = \sum_{k=0}^{\infty} \left[ \frac{1}{1-T_k(z)} + \gamma S_1(T_k(z)) \right] \frac{T_k(z)}{\xi T_{k+1}(z)} \left(\frac{\xi}{\gamma + \xi}\right)^{k+1} \frac{T_{k+1}(z)}{z}.
\] (4.17)
Observing that
\[
\frac{1}{1 - T_k(z)} = \frac{1 - (1 - q^k)z}{1 - z},
\]
\[
S_1(T_k(z)) = \frac{1 - (1 - q^k)z}{\eta} \left( \frac{1}{1 - z} - \frac{\mu}{\eta(1 - (1 - q^k)z) + \mu(1 - z)} \right),
\]
we have that (4.17) is written in the form
\[
S_0(z) = \frac{1}{\gamma + \xi} \sum_{k=0}^{\infty} \left[ \frac{1}{1 - z} + \frac{\gamma}{\eta} \left( \frac{1}{1 - z} - \frac{\mu(1 - z) + \eta(1 - (1 - q^k)z)}{\eta(1 - (1 - q^k)z) + \mu(1 - z)} \right) \right] \left( \frac{\xi q}{\gamma + \xi} \right)^k
\]
\[
= \frac{1}{(\gamma + \xi)(1 - z)} \sum_{k=0}^{\infty} \left[ \frac{\gamma + \eta}{\eta} - \frac{\gamma \mu}{\eta(\mu + \eta)} \right] \left( \frac{\xi q}{\gamma + \xi} \right)^k
\]
\[
= \frac{1}{(\gamma + \xi(1 - q))(1 - z)} \frac{\gamma + \eta}{\eta(\gamma + \xi)} \frac{\gamma \mu}{\eta(\mu + \eta)} \sum_{k=0}^{\infty} \left( \frac{\eta}{\eta + \mu} \right)^k \left( \frac{1}{1 - z} \right)^k
\]
which can be put in the standard \(q\)-series notation as
\[
S_0(z) = \frac{1}{(\gamma + \xi(1 - q))(1 - z)} \frac{\gamma + \eta}{\eta} \frac{\gamma \mu}{\eta(\mu + \eta)} 2\phi_1 \left( q, \frac{\eta}{\eta + \mu} \frac{z}{1 - z} ; q, \frac{\xi q}{\gamma + \xi} \right),
\]
which is easily seen to converge in the open unit disk (indeed the values of \(z\) where the denominators vanish are \(z = 1\) and \(z = \frac{\mu + \eta}{\eta + \mu} \) that lie outside the open unit disk while the singularities of the \(2\phi_1\) series for \(z\) such that \(- \frac{\eta q}{\eta q + 1 - z} = q^{-m}, m = 0, 1, 2, \ldots \) lie also outside the open unit disk). Using Heine’s transformation formula (3.19) yields
\[
S_0(z) = \frac{1}{(\gamma + \xi(1 - q))(1 - z)} \frac{\gamma + \eta}{\eta} \frac{\gamma \mu}{\eta(\mu + \eta)} \sum_{k=0}^{\infty} \left( \frac{1}{\gamma + \xi(1 - q^k + 1)} \right) \left( \frac{z}{1 - z} \right)^k.
\]
and by expanding the \(q\)-series we obtain
\[
S_0(z) = \frac{1}{(\gamma + \xi(1 - q))(1 - z)} \frac{\gamma + \eta}{\eta} \frac{\gamma \mu}{\eta(\mu + \eta)} \sum_{k=0}^{\infty} \frac{1}{\gamma + \xi(1 - q^k + 1)} \left( \frac{z}{1 - z} \right)^k.
\]
Using (4.9) to expand the term \(\frac{z^k}{(1-z)^{k+1}}\) in the right side of (4.19) in powers of \(z\) yields
\[
S_0(z) = \frac{1}{(\gamma + \xi(1 - q))(1 - z)} \frac{\gamma + \eta}{\eta} \sum_{n=0}^{\infty} z^n
\]
\[
- \frac{\gamma \mu}{\eta(\mu + \eta)} \sum_{k=0}^{\infty} \frac{1}{\gamma + \xi(1 - q^k + 1)} \left( \frac{\eta}{\eta + \mu} \right)^k \sum_{n=0}^{\infty} \left( \frac{n}{k} \right) z^n
\]
\[
= \sum_{n=0}^{\infty} \left[ \frac{1}{(\gamma + \xi(1 - q))} \frac{\gamma + \eta}{\eta} - \frac{\gamma \mu}{\eta(\mu + \eta)} \sum_{k=0}^{n} \frac{1}{\gamma + \xi(1 - q^k + 1)} \left( \frac{\eta}{\eta + \mu} \right)^k \left( \frac{n}{k} \right) \right] z^n.
\]
Now (4.20) implies readily (4.4).

In the two limiting regimes that we have considered in the previous section, where \( \lambda, \mu, \gamma, \eta \) and \( \xi^* \) are kept fixed, we can proceed a bit further and give the results for the conditional expected total sojourn times in the case of no synchronization \((p \to 0^+)\) and full synchronization \((p \to 1^-)\). The results are immediate by taking the appropriate limits in (4.4). More specifically we have the following theorems.

**Theorem 4.2** Consider a system with arrival rate \( \lambda \), service rate \( \mu \), effective abandonment rate per customer \( \xi^* \), setup rate \( \gamma \) and catastrophe rate \( \eta \). In the limiting case of no synchronization \((\xi = \xi^*/(1-q), q \to 1^-)\), the conditional expected total sojourn time of a tagged customer in the system \( S^{(1)}_{(n,i)} \), given that upon his arrival he finds the system in state \((n,i)\), is given as follows

\[
E[S^{(1)}_{(n,0)}] = \frac{1}{(\gamma + \xi^*)} \frac{\gamma + \eta}{\eta} - \frac{\gamma \mu}{\eta(\mu + \eta)} \sum_{k=0}^{n} \frac{1}{\gamma + \xi^*(k+1)} \left( -\frac{\eta}{\eta + \mu} \right)^k \binom{n}{k}, \quad n \geq 0,\tag{4.21}
\]

\[
E[S^{(1)}_{(n,1)}] = \frac{1}{\eta} \left( 1 - \left( \frac{\mu}{\mu + \eta} \right)^{n+1} \right), \quad n \geq 0.\tag{4.22}
\]

**Theorem 4.3** Consider a system with arrival rate \( \lambda \), service rate \( \mu \), effective abandonment rate per customer \( \xi^* \), setup rate \( \gamma \) and catastrophe rate \( \eta \). In the limiting case of full synchronization \((\xi = \xi^*/(1-q), q \to 0^+)\), the conditional expected total sojourn time of a tagged customer in the system \( S^{(2)}_{(n,i)} \), given that upon his arrival he finds the system in state \((n,i)\), is given as follows

\[
E[S^{(2)}_{(n,0)}] = \frac{\gamma + \eta}{(\gamma + \xi^*)\eta} \left( 1 - \frac{\gamma}{\gamma + \eta} \left( \frac{\mu}{\eta + \mu} \right)^{n+1} \right), \quad n \geq 0,\tag{4.23}
\]

\[
E[S^{(2)}_{(n,1)}] = \frac{1}{\eta} \left( 1 - \left( \frac{\mu}{\mu + \eta} \right)^{n+1} \right), \quad n \geq 0.\tag{4.24}
\]

### 5 System busy period

We now study the busy period of the model, i.e. the time from the arrival of a customer at an empty system till the next epoch that the system is empty again. Let \( L_{(n,i)} \), \( i = 0, 1 \) and \( n \geq 0 \) be a generic random variable representing a first passage time to one of the states in \( \{(0,0), (0,1)\} \) starting from \((n,i)\) and denote by \( \psi_{(n,i)}(s) = E[e^{-sL_{(n,i)}}] \) its Laplace Stieltjes transform. The busy period \( L \) is equal to \( L_{(1,0)} \) with probability \( \frac{\pi(0,0)}{\pi(0,0) + \pi(0,1)} \) and equal to \( L_{(1,1)} \) with probability \( \frac{\pi(0,1)}{\pi(0,0) + \pi(0,1)} \). Therefore the Laplace Stieltjes transform of the busy period \( \psi_L(s) \) is given by

\[
\psi_L(s) = \frac{\pi(0,0)}{\pi(0,0) + \pi(0,1)} \psi_{(1,0)} + \frac{\pi(0,1)}{\pi(0,0) + \pi(0,1)} \psi_{(1,1)}.
\]

The probabilities of an empty system \( \pi(0,0) \) and \( \pi(0,1) \) are calculated by setting \( z = 0 \) in equations (3.5) and (3.6), respectively. Moreover, by conditioning on the time of the next event
has two roots.

We define the mixed transforms \( \Phi_i(s, z) \), \( i = 0, 1 \), by

\[
\Phi_i(s, z) = \sum_{n=0}^{\infty} \varphi_{(n, i)}(s) z^n, \quad i = 0, 1, \ s \geq 0, \ |z| < 1.
\]

These mixed transforms \( \Phi_i(s, z) = \sum_{n=0}^{\infty} \varphi_{(n, i)}(s) z^n, \ i = 0, 1 \) do converge for \( s \geq 0 \) and \( |z| < 1 \).

Indeed the LST \( \varphi_{(n, j)}(s) \) are well-defined for \( s \geq 0 \). Moreover, for \( s \geq 0 \) we have that \( \varphi_{(n, j)}(s) = E[e^{-sL(n, j)}] \leq 1 \), hence \( |\Phi_i(s, z)| \leq \sum_{n=0}^{\infty} |\varphi_{(n, j)}(s)||z|^n \leq \sum_{n=0}^{\infty} |z|^n < \infty \).

The mixed transforms \( \Phi_i(s, z) \) carry information for all first passage times distributions from an arbitrary state to one of the states in \( \{(0, 0), (0, 1)\} \) and in particular for the busy period of the system. We use the \( \Phi_i(s, z) \) transforms, \( i = 0, 1 \) to calculate \( \varphi_{(1, j)}(s) \), \( i = 0, 1 \).

Multiplying (5.3) with \((\lambda + \mu + \eta + s)z^n\) and adding for all \( n \geq 1 \) results after some manipulations to

\[
[(\lambda + \mu + \eta + s)z - \lambda - \mu z^2] \Phi_1(s, z) = (\lambda + \mu + \eta + s)z - \lambda - \lambda z \varphi_{(1, 1)}(s) + \eta \frac{z^2}{1 - z}.
\]

We observe that the quadratic polynomial

\[
g(s, z) = (\lambda + \mu + \eta + s)z - \mu z^2 - \lambda
\]

has two roots

\[
z_0(s) = \frac{\lambda + \mu + \eta + s - \sqrt{(\lambda + \mu + \eta + s)^2 - 4\lambda \mu}}{2\mu} \in (0, 1), \quad z_0'(s) = \frac{\lambda + \mu + \eta + s + \sqrt{(\lambda + \mu + \eta + s)^2 - 4\lambda \mu}}{2\mu} \in (1, \infty).
\]

Setting \( z = z_0(s) \) in equation (5.5) (note that \( \Phi_1(s, z_0(s)) \) converges since \( z_0(s) \in (0, 1) \)) we obtain

\[
\varphi_{(1, 1)}(s) = \frac{z_0(s) [\mu + \eta - \mu z_0(s)]}{\lambda (1 - z_0(s))}.
\]

Substituting (5.9) into (5.5) and taking into account (5.6)–(5.8) we obtain

\[
\mu(z - z_0(s))(z_0'(s) - z) \Phi_1(s, z) = (z - z_0(s)) \left[ \mu z_0'(s) + \eta \frac{z}{(1 - z)(1 - z_0(s))} \right]
\]
which gives that
\[
\Phi_1(s, z) = \frac{1 - \frac{\mu z(s)(1 - z_0(s)) - 1}{\mu z(s)(1 - z_0(s))}}{1 - z(1 - \frac{1}{u_0(s)z})} = 1 - \frac{s + \mu(1 - z_0(s))}{\eta + s + \mu(1 - z_0(s))} \frac{z}{1 - \frac{1}{\lambda}z}.
\] (5.11)

Multiplying now (5.2) with \((\lambda + \gamma + \xi + s)z^n\) and adding for all \(n \geq 1\) results after some manipulations to
\[
[(\lambda + \gamma + \xi + s)z - \lambda] \Phi_0(s, z) = (\lambda + s)z - \lambda - \lambda z \psi_{(1,0)}(s) + \gamma z \Phi_1(s, z) + \frac{\xi z}{1 - p z} \Phi_0(s, T(z)),
\] (5.12)
where \(T(z)\) is given by (4.12). We observe that equation (5.12) is of the form
\[
A(s, z) \Phi_0(s, z) = B(z) \Phi_0(s, T(z)) + C(s, z),
\] (5.13)
where
\[
A(s, z) = (\lambda + \gamma + \xi + s)z - \lambda,
\]
\[
B(z) = \frac{\xi z}{1 - p z},
\]
\[
C(s, z) = (\lambda + s)z - \lambda - \lambda z \psi_{(1,0)}(s) + \gamma z \Phi_1(s, z).
\] (5.14)

Therefore, it can be solved by iteration, following the technique that we described in section 4 for the sojourn times. Hereafter, we will suppress the details of the method, as it has been described in detail above. By iterating equation (5.13) we can prove inductively that
\[
A(s, z) \Phi_0(s, z) = \Phi_0(s, T_{n+1}(z)) B(z) \prod_{i=1}^{n} \frac{B(T_i(z))}{A(s, T_i(z))} + B(z) \sum_{k=0}^{n} \frac{C(s, T_k(z))}{B(T_k(z))} \prod_{i=1}^{k} \frac{B(T_i(z))}{A(s, T_i(z))}
\]
\[
= \Phi_0(s, T_{n+1}(z)) B(z) \prod_{i=1}^{n} \frac{B(T_i(z))}{A(s, T_i(z))} + \sum_{k=0}^{n} C(s, T_k(z)) \prod_{i=1}^{k} \frac{B(T_{i-1}(z))}{A(s, T_i(z))}.\] (5.15)

Taking the limit as the number of iterations \(n \to \infty\) yields
\[
[(\lambda + \gamma + \xi + s)z - \lambda] \Phi_0(s, z) = \sum_{k=0}^{\infty} [(\lambda + s)T_k(z) - \lambda + \gamma T_k(z) \Phi_1(s, T_k(z))] \prod_{i=1}^{k} \frac{\xi T_i(z)}{q(\lambda + \gamma + \xi + s)T_i(z) - \lambda}
\]
\[
- \lambda \psi_{(1,0)}(s) \sum_{k=0}^{\infty} T_k(z) \prod_{i=1}^{k} \frac{\xi T_i(z)}{q(\lambda + \gamma + \xi + s)T_i(z) - \lambda}.\] (5.16)

The absolute convergence of the series can be proved straightforward by applying the ratio test. We can now verify that
\[
a_k(s, z) = \prod_{i=1}^{k} \frac{\xi T_i(z)}{q(\lambda + \gamma + \xi + s)T_i(z) - \lambda} = (-1)^k \left( \frac{\xi z}{\lambda(1 - z)} \right)^k q^{(k)}_{(\frac{s}{\lambda_1 + \xi + s}q)^{\frac{1}{1 - z}}; q_k}, \ k \geq 0.
\] (5.17)
Moreover, we observe that (5.11) yields
\[
(\lambda + s)z - \lambda + \gamma z \Phi_1(s, z) = \frac{a(s)z^3 + b(s)z^2 + c(s)z + d}{(1 - z)(1 - \frac{\mu z_0(s)}{\lambda} z)},
\]
where
\[
a(s) = \frac{\mu z_0(s)}{\lambda}(\lambda + s), \\
b(s) = -((\lambda + s)(1 + \frac{\mu z_0(s)}{\lambda}) - \mu z_0(s)) - \gamma \frac{s + \mu(1 - z_0(s))}{\eta + s + \mu(1 - z_0(s))}, \\
c(s) = 2\lambda + s + \mu z_0(s) + \gamma, \\
d = -\lambda.
\]

We aim to factor the numerator in the rational form of \((\lambda + s)z - \lambda + \gamma z \Phi_1(s, z)\) in terms of the form \(1 - \alpha z\) as its denominator. To this end, let \(z_i(s), i = 1, 2, 3\) be the roots of the cubic polynomial \(a(s)z^3 + b(s)z^2 + c(s)z + d\). Then
\[
a(s)z^3 + b(s)z^2 + c(s)z + d = -\lambda(1 - \frac{1}{z_1(s)z})(1 - \frac{1}{z_2(s)z})(1 - \frac{1}{z_3(s)z}).
\]
Let
\[
c_0(s, z) = \left(\frac{\mu z_0(s)}{\lambda} - 1\right)\frac{z}{1 - z}, \quad c_i(s, z) = (\frac{1}{z_i(s)} - 1)\frac{z}{1 - z}, \quad i = 1, 2, 3, \quad c_4(z) = -\frac{z}{1 - z}, \quad (5.18)
\]
We then have that
\[
(\lambda + s)T_k(z) - \lambda + \gamma T_k(z)\Phi_1(s, T_k(z)) = -\lambda \frac{(1 - c_1(s, z)q^k)(1 - c_2(s, z)q^k)(1 - c_3(s, z)q^k)}{(1 - c_0(s, z)q^k)(1 - c_4(z)q^k)}, \quad (5.19)
\]
a rational function of \(q^k\). Then, we can easily see that \((\lambda + s)T_k(z) - \lambda + \gamma T_k(z)\Phi_1(s, T_k(z))\) can be expressed using \(q\)-factorials in the form
\[
(\lambda + s)T_k(z) - \lambda + \gamma T_k(z)\Phi_1(s, T_k(z)) = -\lambda \frac{(1 - c_1(s, z))(1 - c_2(s, z))(1 - c_3(s, z))}{(1 - c_0(s, z))(1 - c_4(z))} b_k(s, z), \quad (5.20)
\]
where \(b_k(s, z)\) are given by
\[
b_k(s, z) = \frac{(c_0(s, z), c_1(s, z)q, c_2(s, z)q, c_3(s, z)q, c_4(z); q)_k}{(c_0(s, z)q, c_1(s, z), c_2(s, z), c_3(s, z), c_4(z)q; q)_k}, \quad k \geq 0. \quad (5.21)
\]
Then equation (5.16) can be written as
\[
[(\lambda + \gamma + \xi + s)z - \lambda]\Phi_0(s, z) = -\lambda \frac{(1 - c_1(s, z))(1 - c_2(s, z))(1 - c_3(s, z))}{(1 - c_0(s, z))(1 - c_4(z))} \sum_{k=0}^\infty a_k(s, z)b_k(s, z)
\]
\[
-\lambda \varphi_{(1, 0)}(s) \sum_{k=0}^\infty a_k(s, z)T_k(z). \quad (5.22)
\]
We set \( z = \frac{\lambda}{\lambda + \gamma + \xi + s} \) in (5.22) and define \( \hat{c}_i(s) = c_i(s, \frac{\lambda}{\lambda + \gamma + \xi + s}) \), \( i = 0, 1, 2, 3 \) and \( \hat{c}_4(s) = c_4(s, \frac{\lambda}{\lambda + \gamma + \xi + s}) \). Then, after reducing the sums in the canonical form of the \( q \)-series, (5.22) yields

\[
\varphi_{(1,0)}(s) = -\frac{(\gamma + \xi + s + \lambda)(1 - \hat{c}_1(s))(1 - \hat{c}_2(s))(1 - \hat{c}_3(s))}{\lambda(1 - \hat{c}_0(s))(1 - \hat{c}_4(s))} \phi_1(\hat{c}_4(s); c_4(s)q; q, \frac{\xi q}{\gamma + \xi + s}) \\
\times 5\phi_5 \left( \frac{\hat{c}_0(s)q, \hat{c}_1(s)q, \hat{c}_2(s)q, \hat{c}_3(s)q, \hat{c}_4(s)q}{\hat{c}_0(s)q, \hat{c}_1(s)q, \hat{c}_2(s)q, \hat{c}_3(s)q}; q, \frac{\xi}{\gamma + \xi + s} \right).
\]

(5.23)

We can now use (5.9) and (5.23) to obtain \( \varphi_L(s) \). Although the symbolic inversion of \( \varphi_L(s) \) is not possible, one can perform numerical inversion to obtain the distribution of the busy period \( L \) or other associated measures (see e.g. Abate et al. (2000)).

References


