SOLVING THE UNDIRECTED MINIMUM COST FLOW PROBLEM WITH ARBITRARY COSTS

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Abstract

We address the undirected minimum cost flow problem with arbitrary arcs costs. The optimal solutions for this problem is characterized proving that the flow of all arc with negative cost must be equal its capacity. That is, the flow can be non-zero in both directions. This situation implies that the flow can take values that are integer multiple of ½. Therefore, this single commodity flow problem does not satisfy the unimodularity property. However, using a reformulation of the original problem, we develop an easy method for solving it using any classic minimum-cost flow problem algorithm.

Keywords: network flows, undirected minimum cost flow problem.

1. Introduction

Network flow models are required in a wide variety of contexts because physical networks arise in many applications in different types of real systems: communications, hydraulic, mechanical, electronic, and transportation. In addition, network flow problems take place in optimization problems that in appearance do not involve networks at all. A wide list of these applications has been compiled in Ahuja et al [1]. The classical minimum cost flow problem, the most fundamental among network flow problems, has been studied in an exhaustive way. Comprehensive lists of references are provided in specialized books, for example: Ahuja et al. [1], Glover et al. [3], Murty [4] and Rockafellar [5]. The undirected case of the minimum cost flow problem has been deserved less attention.

The undirected minimum cost flow problem, in which all arc costs are non-negative, reduces easily to a directed minimum cost flow problem (see for example, Ahuja et al. [1], Chardaire and Lisser [2]). In this note, we show how to reduce the undirected minimum cost flow problem in which arc costs may be negative to a directed minimum cost flow problem. We show that the arc flows on arcs with non-negative costs are integral, and the arc flows on arcs with negative cost arcs may be half-integral or integral, assuming integral data.

In this paper, we study the undirected minimum cost flow problem with arbitrary costs and characterize the optimal solution for this problem. We prove that in the optimal solution, the flow of any arc with negative cost must be equal to its capacity. We will establish that, therefore, the optimal flows can take values that are integer multiple of ½. Consequently, the problem we deal with is a single flow problem with an optimal solution, generally, non-integer. However, in the actual state of art, a specialized network method to find non-integer flows has not been introduced yet.

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We will give a simple network flows methodology to solve the undirected minimum cost flow problem with arbitrary costs. For this, we will prove that this problem can be solved using any minimum cost flow algorithm, and later, from its output, the final vector flow is appropriately identified.

The rest of the paper is organized as follows. In section 2, the mathematical notation and statement of the problem are introduced. Also in this section, we present a result about arcs with negative costs, which characterizes the optimal solution for this problem. From this result, the most efficient way to solve the problem is indicated.

2. Notation and mathematical formalization

The usual formulation of the undirected minimum cost flow problem is stated from a symmetric directed network $\mathcal{G} = (V, A)$, where $V$ is the set of $n$ nodes and $A$ is the set of $m$ arcs. For symmetric we means that $(i, j) \in A$ and $(j, i) \in A$. This directed network $\mathcal{G} = (V, A)$ is obtained from the undirected network $\mathcal{G} = (V, E)$ when directions on arcs in $\mathcal{G}$ are considered. For notational convenience, we refer to the undirected arc $(i, j)$ as $\{i, j\}$. Let an integer $b_i$ be the supply/demand rate of node $i$. Each arc $(i, j) \in A$ and, its symmetric $(j, i) \in A$, have associated the following values: $u_{ij}$, the upper bound on net flow through arcs $(i, j)$ and $(j, i)$ and, $c_{ij}$, the cost per unit of flow on arc $(i, j)$ or $(j, i)$ in the objective function. We denote by $U = \max\{u_{ij} \| (i, j) \in A\}$.

If $y_{ij}$ denotes the amount of flow on an arc $(i, j)$, the undirected minimum cost flow with arbitrary costs (UMCF) problem can be stated as follows:

$$\begin{align*}
\text{minimize} \quad & z = \sum_{(i,j) \in A} c_{ij}y_{ij} \\
\text{subject to:} \quad & \sum_{(j,i) \in A} y_{ij} - \sum_{(i,j) \in A} y_{ji} = b_i, \quad \forall i \in V \\
& y_{ij} + y_{ji} \leq u_{ij}, \quad \forall \{i, j\} \in E \\
& 0 \leq y_{ij} \leq u_{ij}, \quad \forall (i,j) \in A
\end{align*}$$

The constraints (1.1) are called flow-balance constraints, and equations (1.2) are known as capacity constraints. The upper bounds in (1.3) are redundant, but they will play a role soon. Notice that the unimodularity property of the minimum cost flow problem (MCF) with a single flow is no longer fulfilled due to constraints (1.2). Thus, even assuming networks with integer capacities, the flow could be non-integer. The following example shows this situation using a simple network with only two arcs, $(i, j)$ and $(j, i)$, that share a capacity equal to 4. Note, from Figure 1, that in the optimal solution, the flow values are integer multiple of $\frac{1}{2}$. 

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The above example indicates us intuitively that in the optimal solution, every undirected arc with negative cost must satisfy $y_{ij} + y_{ji} = u_{ij}$. This is a general result that we prove taken into account the optimality conditions for the UMCF problem.

**Lemma 1. (Negative cost arcs)** If $c_{ij} < 0$ then, in every optimal solution of the UMCF problem, $y_{ij} + y_{ji} = u_{ij}$.

**Proof.** Let $y'$ be an optimal flow. If $c_{ij} < 0$, and $y'_{ij} + y'_{ji} < u_{ij}$, then the flow is not optimum because the objective function value can be improved by increasing the flows $y'_{ij}$ and $y'_{ji}$ in $\delta = (u_{ij} - y'_{ij} - y'_{ji})/2$ units. Note that these changes preserve the mass balance constraints and satisfy $y'_{ij} + y'_{ji} = u_{ij}$.

Let us define the set $E^- = \{ (i, j) \in E : c_{ij} < 0 \}$. For any real number $r$, let $r^+ = \max(r, 0)$.

**Corollary 1.** If $y^*$ is an optimal flow of the UCMF problem, then

$$\sum_{(i, j) \in E^-} c_{ij} y^*_{ij} = \sum_{(i, j) \in E^-} c_{ij} y^*_{ij} + \sum_{(i, j) \in E^+} c_{ij} u_{ij}$$

**Proof.** If $(i, j) \in E^-$, then $c_{ij}(y^*_{ij} + y^*_{ji}) = c_{ij} u_{ij}$ and $c_{ij}(y^*_{ij} + y^*_{ji}) = 0$.

Moreover, the sum $\sum_{(i, j) \in E^-} c_{ij} u_{ij}$ of the above equality is a constant term, and therefore, to minimize $z$ is equivalent to minimize $\sum_{(i, j) \in E^-} c_{ij} y^*_{ij}$. Then we can conclude that in the formulation of UMCF problem, we can consider the cost arcs $c_{ij}^+$ instead of $c_{ij}$, with the property $y_{ij} + y_{ji} = u_{ij}$ for all $(i, j) \in E^-$. Using these previous ideas, we will develop a methodology to solve efficiently the UMCF problem.

Let us consider the following directed minimum cost flow problem ($R$ problem for short):

\[
\begin{align*}
\text{minimize} \quad & w = \sum_{(i, j) \in E} c_{ij} y_{ij} \\
\text{subject to:} \quad & \sum_{(i, j) \in E} y_{ij} - \sum_{(i, j) \in E} y_{ji} = b_i, \quad \forall i \in V \\
& 0 \leq y_{ij} \leq u_{ij}, \quad \forall (i, j) \in A
\end{align*}
\]
Let \( y' \) be an optimal basic solution for the above problem. Then \( y' \) is integral, and verifies that either \( y'_y = 0 \) or \( y'_j = 0 \) or both (see for example, Ahuja et al. [1]). Moreover, \( y_y + y_j \leq u_y \) for all arc \( (i, j) \in E \), that is, the optimal solution \( y' \) satisfies (1.2). Let \( y^* \) be obtained from \( y' \) as follows:

\[
y^*_y = \begin{cases} 
y'_y & \text{if } c_y \geq 0 \\ y_y' + (u_y - y'_y - y'_j)/2 & \text{if } c_y < 0, \forall (i, j) \in A 
\end{cases}
\]

Then \( y^* \) is feasible for both \( R \) and UMCF problem, it is half-integral, and \( y^* \) is optimal for \( R \) problem because it has the same cost as \( y' \). Note that increasing the flows on arcs with negative costs does not alter the optimal value \( w \) of the \( R \) problem. Then, from corollary 1, \( y^* \) is optimal for UMCF problem.

Using these previous ideas, an algorithm to solve the UMCF problem is stated as follows:

**Algorithm for the UMCF problem**

*Obtain an optimal basic solution \( y' \) for \( R \) problem using any minimum cost flow algorithm;*

*Then an optimal solution for UMCF problem is \( y^*_y = \begin{cases} y'_y & \text{if } c_y \geq 0 \\ y'_y + (u_y - y'_y - y'_j)/2 & \text{if } c_y < 0, \forall (i, j) \in A 
\end{cases} \).*

**Acknowledgments**

The authors wish to thank the anonymous associate editor for their valuable keys and comments which it have improved and shorted this paper.

**References**


