Set membership localization of mobile robots via angle measurements

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Abstract—This paper addresses the localization problem for a mobile robot navigating in an unstructured outdoor environment. A new technique is introduced, for computing an estimate of the position of the robot and the related uncertainty region, in the presence of visual angle measurements affected by bounded errors. The proposed set membership estimation procedure exploits the structure of the static set estimator, to solve recursively the dynamic localization problem.

Keywords—Mobile robotics, localization, landmarks, angle measurements, set membership.

I. INTRODUCTION

One of the important issues for autonomous navigation in an unstructured outdoor environment is the localization problem (see [1], [2], [3] and references therein). Environmental conditions, sensor limitations and map inaccuracies require that the navigating robot periodically computes its coordinates in the reference system by exploiting geometric properties and measurements based on landmarks distributed in the environment. In the present paper, the relative bearing approach to the map localization problem for a mobile robot is considered. In this setting, it is assumed that the robot moves on a terrain map and is able to perform angular measurements with respect to known landmarks (see [4], [5], [6], [7], [8]). Three main sources of uncertainty characterize the localization problem in this context: a) measurement errors of the visual angles from the robot standpoint; b) landmark misidentification; c) environment map errors. The fact that the dominant component of uncertainty is of the first type, motivated the introduction of estimation approaches explicitly accounting for angle measurement errors ([9], [5], [7]). Batch or recursive techniques have been used, according to the static or dynamic setting of the position estimation problem. However, most of the results available in the literature refer to the static problem, while relatively little attention has been devoted to the dynamic problem [10], [11].

Different approaches have been adopted in the literature to tackle the localization problem. Most popular techniques rely on statistical assumptions about the error and adopt standard estimation tools to compute the robot position. For example, the triangulation problem is solved via linear least-squares algorithms in [7] and [12], in two different contexts. Extended Kalman filtering is generally adopted when facing the dynamic localization problem (see e.g. [10], [11], [13]). Other localization techniques based on different probabilistic paradigms, like Bayesian filtering [14] and maximum-likelihood estimation [15], have been recently introduced.

A quite different way to tackle the problem is to assume that all the uncertainty sources generate errors that are unknown but bounded. This has given rise to localization methods based on geometric properties of the measurement model [5], or on the computation of feasibility regions where the robot is known to lie, the so-called set membership approach [16], [6]. Properties of statistical and set-theoretic techniques for sensor data fusion have been analyzed in [17].

The contribution of the present paper is twofold. First, it introduces a new estimation technique which allows to derive an estimate of the position of the robot and the related uncertainty region, in the presence of angle measurements affected by bounded errors. No statistical assumption is made on the errors; the only assumption is that they are bounded in norm by some known quantity. The second objective is to provide a procedure which exploits the structure of the static set estimator to solve recursively the dynamic localization problem. The basic reference theory for the technical development of the paper relies on set theoretic concepts which have been recently investigated in the set membership estimation area of research (see e.g., [18], [19], [20]). Although most of the general theory developed in these references refers to linear estimation problems, the specific nonlinear structure of the admissible position set allows for an efficient solution of the uncertainty region recursive updating problem. The proposed recursive approximation algorithms, based on two different classes of approximating regions, exhibit a low computational complexity which is suitable for online implementation. Moreover, when the bounded-error assumption is satisfied, a guaranteed outer approximation is provided, i.e. the approximating region always contains the admissible position set.

The paper is structured as follows. Section II introduces the localization problem and the set theoretic approach. Section III provides the main contribution on recursive approximation of the position uncertainty set. A discussion on the computational complexity of the proposed technique and of relevant extensions is provided in Section IV. Section V reports the results of extensive numerical simulation experiments conducted in order to assess the validity of the proposed procedures. Some concluding remarks are given in Section VI.

Notation

Let us introduce the notation adopted throughout the paper. Prime denotes transpose; $0_n$ and $I_n$ are the $n \times n$
zero and identity matrices. For a vector \( v \in \mathbb{R}^n \), \( \text{Diag}\{v\} \) denotes the \( n \times n \) matrix with \( v_i, i = 1, \ldots, n \), on the diagonal and zeros elsewhere. The \( \ell_\infty \) and \( \ell_1 \) vector norms are defined in the usual way: \( \|v\|_\infty = \max_{i=1, \ldots, n} |v_i| \), \( \|v\|_1 = \sum_{i=1}^n |v_i| \). The unit ball in the \( \ell_\infty \) norm is denoted by \( B_\infty \). The weighted \( \ell_\infty \) norm of a vector \( v \in \mathbb{R}^n \) is defined as \( \|v\|_{\ell_\infty} = \max_{i=1, \ldots, n} |w_i v_i| \), where \( w_i, i = 1, \ldots, n \) are positive scalars. The Euclidean norm is denoted by \( \|v\| \). Projection matrices from \( \mathbb{R}^m \) to \( \mathbb{R}^2 \) are defined as \( \Pi = [I_2 \ 0_2] \) and \( \bar{\Pi} = [0_2 \ I_2] \).

Operations between sets are defined in the standard way. The vector sum of two sets \( Z_1 \) and \( Z_2 \) is given by \( Z_1 + Z_2 \) if \( \{s + s' : s \in Z_1, s' \in Z_2\} \), while \( MZ \) if \( \{q : q = Ms, s \in Z\} \) is the image of a set \( Z \) according to the linear transformation \( M \). The cartesian product of two sets is \( \{1, \ldots, n\} \), and let \( T \) be the coordinates of the vehicle at time \( t \).

A parpolygon is the vector sum of \( n \) displacement measurements provided by odometers, \( w(k) \) is the errors of such sensors and velocities are not included in the state vector. In this case, \( \xi(k) = p(k) \in \mathbb{R}^2 \) and model (7) boils down to (8).

The random walk model (9) provides a good approximation of the vehicle dynamics as long as \( T \) is sufficiently small. A simpler model can be considered if the inputs \( u(k) \) represent the \( x \) and \( y \) displacement measurements provided by odometers, \( w(k) \) is the errors of such sensors and velocities are not included in the state vector. In this case, \( \xi(k+1) = \xi(k) + u(k) + w(k) \).

Due to the presence of uncertainties in the model, the navigating vehicle must collect measurements from the environment to localize itself and to dynamically update its estimated position. It is assumed that a map of the environment is available and \( n \) landmarks \( l_i, i = 1, \ldots, n \) have been identified. Landmark coordinates are known.

II. SET MEMBERSHIP LOCALIZATION PROBLEM

Let us consider a vehicle navigating in a 2D environment, and let \( p(t) = [x(t) \ y(t)]' \) be the coordinates of the vehicle location at time \( t \). The vehicle dynamics is described by the linear time-varying model

\[
\dot{\xi}(t) = A_c(t)\xi(t) + B_c(t)u(t) + G_c(t)w(t)
\]

where \( \xi(t) = [x(t) \ y(t) \ \dot{x}(t) \ y(t)]' \) is the state vector, containing position and velocity of the vehicle, \( u(t) \in \mathbb{R}^2 \) is a known input arising from external driving commands, and the noise term \( w(t) \in \mathbb{R}^2 \) accounts for unknown forces acting on the vehicle, undesired effects (like wheel slippage or small terrain unevenness) and other unmodeled dynamics. Matrices \( A_c(t), B_c(t), G_c(t) \) in (6) may originate from the linearization of a nonlinear model with respect to the current vehicle state (position and velocity). Loosely speaking, these matrices reflect the a priori knowledge on the vehicle dynamics (see e.g. [21]). Since measurements of the environment are collected at finite time instants, it is customary to consider the discretized model

\[
\xi(k+1) = A(k)\xi(k) + B(k)u(k) + G(k)w(k)
\]
where the error term \( v_{ij} \) accounts for measurement noise and other uncertainties (for example, errors in landmark positions). The measurement process can be repeated at different time instants. Exploiting the projection matrix \( \Pi \), the vehicle position at time \( k \) is given by \( p(k) = \Pi \xi(k) \), and hence the measurement equation can be written as

\[
\theta_{ij}(k) = \eta_{ij}(\Pi \xi(k)) + v_{ij}(k) \quad (11)
\]

for \( i, j = 1, \ldots, n, i < j \), and \( k = 0, 1, \ldots \). Notice that at each time \( k \), there are in principle \( m = n(n-1)/2 \) possible visual angle measurements, one for each pair of landmarks. Within the framework outlined above, we introduce the general localization problem.

**Localization Problem:** Let \( \xi(0) \) be an estimate of the initial position and velocity of the vehicle. Given the dynamic model (7) and the measurement equation (11), construct an estimator of the vehicle position \( \hat{p}(k) = \Pi \hat{\xi}(k) \) at each time instant \( k = 1, 2, \ldots \).

This problem can be tackled in many different ways, depending on the hypotheses on the unknown disturbances \( w(k) \) and \( v_{ij}(k) \) in eqns. (7) and (11). Estimates of the robot position are commonly computed in the literature through the extended Kalman filter [10], [11], [13]. In this paper, a different approach is presented, based on set membership hypotheses on the uncertainties. In particular, it is assumed that the disturbances are unknown-but-bounded, i.e.

\[
|w_i(k)| \leq \varepsilon_i^w(k) \quad i = 1, 2 \quad (12)
\]

\[
|v_{ij}(k)| \leq \varepsilon_{ij}^v(k) \quad i, j = 1, \ldots, n, i < j \quad (13)
\]

where \( \varepsilon_i^w(k) \), \( \varepsilon_{ij}^v(k) \) are known scalars. Eqns. (12)-(13) can be written in compact form as \( \|w(k)\|_{\infty}^w \leq 1 \), \( \|v(k)\|_{\infty}^v \leq 1 \) where \( w = [v_{12} \ v_{13} \ ... \ v_{ij} \ ...] \), \( v^w = [\varepsilon_{12}^w \ \varepsilon_{23}^w \ ... \ \varepsilon_{ij}^w \ ...] \), \( v^v = [\varepsilon_{12}^v \ ... \ \varepsilon_{ij}^v \ ...] \). The use of the weighted \( \ell_{\infty} \) norm gives the possibility of incorporating information on each visual angle measurement in the error bound. Notice that the disturbance bounds are time-varying, to explicitly account for known changes in the environment conditions or in the vehicle dynamics.

Assumption (13) allows one to define a bounded set where the navigator is allowed to lie, for each visual angle measurement \( \theta_{ij} \). This set is given by

\[
C_{ij} = C(\theta_{ij}, \varepsilon_{ij}^v) = \{ p \in \mathbb{R}^2 : \theta_{ij} - \varepsilon_{ij}^v \leq \eta_{ij}(p) \leq \theta_{ij} + \varepsilon_{ij}^v \}, \quad (14)
\]

where the dependence on \( k \) has been omitted for notational convenience. From a geometrical point of view, \( C_{ij} \) is a “thickened ring”, i.e. the region between two circular arcs with the same extreme points, the landmarks \( l_i, l_j \) [9], [5] (an example is reported in Fig. 1).

Therefore, at each time \( k \), the visual angle measurements constrain the navigator to the intersection of \( m \) thickened rings, which defines the measurement set

\[
\mathcal{M}(k) = \bigcap_{v_{ij}=1}^{n} C_{ij}(k). \quad (15)
\]

This leads to the formulation of a localization problem in which the dynamic estimate of the position can be expressed in terms of bounded sets.

**Set Membership Localization Problem:** Let \( \Xi(0) \subset \mathbb{R}^3 \) be a set containing the initial position and velocity vector \( \xi(0) \). Given the dynamic model (7) and the measurement equation (11), find a set \( \mathcal{L}(k) \) containing the vehicle position at time \( k \), i.e. \( p(k) \in \mathcal{L}(k), \ k = 1, 2, \ldots \)

The solution of the above problem can be obtained from the following recursion, which follows directly from equations (7) and (11)

\[
\Xi(0|0) = \Xi(0), 
\Xi(k|1) = A(k-1)\Xi(k-1|k-1) + B(k-1)u(k-1)
+ G(k-1)\text{Diag}(\varepsilon^w(k-1))B_\infty, 
\Xi(k|k) = \Xi(k|k-1) \cap \{ \{ \xi : \Pi \xi \in \mathcal{M}(k) \} \}, 
\mathcal{L}(k) = \Pi \Xi(k|k). \quad (16, 17, 18, 19)
\]

The sets \( \Xi(k|k-1) \) and \( \Xi(k|k) \) are the feasible state sets, containing all the positions and velocities that are compatible with the dynamics of the vehicle and the available measurements, up to time \( k - 1 \) and \( k \) respectively. Similarly, \( \mathcal{L}(k) \) is the feasible position set, that includes all possible positions of the navigator at time \( k \) (regardless of the velocity). Fig. 2 reports an example of sets \( \Xi(k|k-1) \) and \( \Xi(k|k) \), for a 2-dimensional model \( \xi(k) \in \mathbb{R}^2 \) (\( \Xi(k-1|k-1) \) is assumed to be a box). Clearly, in this case \( \mathcal{L}(k) \) coincides with \( \Xi(k|k) \).

Unfortunately, exact computation of the sets \( \Xi(k|k-1) \) and \( \Xi(k|k) \) in (17)-(18) is a prohibitive task, as the set \( \mathcal{M}(k) \) is nonconvex and bounded by nonlinear curves. Since we are looking for regions that are guaranteed to contain the vehicle position \( p(k) \), outer approximations of the sets \( \mathcal{L}(k) \) are sought. In the next section, a recursive approximation strategy exploiting the particular structure of the measurement set \( \mathcal{M}(k) \) will be presented.

**III. Recursive approximations of the feasible position set**

In order to reduce the computational burden of the localization algorithm and obtain tractable analytical expressions for the uncertainty regions, simple structure sets containing the exact feasible sets will be considered. In particular, if a simple set \( \mathcal{R}(k|k-1) \) containing \( \Xi(k|k-1) \)
The following proposition states that the set $R_{II}(k|k)$ is the desired outer approximation of the feasible position set.

**Proposition 1:** Let $R_{II}(k|k)$ satisfy the recursion (20)-(23). Then, $p(k) \in \mathcal{L}(k) \subseteq R_{II}(k|k)$ for all $k = 1, 2, \ldots$. 

**Proof.** See Appendix.

Two recursive algorithms for computing outer approximations of the set $\mathcal{L}(k)$, based on the recursion (20)-(23), will be described in the following. They adopt two different classes of sets as approximating regions $\mathcal{R}$: boxes and parallelographeopes (see (1)-(2)).

### A. Approximation through boxes

In this subsection, the approximating sets $\mathcal{R}$ and $R_{II}$ will be boxes in $\mathbb{R}^4$ and $\mathbb{R}^2$ respectively. First, notice from (1)-(2) that premultiplication of a box by a square matrix gives a parallelopotope. Moreover, for a box $B \subset \mathbb{R}^4$, $\Pi B$ is a box in $\mathbb{R}^2$. Hence, according to the recursion (20)-(23), one must solve the following set approximation problems:

**B1** compute the minimum volume box containing the vector sum of two parallelopohtopes (see eqn. (21));

**B2** compute the minimum area box, containing the intersection of a box in $\mathbb{R}^2$ with the measurement set $\mathcal{M}(k)$ (eqn. (22));

**B3** compute the minimum volume box containing the intersection of a box with the cartesian product of two boxes (eqn. (23)).

According to the notation introduced above, $\overline{B}(\mathcal{Z})$ denotes the minimum volume box containing the set $\mathcal{Z}$. Optimal solutions of Problems B1 and B3 are provided by the following propositions.

**Proposition 2:** Let $R(k-1|k-1) = B(b, c) \subset \mathbb{R}^4$. Then

$$\overline{B} \left( A(k-1)B(b, c) + B(k-1)u(k-1) + G(k-1)\text{Diag} \{\varepsilon^w(k-1)\}B_{\infty} \right) = B(\overline{b}, \overline{c})$$

where

$$\overline{c} = A(k-1)c + B(k-1)u(k-1), \quad \overline{b}_i = \| [A(k-1)\text{Diag}b]^T \cdot G(k-1)\text{Diag} \{\varepsilon^w(k-1)\} \| e_i\|_1,$$

for $i = 1, \ldots, 4$, and $e_i$ denotes the ith column of identity matrix $I_4$.

**Proof.** See Appendix.

A 2-dimensional example of Proposition 2 is depicted in Fig. 3.

**Proposition 3:** Let $R(k|k-1) = B(b, c) \subset \mathbb{R}^4$ and $R_{II}(k|k) = B(\tilde{b}, \tilde{c}) \subset \mathbb{R}^2$. Then

$$\overline{B} \left( \left\{ R_{II}(k|k) \times \Pi R(k|k-1) \right\} \cap R(k|k-1) \right) = R_{II}(k|k) \times \Pi R(k|k-1) = B(\overline{b}, \overline{c})$$

where

$$\overline{c} = \left[ \begin{array}{c} \hat{c} \\ \Pi c \end{array} \right], \quad \overline{b} = \left[ \begin{array}{c} \hat{b} \\ \Pi b \end{array} \right].$$

**Proof.** See Appendix.

Proposition 3 just states that when boxes are used as approximating region, equation (23) boils down to the cartesian product of two 2D boxes in $\mathbb{R}^2$. 

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Fig. 2. (a) Time update, eq. (17); (b) Measurement update, eq. (18).
Fig. 3. Box approximation of time update according to Proposition 2 (Problem B1).

Now, let us turn to problem B2. This is the most involved task, since the set \( \mathcal{M}(k) \) is the intersection of \( m \) thickened rings \( \mathcal{C}_{ij} \). In order to obtain a computationally tractable algorithm, a suboptimal solution based on recursive minimum area approximations is pursued: for each pair \( i, j \), the minimum area box containing the intersection of \( \mathcal{C}_{ij} \) with the current approximating box \( \mathcal{B} \) is computed. Hence, problem B2 boils down to computing \( \mathcal{B}(\mathcal{B} \cap \mathcal{C}_{ij}) \) for each pair of landmarks \( l_i, l_j \). Let us denote by \( \delta \mathcal{C}_{ij} \) the boundary of \( \mathcal{C}_{ij} \) in (14). Notice that \( \delta \mathcal{C}_{ij} = \delta \mathcal{C}_{ij}^+ \cup \delta \mathcal{C}_{ij}^- \), where \( \delta \mathcal{C}_{ij}^+ = \mathcal{C}(\theta_{ij} + \varepsilon_{ij}^+, 0) \) and \( \delta \mathcal{C}_{ij}^- = \mathcal{C}(\theta_{ij} - \varepsilon_{ij}^-, 0) \). Moreover, let \( \mathcal{V}(\mathcal{B}) \) be the set of vertices of the box \( \mathcal{B} \). The following result allows one to compute the minimum area box containing \( \mathcal{B} \cap \mathcal{C}_{ij} \).

**Proposition 4:** Let \( \mathcal{C}_{ij} \) be given by (14), and \( \mathcal{B} \) be an assigned box. Then

\[
\mathcal{B}(\mathcal{B} \cap \mathcal{C}_{ij}) = \mathcal{B}(\mathcal{E})
\]

where \( \mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \), and

\[
\begin{align*}
\mathcal{E}_1 &= \mathcal{V}(\mathcal{B}) \cap \mathcal{C}_{ij}, \\
\mathcal{E}_2 &= \delta \mathcal{B} \cap \delta \mathcal{C}_{ij}, \\
\mathcal{E}_3 &= \{l_i, l_j\} \cap \mathcal{B}, \\
\mathcal{E}_4 &= \delta(\mathcal{B}(\delta \mathcal{C}_{ij}^-)) \cap \delta \mathcal{C}_{ij} \cap \mathcal{B}.
\end{align*}
\]

**Proof.** See Appendix.

We remark that the sets \( \mathcal{E}_i, i = 1, \ldots, 4 \) in Proposition 4 contain a finite number of points, which are very easy to compute. For example, the points in \( \mathcal{E}_2 \) are given by the intersection of a circle and a segment, while \( \mathcal{E}_4 \) requires the computation of the tangency point between a line and a circle. An example in which all the four sets \( \mathcal{E}_i \) are not empty is reported in Fig. 4.

The important implication of Proposition 4 is that the finite set of points \( \mathcal{E} \) is the only thing one needs to know in order to compute the minimum area box containing \( \mathcal{B} \cap \mathcal{C}_{ij} \). Therefore, if \( \Pi \mathcal{R}(k|k-1) = \mathcal{B}(b, c) \subset \mathbb{R}^2 \), \( \mathcal{C}_{ij}(k) \) is given by (14) and \( \mathcal{E} \) is defined as in Proposition 4, then

\[
\mathcal{B}(\Pi \mathcal{R}(k|k-1) \cap \mathcal{C}_{ij}(k)) = \mathcal{B}(\tilde{b}, \tilde{\tau})
\]

where

\[
\begin{align*}
\tilde{\tau}_1 &= \frac{x + \bar{x}}{2}, & \tilde{\tau}_2 &= \frac{y + \bar{y}}{2} \\
\tilde{b}_1 &= \frac{x - \bar{x}}{2}, & \tilde{b}_2 &= \frac{y - \bar{y}}{2}
\end{align*}
\]

and \( \bar{x} = \max_{p \in \mathcal{E}} x \), \( \bar{x} = \min_{p \in \mathcal{E}} x \), \( \bar{y} = \max_{p \in \mathcal{E}} y \), \( \bar{y} = \min_{p \in \mathcal{E}} y \).

According to the recursive strategy proposed for the suboptimal solution of problem B2, the set \( \mathcal{R}_\Pi(k|k) \) can be computed in the following way:

- set \( \mathcal{B} = \Pi \mathcal{R}(k|k-1) \);
- for \( i, j = 1, \ldots, n, i < j \), set \( \mathcal{B} = \mathcal{B}(\mathcal{B} \cap \mathcal{C}_{ij}) \) using (28)-(29);
- set \( \mathcal{R}_\Pi(k|k) = \mathcal{B} \).

**B. Approximation through parallelotopes**

A more accurate and still viable alternative to box approximations, is to choose parallelotopes in \( \mathbb{R}^4 \) and \( \mathbb{R}^2 \) as approximating sets \( \mathcal{R} \) and \( \mathcal{R}_\Pi \). The main advantage of using parallelotopes instead of boxes is that the orientation of the approximating region can be changed: this allows one to reduce the size of the uncertainty set, especially when the true feasible sets stretches along some direction in the state space (or in the position plane). The overall strategy is the same as in subsection III-A. The criterion for the selection of the approximating parallelotopes is still minimum volume (or area).

From (2) it is easy to see that premultiplication of a parallelotope by a square matrix gives a parallelotope. Moreover, for a parallelotope \( \mathcal{P} \subset \mathbb{R}^4 \), \( \Pi \mathcal{P} \) is a 4-segment parapolygon in \( \mathbb{R}^2 \). Hence, the approximation problems arising from recursion (20)-(23) with parallelotopes as sets \( \mathcal{R} \), are as follows:

- **P1** compute the minimum volume parallelotope containing the vector sum of two parallelotopes (see eqn. (21));
- **P2** compute the minimum area parallelotope, containing the intersection of a parapolygon in \( \mathbb{R}^2 \) with \( \mathcal{M}(k) \) (eqn. (22));
- **P3** compute the minimum volume parallelotope containing the intersection of a parallelotope with the cartesian product of a parallelotope and a parapolygon (eqn. (23)).

Approximate solutions of problems P1 and P3 can be obtained by exploiting the techniques proposed in [22], [23].
In particular, the following results are needed.

**Theorem 1**: Consider the parpolygon

\[ T = c + \sum_{i=1}^{n+1} T(t_i, 0) , \quad t_i \in \mathbb{R}^n , \]

and let the matrix \([t_1, \ldots, t_n] \) be nonsingular, with \([p'_1, \ldots, p'_n] = [t_1, \ldots, t_n]^{-1} \). Then the parallelotope of minimum volume outbounding \( T \) is given by

\[ \mathcal{P}(T) = c + \sum_{i \neq i^*}^{n+1} T(t_i, 0_n) = \mathcal{P}(T^*, c) \]

where \( T^* \) is obtained by replacing the \( i^* \)-th column of \([t_1, \ldots, t_n] \) by \( t_{n+1} \), and

\[ i^* = \arg \max_{1 \leq i \leq n+1} |p_{i}t_{n+1}| \quad \text{with} \quad p_{n+1}t_{n+1} \neq 1 \]

\[ t_i = r_it_i , \quad 1 \leq i \leq n+1 , \quad i \neq i^* \]

\[ r_i = 1 + (|p_{i}t_{n+1}|/|p_{i}t_{n+1}|) , \quad 1 \leq i \leq n+1 , \quad i \neq i^*. \]

**Theorem 2**: Consider the convex polytope

\[ X = \bigcap_{i=1}^{n+1} S(p_i, s_i) , \]

and let the matrix \([p'_1, \ldots, p'_n] \) be nonsingular, with \([t_1, \ldots, t_n] = ([p'_1, \ldots, p'_n])^{-1} \). Then, the parallelotope of minimum volume outbounding \( X \) is given by

\[ \mathcal{P}(X) = \bigcap_{i \neq i^*}^{n+1} S(p_i, s_i) = \mathcal{P}(P^*, c^*) \]

where \( P^* = (P^*)^{-1} \), with \( P^* \) obtained by replacing the \( i^* \)-th row of \([p'_1, \ldots, p'_n] \) by \( p_{n+1} , \quad c^* = T^*x \), and

\[ i^* = \arg \max_{1 \leq i \leq n+1} |p_{n+1}t_i| \quad \text{with} \quad p_{n+1}t_i \neq 1 \]

\[ \mathcal{P}_i = \frac{2}{r_i^+ - r_i^-} p_i \]

\[ \mathcal{S}_i = \frac{2}{r_i^+ - r_i^-} \left( s_i + \frac{r_i^+ + r_i^-}{2} \text{sign}(p_{n+1}t_i) \right) \]

\[ r_i^+ = \begin{cases} \min(-1 + \frac{1 - s_{n+1}}{|p_{n+1}t_i|}, 1) & \text{if} \quad p_{n+1}t_i \neq 0 \\ 1 & \text{if} \quad p_{n+1}t_i = 0 \end{cases} \]

\[ r_i^- = \begin{cases} \max(1 - \frac{1 + s_{n+1}}{|p_{n+1}t_i|}, -1) & \text{if} \quad p_{n+1}t_i \neq 0 \\ -1 & \text{if} \quad p_{n+1}t_i = 0 \end{cases} \]

for \( i = 1, \ldots, n \)

\[ \epsilon_{n+1}^+ = \min(t_{n+1}^+, 1) \]

\[ \epsilon_{n+1}^- = \max(1, -1) \]

\[ c_{n+1}^+ = (p_{n+1}x - s_{n+1}) + \sum_{i=1}^{n} |p_{n+1}t_i| \]

\[ c_{n+1}^- = (p_{n+1}x - s_{n+1}) - \sum_{i=1}^{n} |p_{n+1}t_i| . \]

Examples of the approximations provided by Theorems 1 and 2 are presented in Fig. 5, for the case \( n = 2 \).

Now, we show how these results can be exploited to compute recursively approximate solutions of problems P1 and P3. Let us consider problem P1 and assume \( R(k-1)[k-1] = \mathcal{P}(T, c) \in \mathbb{R}^4 \). According to (5), equation (21) can be rewritten as

\[ R(k)[k-1] = \mathcal{P}(T, c) \]

where

\[ T_8 = T \begin{bmatrix} A(k-1)T & G(k-1) \text{Diag}\{\varepsilon(w(k-1))\} \end{bmatrix} , \]

\[ A(k-1)c + B(k-1)u(k-1) \]

and as usual \( \mathcal{P}(Z) \) denotes the minimum volume parallelotope containing \( Z \). A suboptimal solution of problem (30) can be computed by recursively applying Theorem 1 to \( T_8 \). The idea is as follows:

- let \( \bar{c} = A(k-1)c + B(k-1)u(k-1) \) and set

\[ \bar{T}^{(0)} = A(k-1)T , \quad [t_1 t_2 t_3 t_4] = G(k-1)\text{Diag}\{\varepsilon(w(k-1))\} ; \]

- for \( i = 1, \ldots, 4 \), compute

\[ \mathcal{P}(\bar{T}^{(i)}, \bar{c}) = \mathcal{P}(T \begin{bmatrix} [\bar{T}^{(i-1)}] & t_i \end{bmatrix} , \bar{c}) \]

using Theorem 1.

- set \( R(k)[k-1] = \mathcal{P}(\bar{T}^{(4)}, \bar{c}) \).

Clearly, the above construction guarantees that

\[ R(k)[k-1] \supseteq A(k-1)R(k-1)[k-1] + B(k-1)u(k-1) + G(k-1)\text{Diag}\{\varepsilon(w(k-1))\}B_{\infty} . \]
Turning to problem P3, let us assume \( R(k|k-1) = P(T,c) \subset \mathbb{R}^2 \) and \( R_{\Pi}(k|k) = P(\hat{T},\hat{c}) \subset \mathbb{R}^2 \). Then, equation (23) boils down to

\[
R(k|k) = \overline{P}(T_6 \cap P(T,c)) \tag{31}
\]

where

\[
T_6 = T \left( \left[ \begin{array}{ccc} \hat{T} & 0 & 0 \\ 0 & \Pi_T & 0 \\ 0 & 0 & \Pi_c \end{array} \right] \right).
\]

A suboptimal solution of problem (31) can be computed in two steps:

1. Apply Theorem 1 to \( T_6 \), as it has been done above for \( T_8 \) in problem P1, thus obtaining \( P(\hat{T},\hat{c}) \supset T_6 \).

2. Recursively apply Theorem 2 to the intersection between \( P(\hat{T},\hat{c}) \) and \( P(T,c) \). This can be done in the following way:

- set \( T^{(0)} = \hat{T}, \hat{c}^{(0)} = \hat{c} \), and \([p'_1 \ldots p'_4]' = T^{-1}, s = T^{-1}c;\)
- for \( i = 1, \ldots, 4 \), use Theorem 2 to compute

\[
P(\hat{T}^{(i)}, \hat{c}^{(i)}) = \overline{P}(P(\hat{T}^{(i-1)}, \hat{c}^{(i-1)}) \cap S(p_i, s_i))
\]

- let \( R(k|k) = P(\hat{T}^{(4)}, \hat{c}^{(4)}) \)

Once again, the above procedure provides a guaranteed approximation of the feasible set, i.e.

\[
R(k|k) = P(\hat{T}^{(4)}, \hat{c}^{(4)}) \supset \{ R_{\Pi}(k|k) \times \Pi_R(k|k-1) \} \cap R(k|k-1).
\]

A suboptimal solution of Problem P2 can be obtained adopting a recursive strategy similar to that proposed for boxes in Subsection III-A. Recall from (22) that one has to find a parallelotope containing the intersection of \( M(k) \) with the parpolygon \( \Pi R(k|k-1) \), where \( R(k|k-1) \) is given by (30). One can first exploit the same technique adopted to solve problem P1 (based on Theorem 1), to find a parallelotope containing \( \Pi R(k|k-1) \). Then, the main idea is that for each pair \( i, j \), the minimum area parallelotope containing the intersection of the thickened ring \( C_{ij} \) with the current approximating parallelotope \( P \) must be computed. Because parallelotopes have more degrees of freedom than boxes, the intersection with \( C_{ij} \) can not only shrink the approximating parallelotope, but also change its shape (i.e., the orientation of the strips). We consider shrinking and reshaping separately.

First, let us denote by \( \overline{P}_T(Z) \) the minimum parallelotope of fixed shape \( T \) containing the set \( Z \), i.e. \( \overline{P}_T(Z) = P(TD^*, c^*) \) where

\[
\{D^*, c^*\} = \arg \min \begin{cases} \det(TD) \\ \text{s.t.} \\ D \text{ diagonal, } c \in \mathbb{R}^2 \\ P(T,c) \supset Z \end{cases} \tag{32}
\]

Then, we have the following result, which is a straightforward extension of Proposition 4.

**Proposition 5:** Let \( C_{ij} \) be given by (14), and \( P = P(T,c) \) be an assigned parallelotope. Then

\[
\overline{P}_T(P \cap C_{ij}) = \overline{P}(F)
\]

where \( F = F_1 \cup F_2 \cup F_3 \cup F_4 \), and

\[
F_1 = \delta P \cap \delta C_{ij},
\]

\[
F_2 = \delta P \cap \delta C_{ij},
\]

\[
F_3 = \{i, j\} \cap P,
\]

\[
F_4 = \delta(\overline{P}_T(\delta C_{ij})) \cap \delta C_{ij} \cap P.
\]

As for Proposition 4, the sets \( F_i, i = 1, \ldots, 4 \), contain a finite number of points that are very easy to compute (see Fig. 6 below). Using Proposition 5, it is possible to give an explicit characterization of the set \( \overline{P}_T(P \cap C_{ij}) \). Let a parallelotope \( P = P(T,c) \subset \mathbb{R}^2 \) and a thickened ring \( C_{ij} \) be given, and let \([p'_1 p'_2] = T^{-1}\). Moreover, let \( F \) be defined as in Proposition 5.

\[
\overline{P}_T(P \cap C_{ij}) = \bigcup_{i=1}^{2} S(p_i, s_i) \tag{37}
\]

where

\[
p_i = \frac{2}{\gamma_i} p_i, \quad s_i = \frac{\gamma_i^+ + \gamma_i^-}{\gamma_i}, \quad \gamma_i^+ = \max_{q \in F} p_q q, \quad \gamma_i^- = \max_{q \in F} p_q q.
\]

Notice that the optimal solution \( D^*, c^* \) of problem (32) can be easily obtained from \( p_i, s_i, i = 1, 2 \) in (38), using parallelotope definitions (2)-(3) and equations (4).

Now, let \( \overline{P}_T \) denote the tightened parallelotope \( \overline{P}_T(P \cap C_{ij}) \) and let us try to further reduce the volume of \( \overline{P}_T \) by changing its shape, i.e. by substituting new strips to the \( S(p_i, s_i), i = 1, 2 \), in (37)-(38). Let us consider the set

\[
F_s = F_3 \cup \{ \delta C_{ij} \cap \overline{P}_T \}
\]

Recall that \( F_3 \) is the set of landmarks lying inside the initial parallelotope \( P \) (and hence also inside \( \overline{P}_T \)), while \( \delta C_{ij}^+ \) is the inner circle bounding the thickened ring \( C_{ij} \). It is clear that \( F_s \) is made of a finite set of points; more precisely, it can contain from zero up to eight points (the maximum number of intersections between a circle and a parallelotope in \( \mathbb{R}^2 \)). However, the most common situations are the following: (i) \( F_s \) is empty; (ii) \( F_s \) contains two points. When \( F_s \) is empty it is easy to check that reshaping cannot reduce the size of the approximating parallelotope, i.e.

\[
\overline{P}_T(P(T,c) \cap C_{ij}) = \overline{P}_T.
\]

Let us consider case (ii) and set \( F_s = \{ q_1, q_2 \} \). Let \( v \) be a row vector orthogonal to \( q_1 - q_2 \) (i.e., satisfying \( v(q_1 - q_2) = 0 \)), and denote by \( S_v(Z) \) the minimum width strip orthogonal to \( v \) containing the set \( Z \). From the definition of strip, this is given by \( S(\alpha^+ v, s^*) \), where \( \alpha^+ \) is picked as the maximum positive \( \alpha \) such that \( S(\alpha v, s^*) \supset C_{ij} \) for some \( s^* \in \mathbb{R} \). The computation of \( S(\alpha^+ v, s^*) \) simply requires the computation of the tangency point between a line of direction \( q_1 - q_2 \) and the circle \( \delta C_{ij} \) (the outer circle bounding \( C_{ij} \)). The strip \( S(\alpha^+ v, s^*) \) is used as a candidate
strip for replacing one of the two strips of $\mathcal{P}_T$. The minimum volume parallelotope $\mathcal{P}(\hat{T}, \hat{c})$ containing the intersection of the three strips $S(p_1, \bar{s}_1), S(p_2, \bar{s}_2)$ and $S(\alpha^*, s^*)$, can be easily obtained by applying Theorem 2. Notice that by construction we have

$$\mathcal{P}(\hat{T}, \hat{c}) = \mathcal{P}(S(\alpha^*, s^*) \cap S(p_1, \bar{s}_1) \cap S(p_2, \bar{s}_2)) \supset \{\mathcal{P}(T, c) \cap \mathcal{C}_{ij}\}.$$ (40)

Fig. 6 shows two examples of the parallelotopic approximation of $\mathcal{P}(T, c) \cap \mathcal{C}_{ij}$, computed according to the above procedure. In Fig. 6a, the set $\mathcal{F}_s$ is empty because there are no landmarks inside the initial parallelotope $\mathcal{P}(T, c)$ ($\mathcal{F}_s = \emptyset$) and the inner bound of the thickened ring does not intersect the tightened parallelotope $\mathcal{T}_T$ (the thick one in the figure). Hence, the initial parallelotope is only thickened according to Proposition 5, but its shape does not change. On the contrary, Fig. 6b shows a case in which $\mathcal{F}_s$ is not empty, as $\delta \mathcal{C}_{ij}$ cuts the tightened parallelotope $\mathcal{P}_T$ (dashed) in the points $p_4, p_5$. Notice that the resulting parallelotope $\mathcal{P}(\hat{T}, \hat{c})$ (thick) is much smaller than the tightened one.

Finally, when $\mathcal{F}_s$ contains more than two points, the intersection $\mathcal{P} \cap \mathcal{C}_{ij}$ is made up of two or more disjoint subsets. A possible strategy may be to approximate each subset by a new parallelotope. A conservative but less computationally demanding alternative is to keep the tightened parallelotope $\mathcal{T}_T (\mathcal{P} \cap \mathcal{C}_{ij})$ as approximating parallelotope containing $\mathcal{P} \cap \mathcal{C}_{ij}$, without selecting any new strip. This does not cause any problem in most practical situations, as the ambiguity between disjoint feasible subsets are usually resolved by the intersection with thickened rings associated to other measurements.

Summing up, the following recursive procedure can be employed for the computation of the set $\mathcal{R}_\Pi(k|k)$ in problem P2:
- compute the parpolygon $\mathcal{T}_4 = \Pi \mathcal{R}(k|k-1)$;
- set $\mathcal{P} = \mathcal{P}(\mathcal{T}_4)$ (using the same technique, based on Theorem 1, adopted for computing (30));
- for $i, j = 1, \ldots, n, i < j$, set
  $$\mathcal{P} = \begin{cases} \mathcal{P}(\hat{T}, \hat{c}) \text{ in (40)}, & \text{if } \mathcal{F}_s \text{ contains two points,} \\ \mathcal{P}_T (\mathcal{P} \cap \mathcal{C}_{ij}), & \text{otherwise;} \end{cases}$$
- set $\mathcal{R}_\Pi(k|k) = \mathcal{P}$.

IV. DISCUSSION OF SET APPROXIMATION ALGORITHMS

In this section, the computational complexity of the set membership localization technique is analyzed. Moreover, some relevant extensions of the proposed approach are briefly discussed.

A. Computational complexity

The set membership localization algorithms presented in Section III enjoy a low computational complexity, thanks to the fact that the true feasible sets are recursively approximated via simple sets, like boxes or parallelotopes. Consider for example the box-based approximation procedure described in Subsection III-A. Tasks B1 and B3 can be accomplished according to Propositions 2 and 3 and require very few operations (products of matrices and vectors in a space of dimension 4). The most complex task is B2, whose suboptimal recursive solution is based on the computation of the smallest box containing the intersection of a box and a thickened ring, for each landmark pair. This can be obtained via Proposition 4, which requires the computation of a small number of points, as intersections of segments and circles in $\mathbb{R}^2$. Hence, the overall computational burden of recursion (21)-(23) at time $k$ basically depends on the number of considered landmark pairs. For $n$ landmarks, this can be at most $n(n-1)/2$ and therefore the computational complexity of the localization landmark turns out to be $O(n^2)$. If the robot is equipped with a panoramic vision system [24] or a rotating laser scanner [2], performing one 360° circular scan of the environment and collecting measurements of visual angles between pairs of consecutive landmarks, one has $n$ thickened rings and the complexity reduces to $O(n)$.

Analogue observations apply to the parallelotope-based algorithm of Subsection III-B, whose only additional computations required are those relative to the set $\mathcal{F}_s$ in (39),
which again contains points given by intersections of segments and circles in $\mathbb{R}^2$.

In general, the computational burden of the set membership localization algorithm depends on the number of landmarks detected at time $k$. If the robot is able to scan only a limited part of the environment (for example, due to use of a camera with restricted vision field) the average complexity will be $O(m)$, where $m$ is the average number of visual angle measurements processed at each time instant. Clearly, if very few landmarks are detected, the localization will be more rough, resulting in larger feasible sets.

It is also worth observing that due to the approximations involved in the recursive procedure, re-processing of measurements may be useful to further reduce the size of the approximate feasible sets, as it is well known in set membership estimation literature (see e.g. [25]). In this respect, an objective to be pursued is to suitably trade-off the increase of the computational burden due to data re-processing, and the uncertainty reduction in the approximation of feasible sets.

B. Extensions

The proposed set membership technique can be extended to more complex scenarios, without affecting the basic approach and with a limited increase of the computational burden.

The assumption that landmarks can be modeled as points in $\mathbb{R}^2$ can be easily relaxed. Indeed, non negligible landmark size can be treated by enlarging the bounds $c_{ij}^v$ on visual angle measurement errors. In principle, this can be done also to account for small uncertainties in the position of landmarks, which can be supposed to be bounded by a fixed quantity.

A much more challenging problem arises when the position of landmarks is unknown, and the robot has to both localize the landmarks and estimate its own position with respect to them. This is the so-called simultaneous localization and map building problem, which has been recently addressed in the literature in the classical probabilistic framework [3], [26], and also via a set membership approach [27]. The greatest difficulty when facing this problem originates from the fact that the state vector to be estimated includes all landmark positions, and therefore its dimension in no less than $2(n+1)$ no matter which motion model is adopted.

In this paper, only measurements of visual angles between a pair of landmarks are considered, according to what is done in many papers in the literature (see e.g. [4], [6], [5], [7], [8]). However, the proposed set membership approach is able to cope with different type of measurements, like e.g.

- distance measurements between the robot and a landmark, provided by a laser rangefinder or a stereo couple;
- measurements of absolute orientation, i.e. of the angle between the direction from the robot to a landmark and a fixed direction; the latter can be defined by the current robot heading or by a fixed direction detected by absolute orientation sensors (like a compass or a solar sensor).

Assuming bounded additive errors for the above measurements, one can define the corresponding measurement set, similarly to what has been done in (14)-(15). Specifically, each distance measurement defines a feasible circular corona in $\mathbb{R}^2$, while an absolute orientation measurement gives a feasible conic sector. The recursive set approximation techniques in Sect. III can be easily extended to these measurements sets.

Another relevant extension concerns the estimation of robot orientation. This can be included in the state vector, whose dimension turns out to be 6 (reducing to 3 for the simplified motion model that does not consider velocities). Simultaneous estimation of position and orientation leads to feasible state sets of higher dimension, which slightly complicates the set approximation procedure. A viable alternative may consist of estimating 2-dimensional feasible position sets and 1-dimensional feasible orientation interval. The correlation between position and orientation can be exploited by iteratively refining the position set with respect to the current admissible orientations, and vice versa (see [28] for details).

V. Numerical simulation experiments

In this section, simulation results of some localization experiments are reported. The set membership localization algorithms providing box and paralleloptopic approximations of the feasible position set have been tested in both static and dynamic setting.

A. Static setting

First, we consider a situation in which the robot is still and has to locate itself with respect to the landmarks. At a fixed time $k$, the navigating vehicle is supposed to be located at the center of a square room, of 20 meters side. This means that $\Pi R(k|k-1) = \Xi(k|k-1) = B([10, 10]', [0, 0]')$, in a reference system centered at the vehicle position (see Fig. 7). The sets $\mathcal{R}_\Pi(k|k)$ defined in (22) are computed according to the recursive strategies for the solutions of problems B2 and P2, outlined in Subsections III-A and III-B. The centers of boxes and paralleloptopes have been considered as nominal estimates $\hat{p}$ of the vehicle position.

![Fig. 7. The simulation environment (thick box: the room; *: robot position; x: landmarks).](image)

In a first set of experiments, it is assumed that 5 landmarks are identified in the scene. The visual angle measure-
ments are corrupted by additive noise \(v_{ij}(k)\), generated as an independent uniformly distributed signal satisfying (13) with constant bound \(\varepsilon_v = \varepsilon_v^\forall\). Error bounds for commercial sensors range from 0.5° ± 1° of popular rotating laser scanners, to much smaller errors if sophisticated panoramic vision systems are employed. Nevertheless, also larger error bounds have been considered in the simulations, in order to account for non negligible landmark size and errors in landmark positions, as explained in Sect. IV-B. The nominal localization errors \(\|\hat{p} - p\|\) (in meters), obtained for different values of \(\varepsilon_v^\forall\) ranging from 0.5 to 5 degrees, are reported in Fig. 8(a). Results are averaged over 1000 different landmark configurations in the square room. In Fig. 8(b), the area of the approximating boxes and parallelotopes are compared with that of the minimum area box containing the exact feasible position set \(L(k)\), i.e. \(B(L(k)) = B(\Pi\Xi(k)|k - 1) \cap M(k)\)). Notice that the parallelotopic uncertainty sets are smaller than the corresponding boxes, and that their area is quite close to that of the minimum outer box \(B(L(k))\) (whose computation has been performed by gridding a sufficiently large area containing the true robot position: notice that this has required a much higher computational burden, compared to the proposed approximation algorithms). The same experiment is repeated for the case of 10 landmarks, in the above setting. Results concerning nominal position errors and uncertainty set areas are reported in Figs. 9. It can be noticed that nominal position errors and areas of the uncertainty sets have been remarkably reduced.

Another set of experiments is performed, assuming a relative visual angle error bound. In these simulations, \(\varepsilon_v^\forall\) is set to 20% of the current visual angle measure \(\theta_{ij}\). The estimated nominal positions for 100 different settings including 5 landmarks are reported in Fig. 10. It can be observed that, despite the large measurement error, the “cloud” of nominal estimates is not too scattered. The average nominal position error is 0.49m for parallelotopes and 0.51m for boxes. Average uncertainty areas are respectively 2.32m² and 5.42m². Once again, nominal errors provided by the two algorithms are similar, but the parallelotopic approximation gives smaller uncertainty sets.

The same experiment is repeated placing the vehicle almost at one corner of the square room. In this case, the visual angles range only from 0 to 90 degrees, thus reducing the maximum size of measurement noise. The results are depicted in Fig. 11, where average nominal position errors are 0.45m for parallelotopes and 0.51m for boxes. Average uncertainty areas are reduced to 1.77m² and 2.07m²,
respectively. The results obtained in all the performed experiments appear of good quality, when compared to those presented in recent literature addressing the localization problem in a similar setting (see [5], [7], where localization from angle measurements with bounded errors is performed).

B. Dynamic setting

In the following set of experiments, the vehicle follows a given trajectory inside a square room. The dynamic model \( p(k + 1) = p(k) + u(k) + w(k) \) is considered for the position of the vehicle, with \( k = 0, \ldots, 15 \). The driving input \( u(k) \) describes the required trajectory, while the disturbance \( w(k) \) is assumed to satisfy (12), with constant bound \( \varepsilon_i w(k) = 0.2, \ i = 1, 2, \forall k \).

A first set of experiments is performed with 5 landmarks in the room and visual angles corrupted by bounded noise, with absolute noise bound equal to 5 degrees. A typical run employing boxes or parallelogotopes as approximating regions is reported in Fig. 12. Fig. 13 shows the time variation of uncertainty set areas \( R(k|k) \) and position errors \( \|p(k) - \hat{p}(k)\| \) (where \( \hat{p}(k) \) is the center of the approximating box or parallelogotope). Here the results are averaged over 100 different noise realizations, for the same trajectory and landmark configuration in the room.

Another set of experiments is performed for the case of 20% relative noise bound (i.e., \( \varepsilon^v_i(k) = 0.2 \theta_i(k) \) for each visual angle measurement, at each time instant \( k \)). This time, 8 landmarks in the room are used for dynamic localization. A typical run is reported in Fig. 14, while average uncertainty set areas and position errors are shown in Fig. 15.

Figs. 12-15 show that the set membership localization algorithms are able to remarkably reduce localization error and related uncertainty, also in the presence of disturbances in the vehicle dynamics (the term \( G(k)w(k) \) in equation (7)). It can be observed that parallelogotopes generally give better approximations of the feasible position set (especially during transients) with respect to boxes, while the two algorithms show similar performance in terms of nominal position error. The large reduction of uncertainty achieved during the first half of the trajectory in Fig. 14 is due to the fact that the robot reaches a location where most of the visual angles are small, and therefore small relative errors affect the visual angle measurements.

It is worth noting that, due to the approximation introduced by recursive bounding, the size of uncertainty sets can be further reduced by reprocessing the same measurements several times. In other words, the quality of set approximation can be trade-off with the required computational power in order to obtain the maximum uncertainty reduction allowed by the available bandwidth.
VI. Conclusions

In this paper, an approach to the localization problem of a mobile robot based on angle measurements has been presented. A set membership framework has been adopted to deal with uncertainty affecting both measurements and the mobile robot dynamics equations. The techniques devised provide recursive estimates for the robot position uncertainty set. These algorithms can be applied successfully in real-time localization and navigation problems, due to the reduced computational complexity of the involved numerical procedures. Several numerical simulation examples show that the proposed approach provides encouraging results.

Although the problem dealt with in the present paper is formulated in a quite simple setting, the set membership approach can be easily extended to include estimation of the robot orientation, landmarks uncertainty and different measurement sources. A more challenging problem, which is a subject of great interest in exploration, concerns the localization problem for the case when the landmarks must also be localized in the considered environment. This is the typical scenario in team-based exploration tasks, where the robots of a sub-team act as landmarks for robots of another sub-team, and all of them concur to the global localization problem and to the construction and updating of the map of the region of interest. Tackling this problem by means of set membership techniques will be the subject of future work.

REFERENCES

Appendix

Proof of Proposition 1

Let $\Upsilon = \{ \xi \in \mathcal{M}(k) \}$. First, we show that $\mathcal{R}(k|k-1) \supseteq \Xi(k|k-1)$ and $\mathcal{R}(k|k) \supseteq \Xi(k|k)$. From (20), the claim holds for $k=0$. By induction, let $\mathcal{R}(k-1|k-1) \supseteq \Xi(k-1|k-1)$. From (17) and (21), one has immediately $\mathcal{R}(k|k-1) \supseteq \Xi(k|k-1)$. Then, substituting (22) in (23) and exploiting some straightforward set inclusions, one gets

$$\mathcal{R}(k|k) \supseteq \{ [\Pi \mathcal{R}(k|k-1) \cap \mathcal{M}(k)] \times \Pi \mathcal{R}(k|k-1) \} \cap \mathcal{R}(k|k-1)$$

$$= \{ [\Pi \mathcal{R}(k|k-1) \cap \mathcal{M}(k)] \times \Pi \mathcal{R}(k|k-1) \cap \mathcal{R}(k|k-1) \} \cap \mathcal{R}(k|k-1)$$

$$= \{ \mathcal{M}(k) \times \mathbb{R}^2 \} \cap \mathcal{R}(k|k-1)$$

$$\supseteq \Upsilon \cap \Xi(k|k-1)$$

$$= \Xi(k|k).$$

Hence, from (22) one has

$$\mathcal{R}_\Pi(k|k) \supseteq \Xi(k|k) \supseteq \Xi(k|k-1) \cap \mathcal{M}(k) = \Xi(k|k-1) \cap \Pi \Upsilon$$

$$\supseteq \Pi \{ \Xi(k|k-1) \cap \Upsilon \} = \Xi(k|k) = \mathcal{L}(k)$$

which concludes the proof.

Proof of Proposition 2

From the definition of box, one has

$$A(k-1)B(b,c) + B(k-1)u(k-1)$$

$$+ G(k-1)\Diag\{e^w(k-1)\}B_\infty =$$

$$\{ \xi \in \mathbb{R}^4 : \xi = (A(k-1) + A(k-1)\Diag\{b\})\alpha + B(k-1)u(k-1) + G(k-1)\Diag\{e^w(k-1)\}\beta, \alpha \in \mathbb{R}^4, \beta \in \mathbb{R}^2, \|\alpha\|_{\infty} \leq 1, \|\beta\|_{\infty} \leq 1 \} =$$

$$\{ \xi \in \mathbb{R}^4 : \xi = (A(k-1)c + B(k-1)u(k-1) + [A(k-1)\Diag\{b\} G(k-1)\Diag\{e^w(k-1)\}]\gamma, \gamma \in \mathbb{R}^6, \|\gamma\|_{\infty} \leq 1 \}.$$
(the same reasoning can be repeated for the corresponding minimization problem, and for the other coordinate $y$). Since $\mathcal{E} \subset B \cap \mathcal{C}_{ij}$, one has

$$\max_{p \in \mathcal{E}} x \leq \max_{p \in B \cap \mathcal{C}_{ij}} x.$$ 

Conversely, notice that $B \cap \mathcal{C}_{ij}$ is a compact set (or possibly the union of disjoint compact sets). Therefore, the maximum of the linear functional $x$ is assumed on the boundary of $B \cap \mathcal{C}_{ij}$, which contains segments of $\delta B$ and arcs of circumferences of $\delta C_{ij}^+$ and $\delta C_{ij}^-$. The contact points of these segments and arcs can be vertices of $B$ (points of $\mathcal{E}_1$), intersections of the boundaries of $B$ and $\mathcal{C}_{ij}$ ($\mathcal{E}_2$), or the landmarks contained in $B$ ($\mathcal{E}_3$). If the boundary of $B \cap \mathcal{C}_{ij}$ does not contain arcs of $\delta C_{ij}^-$, then $B \cap \mathcal{C}_{ij} \subset \text{Co}(\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3)$ and hence

$$\max_{p \in B \cap \mathcal{C}_{ij}} x \leq \max_{p \in \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3} x. \quad (42)$$

Otherwise, if the inequality (42) does not hold, the maximum in (41) is assumed on a point of $\delta C_{ij}^-$ which does not belong to $\delta B$. This clearly leads to

$$\max_{p \in B \cap \mathcal{C}_{ij}} x = \max_{p \in \mathcal{E}_4} x. \quad (43)$$

Hence, from (42) and (43)

$$\max_{p \in B \cap \mathcal{C}_{ij}} x \leq \max_{p \in \mathcal{E}} x,$$

which completes the proof.