Locally Polar Geometries with Affine Planes

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In this paper we consider partial linear spaces containing a set of subspaces isomorphic to affine planes, such that the lines and these affine planes on a fixed point form a non-degenerate polar spaces of rank at least 2. We obtain a complete classification, provided that the rank is at least 3.

1. INTRODUCTION

The study of geometries on the absolute points of polarities in projective spaces has been started by Veldkamp [21], who was the first to give a synthetic characterization of these geometries, which he called polar spaces. As part of his work on spherical buildings [19], Tits extended Veldkamp's results to a somewhat larger class of geometries related to pseudo-quadratic forms. In 1974, Buekenhout and Shult proved that most of the axioms used by Veldkamp and Tits can be deduced from the following beautiful axiom only involving points and lines, see [3]:

Let \( p \) be a point and \( l \) a line; then \( p \) is collinear with one or all points incident with \( l \).

We will refer to this axiom as the 'one or all' axiom. Both Veldkamp and Tits assume the polar space to have finite rank. Recently, Johnson extended the results of Tits to the case of infinite rank (see [12] or, for a more elementary approach [9, 13]).

The result of the work mentioned above is that there is a nice and satisfactory characterization of polar spaces. Shortly after simplifying the axioms for polar spaces to the 'one or all' axiom, Shult [18] raised the question of whether it is possible to characterize geometries on points in projective spaces that are the non-absolute points with respect to some given polarity. To indicate that this is possible he proved his well known Cotriangle Theorem, which characterizes geometries on the non-singular points and elliptic lines with respect to some non-degenerate quadratic form defined on a projective space over \( \text{GF}(2) \) (see [18, 11]). The only other result in this direction known to us is provided by the classification of finite irreducible generalized Fischer spaces in [5, 6]. There a characterization of geometries on the non-singular points in orthogonal spaces over \( \text{GF}(3) \) and unitary spaces over \( \text{GF}(2^2) \) is given. It is the purpose of this paper to give a satisfactory answer to Shult's question.

We consider a somewhat larger class of geometries than originally considered by Shult, of which most examples can be described in the following way. Let \( \Pi = (P, L) \) be a polar space embedded in a projective space \( P \). Then the points of \( \Pi \) are called the singular points of \( P \) and the points outside \( \Pi \) the non-singular points of \( P \). A line (or plane) of \( P \) is called a singular line (resp. singular plane) iff all its points are singular, and it is called a tangent line (resp. tangent plane) iff it contains a unique singular point (resp. singular line). In a tangent plane all the singular points are on the unique singular line of the plane. Now consider the geometry with as points the non-singular points of \( P \), as lines the tangent lines, and as planes the tangent planes of \( P \), incidence being symmetrized inclusion. This geometry is in general not connected, but has connected components. The connected components are all members of a class of geometries that we call tangent geometries. They have the property that the lines and
planes on a fixed point form a polar space and that their planes are affine planes. As we already remarked, this does not give us all the examples of tangent geometries we are considering in this paper. In particular, if $\mathbf{P}$ is a projective space over a division ring of even characteristic or has infinite dimension, the above description does not catch all the objects under consideration.

To obtain all the examples of geometries we want to consider in the case in which the characteristic of the underlying division ring is 2 or the projective space is infinite-dimensional, we prefer to look at hyperplanes of the projective space rather than at points.

Let $\mathcal{P}$ be the set of hyperplanes of $\mathbf{P}$ that intersect $\Pi$ in a non-degenerate geometric hyperplane. As lines we take the sets of hyperplanes in $\mathcal{P}$ that contain $p \perp \cap H$ for some point $p \in \Pi$ and $H \in \mathcal{P}$. Let $\mathcal{L}$ denote the set of all these lines. Furthermore, let $\mathcal{A}$ denote the set of planes, where a plane is the set of elements of $\mathcal{P}$ that contain some fixed codimension 3 subspace of $\mathbf{P}$ that intersects $\Pi$ in a proper degenerate subspace, the radical of which is a line of $\Pi$. Now consider the geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{A})$, where incidence is symmetrized inclusion. Every union of connected components of $\Gamma$ is also a member of the class of tangent geometries. Every geometry obtained in this way will be called a projective tangent geometry of $\Pi$.

In Section 2 we give a precise (and synthetic) definition of tangent geometries. The reader is referred to that definition in the following theorems. There we characterize the tangent geometries having the property that for each point the geometry of lines and planes through that point is a non-degenerate polar space of rank at least 2 and planes are affine. (For unexplained notation the reader is referred to Section 2. We warn the reader that the definition of rank that we use in this paper is slightly different from the singular rank; see also [9, 12].)

**Theorem 1.1.** Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{A})$ be a connected geometry consisting of non-empty sets $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{A}$ of points, lines and planes respectively such that the following hold;

(i) the space $(\mathcal{P}, \mathcal{L})$ is a partial linear space;

(ii) the points and lines incident with a plane form an affine plane;

(iii) if $\pi$ is a plane and $l$ a line meeting $\pi$ in a point $x$, then either $l$ is coplanar with all lines of $\pi$ through $x$ or with a unique line of $\pi$;

(iv) for every point $x$ and line $l$ on $x$ there is a line on $x$ not coplanar with $l$.

Then $\Gamma$ is a standard cover of a tangent geometry of a polar space $\Pi$ of rank at least 2, the radical of which is either empty or consists of a unique point.

We obtain this result as a corollary of the following theorem, in which we only consider irreducible geometries. (A geometry $\Gamma$ satisfying the hypothesis of 1.1 is called irreducible if its collinearity graph has diameter at most 2; otherwise it is called reducible.)

**Theorem 1.2.** Let $\Gamma$ be an irreducible geometry satisfying the hypothesis of Theorem 1.1. Then $\Gamma$ is a tangent geometry of a polar space $\Pi$ of rank at least 2, the radical of which is either empty or consists of a unique point.

The following result shows that most tangent geometries satisfying the conditions of Theorem 1.2 can be obtained as the geometries described above.

**Theorem 1.3.** Let $\Gamma$ be an irreducible geometry satisfying the conditions of Theorem 1.1 and having the following property:

(v) there is a plane $\pi$ and a point $x$ not in $\pi$ that is coplanar with all lines in $\pi$.

Then $\Gamma$ is a projective tangent geometry of a polar space $\Pi$ embedded in some projective space.
The special case of the above results when the partial linear space \((\mathcal{P}, \mathcal{L})\) is a gamma space (i.e. every point is collinear with 0, 1 or all points on a line) is covered by the results of Cohen and Shult (see [4]). They prove under the stronger assumption that \((\mathcal{P}, \mathcal{L})\) is a gamma space that \(\Gamma\) is an affine polar space. This is the geometry consisting of the points, lines and planes of a polar space that are not in some fixed geometric hyperplane. Cohen and Shult’s work forms the starting point of this paper. Many of the techniques used in this paper can be found in [4] or in [11]. Other special cases of the above results can be found in [2, 4, 5, 6, 8, 14, 16].

Let \(\Gamma\) be an affine polar space obtained by removing some geometric hyperplane \(H\) from a polar space \(\Pi\), all of the lines of which contain at least 3 points. (In this paper we only consider polar spaces the lines of which contain at least 3 points.) Then \(H\) is called the polar space at infinity of \(\Gamma\). It is shown in [4] that the diameter of the collinearity graph of \(\Gamma\) is at most 3. In the next section we show that the reflexive extension of ‘being at distance 3’ in the collinearity graph of \(\Gamma\) defines an equivalence relation on the points of \(\Gamma\) that extends to an equivalence relation \(*\) on \(\Gamma\) and leads to a quotient geometry \(\Gamma/\ast\), provided that the rank of \(\Pi\) is at least 3. Any quotient geometry of \(\Gamma\) that is a cover of \(\Gamma/\ast\) will be called a standard quotient of \(\Gamma\). As a consequence of the above results and the observation that affine polar spaces are their own universal covers, we obtain a complete classification of the covers of tangent geometries of Theorem 1.1 under the restriction that the geometry can be embedded in a projective space (see Proposition 2.10). In particular, we obtain:

**Theorem 1.4.** Let \(\Gamma\) be a residually connected geometry that satisfies the hypotheses of 1.3. Then \(\Gamma\) is a standard quotient of a uniquely determined affine polar space.

In [7], the case that \(\Gamma\) does not satisfy condition (v) is considered. There it is shown that if \(\Gamma\) is finite then it is a quotient of an affine polar space.

As for diagram geometries, we remark that the geometries of finite rank satisfying (i), (ii) and (iii) of Theorem 1.1 are precisely the (truncations of) Buekenhout geometries described by the following diagram:

\[
\text{Af} \rightarrow \text{---} \rightarrow \text{---}
\]

and satisfying the Intersection Property. This follows from 3.1 and 4.2 of this paper, together with Lemma 1 of [14].

We want to remark that the results 1.1, 1.2 and 1.3 are obtained using only synthetic methods. In the proof of Theorem 1.4, however, we use some non-synthetic methods. In particular, we use the classification of the polar spaces of rank at least 2 that can be embedded in a projective space (see [10, 13]).

Finally, we must make some comments on the organization of this paper. In Section 2 we give some definitions, in particular the synthetic definition of tangent geometries. Furthermore, we discuss the relation between tangent geometries and affine polar spaces. Section 3 contains some preliminary lemmas, and in Section 4 we start the proof of Theorems 1.1 and 1.2. In this section we extend some results of [4] to geometries satisfying the conditions of 1.1. In particular, we show that the transitive extension of the relation of being parallel in some affine plane induces a non-trivial equivalence relation on the set of lines of \(\Gamma\). The equivalence classes of lines of this equivalence relation are shown to be (part of) the point set of a polar space at infinity for \(\Gamma\).

If \(\Gamma\) is irreducible we can identify the points of \(\Gamma\) with geometric hyperplanes in this polar space at infinity, which provides us with a proof of Theorems 1.1 and 1.2. This is
done in Section 5. Finally, the last section is devoted to the proof of Theorem 1.3. In this section we construct a projective space in which we can embed the polar space at infinity of $\Gamma$ and recognize the points of $\Gamma$ as hyperplanes.

2. Quotients of Affine Polar Spaces and Tangent Geometries

In this section we give a precise definition of tangent geometries and discuss the relation between quotients of affine polar spaces and tangent geometries of polar spaces.

But first we will give some definitions and explain the notation used throughout this paper.

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be an incidence system (also called space). It consists of a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of subsets of $\mathcal{P}$ of cardinality at least 2 called lines, incidence being symmetrized inclusion. A subset $X$ of $\mathcal{P}$ is called a subspace if any line meeting $X$ in at least 2 points is contained in $X$. Then $X$ is often identified with the incidence system the point set of which is $X$ and the lines of which are the lines of $\Pi$ which are contained in $X$. For each subset $X$ of $\mathcal{P}$ we denote by $\langle X \rangle$ the subspace generated by $X$, i.e. the intersection of all subspaces containing $X$. A proper subspace of $\Pi$ is called a geometric hyperplane if it meets every line non-trivially.

Two points $x$ and $y$ are called collinear, notation $x \perp y$, if there is a line incident with both of them, or if $x = y$. For any subset $X$ of $\mathcal{P}$ we denote by $X^\perp$ the set $\{ y \in \mathcal{P} \mid y \perp x \text{ for all } x \in X \}$. We write $x^\perp$ for $\{x\}^\perp$, where $x \in \mathcal{P}$. The collinearity graph of $\Pi$ is the graph with vertex set $\mathcal{P}$, two vertices being adjacent iff they are distinct and collinear. We call $\Pi$ connected iff its collinearity graph is connected.

We call $\Pi$ a linear space (resp. partial linear space) iff each pair of distinct points is on exactly (resp. at most) one line. It is called a singular space if all points are collinear.

Let $\mathcal{A}$ be a set of subspaces of $\Pi$ such that each point in an element $x \in \mathcal{A}$ is on at least 2 lines of $\Pi$ and every line in at least one element of $\mathcal{A}$. Then the incidence system $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{A})$ is called a geometry. The elements of $\mathcal{A}$ are called the planes of $\Gamma$. Notice that for every point $x$ of $\Gamma$ the set of lines on $x$ together with the sets of lines on $x$ inside a plane form an incidence system, which we denote as $\text{Res}(x)$.

Let $*$ denote an equivalence relation on $\mathcal{P}$. Then for every pair $X$ and $Y$ of subspaces of $(\mathcal{P}, \mathcal{L})$ we write $X \ast Y$, iff for each point $x \in X$ there is a point $y \in Y$ such that $x \ast y$. We say that $*$ is an equivalence relation on $\Pi$ (or $\Gamma$) if $*$ is an equivalence relation on $\mathcal{L}$ (and $\mathcal{A}$). Furthermore, $*$ is called a standard equivalence relation of $\Pi$ (or $\Gamma$) if it is an equivalence relation on $\Pi$ (or $\Gamma$) and has the following property:

Let $x$ and $y$ be $*$-equivalent points. Then for each point $z$ that is collinear to $x$ there is a unique point in $y^\perp$ $*$-equivalent to $z$.

If $*$ defines a standard equivalence relation on $\Pi$ (or $\Gamma$), then we define the quotient $\Pi/\ast$ (resp. $\Gamma/\ast$) in the following way. For each point $x \in \mathcal{P}$ let $\tilde{x}$ denote the $\ast$-equivalence class of $x$. For each subspace $X$ of $\Pi$ we denote by $\tilde{X}$ the set $\{ \tilde{x} \mid x \in X \}$. Let $\tilde{\mathcal{P}}$, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{A}}$ be the sets of elements $\tilde{X}$, where $X$ runs through $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{A}$ respectively. The quotients $\Pi/\ast$ and $\Gamma/\ast$ are the incidence systems $(\tilde{\mathcal{P}}, \tilde{\mathcal{L}})$ and $(\tilde{\mathcal{P}}, \tilde{\mathcal{L}}, \tilde{\mathcal{A}})$ respectively. They are called standard quotients of $\Pi$ and $\Gamma$ respectively, while $\Pi$ and $\Gamma$ are called standard covers of $\Pi/\ast$ and $\Gamma/\ast$ respectively.

Before continuing, let us state some facts about standard quotients.

Proposition 2.1. Let $*$ be a standard equivalence relation of $\Pi = (\mathcal{P}, \mathcal{L})$. Then the following hold:
Locally polar geometries

(i) If \( x \) and \( y \) are \(*\)-equivalent points, then \( x = y \) or \( x \) and \( y \) have distance at least 3 in the collinearity graph of \( \Pi \).

(ii) If \( X \) is a singular subspace of \( \Pi \), then the map \( x \mapsto \bar{x} \) defines an isomorphism between \( X \) and \( \bar{X} \).

PROOF. Let \( x \) and \( y \) be two points of \( \Pi \) that are \(*\)-equivalent. Suppose that \( z \) is a point collinear to both \( x \) and \( y \). Then \( z \neq z' \) and \( x \) and \( y \) are two points in the \(*\)-equivalence class of \( x \) that are in \( z^+ \)---a contradiction. This proves (i).

Now let \( X \) be a singular subspace of \( \Pi \). Then, by (i), the maps \( x \mapsto \bar{x} \) and \( l \mapsto \bar{l} \) are clearly incidence-preserving bijections. This proves (ii).

From the above proposition it follows that a standard quotient of a partial linear space (resp. geometry) is a partial linear space (resp. geometry).

In the remainder of this section we consider polar spaces and affine polar spaces.

A **polar space** is an incidence system \( \Pi = (\mathcal{P}, \mathcal{L}) \) in which the 'one or all' axiom is satisfied. In this paper we only consider polar spaces the lines of which contain at least 3 points. The **radical** of a polar space \( \Pi \) is the set \( \mathcal{P}^+ \). The space is called **degenerate** if its radical is non-empty and **non-degenerate** otherwise. As follows from [3, 12], non-degenerate polar spaces are partial linear spaces. Furthermore, it is shown in [12] that all singular subspaces of a non-degenerate polar space are projective spaces. The **rank** of a polar space \( \Pi \), denoted by \( rk(\Pi) \), is the least upper bound (possibly \( \infty \)) for the number \( n \) such that there is a chain of singular subspaces \( Rad(\Pi) = X_0 \subset X_1 \subset \cdots \subset X_n \) of \( \Pi \). If the rank of \( \Pi \) is at least 3 and \( \Pi \) is non-degenerate, then the set of singular subspaces of \( \Pi \) that are projective planes provides us with a set of planes for \( \Pi \), which makes \( \Pi \) into a geometry. In this case \( \Pi \) is often tacitly identified with the corresponding geometry of points, lines and projective planes. A geometric hyperplane of a polar space is also a polar space. As geometric hyperplanes of polar spaces play an important role in this paper, we quote some basic facts about them in the next lemma.

The proof of this lemma can be found in [1, 4] (see also [9, Lemma 1]).

**Lemma 2.2.** Let \( \Pi \) be a polar space of rank at least 2 all of the lines of which have at least 3 points. Suppose that \( H \) is a geometric hyperplane of \( \Pi \) and \( X \) is a subspace not contained in \( H \). Then the following hold:

(i) \( X \cap H \) is a geometric hyperplane of \( X \);

(ii) if \( rk(X) \geq 2 \) then \( X \cap H \) is a maximal subspace of \( X \); in particular, \( X = (X \cap H, p) \) for any point \( p \in X \setminus X \cap H \);

(iii) if \( rk(X) \geq 2 \) then the collinearity graph of \( X \setminus X \cap H \) is connected;

(iv) \( rk(X \cap H) \) equals \( rk(X) \) or \( rk(X) - 1 \);

(v) if \( Rad(X) \) is not contained in \( H \) then \( rk(X \cap H) = rk(X) \);

(vi) if \( H_1, H_2 \) and \( H \) are distinct hyperplanes of \( \Pi \) of rank at least 2 with \( H_1 \cap H_2 \subseteq H \), then \( H_1 \cap H_2 = H_1 \cap H \) is a geometric hyperplane of \( H \).

An **affine polar space** (see [4]) is the space obtained by removing all points and lines from a polar space that are contained in a fixed geometric hyperplane of the polar space. Incidence is the incidence inherited from the polar space. Again, if \( \Pi \) is a non-degenerate polar space of rank at least 3, then the projective planes of \( \Pi \) provide us with a set of affine planes for the affine polar spaces obtained from \( \Pi \). In this case, the affine polar spaces are tacitly identified with the corresponding geometries the planes of which are the affine planes obtained from the projective planes of \( \Pi \).

In the remainder of this section we derive some properties of affine polar spaces and their quotients that will be useful in the proof of our main results. In particular, we show how Theorem 1.4 follows from Theorem 1.3.
Let $\Pi = (P, L)$ be a non-degenerate polar space of rank at least 2 and suppose that $H$ is a geometric hyperplane of $\Pi$. Then by $T_0$, we denote the affine polar space obtained by removing the points and lines of $H$ from $\Pi$. As is shown in [4, 14], the collinearity graph of $T$ has diameter at most 3. Furthermore, the reflexive extension of \textquoteleft being at distance 3\textquoteright defines an equivalence relation, denoted by $\ast$, on the point set of $T_0$ (see [14, 15]).

**Lemma 2.3.** Let $p$ and $q$ be two points of $T_0$. Then $p$ and $q$ have distance 3 in the collinearity graph of $T_0$ iff we have $p^\perp \cap q^\perp \subseteq H$ in $\Pi$.

**Proof.** The proof is straightforward.

If $\Pi$ has rank at least 3 then this equivalence relation extends to a standard equivalence relation on the line set of $T_0$.

**Proposition 2.4.** Suppose that $\Pi$ has rank at least 3. Then $\ast$ is a standard equivalence relation on $T_0$.

**Proof.** First we show that $\ast$ induces an equivalence relation on the line set of $T_0$. Let $x$ and $y$ be $\ast$-equivalent points of $T_0$. Let $l$ be a line on $x$. Then $l$ meets $H$ in a point $p$, say. Clearly, $p$ is the unique point on $l$ in $y^\perp$. Let $l'$ be the line through $y$ and $p$. We show that $l'$ is the unique line on $y$ with $l \ast l'$. Now assume that $z$ is a point on $l$ not in $H$ and let $m$ be a line on $z$ not in $x^\perp$. Then $m$ meets $H$ in a point $q$ not collinear to $p$. Let $z'$ be the point $q \cap l'$. We claim that $z' \cap z^\perp \subseteq H$.

Suppose the contrary. Then there is a point $r$ collinear with $z$ and $z'$ but not in $H$. Let $n_1$ and $n_2$ be two lines on $r$ in $z^\perp \cap z'^\perp$ in a singular subspace of $\Pi$. These lines exist as $\Pi$ has rank at least 3. Then $n_i$ meets $H$ in some point $s_i$, $i \in \{1, 2\}$. If $s_i$ is not collinear with $p$ then $p^\perp \cap z^\perp \cap z'^\perp$ contains a point of $n_i$ not in $H$. But that point is also collinear to both $x$ and $y$—a contradiction. Hence $p$ is collinear to both $s_1$ and $s_2$. So in $z^\perp \cap z'^\perp$, which is a non-degenerate polar space, we have $r^\perp \cap H \cap z^\perp \cap z'^\perp \subseteq p^\perp \cap H \cap z^\perp \cap z'^\perp$. But $r^\perp \cap H \cap z^\perp \cap z'^\perp$ is a geometric hyperplane of $H \cap z^\perp \cap z'^\perp$ which contains $p$, $q_1$, $s_1$ and $s_2$, and thus has rank at least 2. Therefore, by 2.2 we have $r^\perp \cap H \cap z^\perp \cap z'^\perp = p^\perp \cap H \cap z^\perp \cap z'^\perp$ or $p^\perp \cap H \cap z^\perp \cap z'^\perp = H \cap z^\perp \cap z'^\perp$. If $p \in r^\perp$ then $x \perp r \perp y$ against the assumption. So $p^\perp \cap q^\perp \subseteq H \cap z^\perp \cap z'^\perp$—again a contradiction.

Thus we have shown that $l \ast l'$ iff $l'$ is the unique line on $y$ that meets $l$ in $l \cap H$. This shows that $\ast$ defines an equivalence relation on the set of lines of $T_0$. Furthermore, as planes are singular subspaces meeting $H$ in a line, it is also clear from the above that $\ast$ defines an equivalence relation on the set of planes in $T_0$. This proves the proposition.

The standard quotient $T_0/\ast$ where $\ast$ is the equivalence relation of \textquoteleft being at distance 3\textquoteright in the collinearity graph of $T_0$, is called the minimal standard quotient of $T_0$ (see [15]).

**Corollary 2.5.** Suppose that $\Pi$ has rank at least 3. Then the affine polar space $T_0$ admits non-trivial standard quotients if and only if its collinearity graph has diameter 3.

Namely, $T_0$ admits non-trivial standard quotients if it is reducible. Furthermore, it will be shown in Section 5 that the minimal standard quotient of $T_0$ is the only irreducible standard quotient of $T_0$.

As is shown in [15], the above definition of a standard quotient of an affine polar space is equivalent to the definition given in [15]. Let $G_0$ be the pointwise stabilizer of...
Locally polar geometries

$H$ in $\text{Aut}(\Gamma_0) \subseteq \text{Aut}(\Pi)$. Then the fact that affine polar spaces are their own universal covers (see [14]) and some general results on universal covers and their quotients, (see [20] or [17]) imply the following.

**Corollary 2.6.** Suppose that $\Gamma_0$ admits standard quotients. Then the affine polar space $\Gamma_0$ is the universal cover of $\Gamma_0/\ast$. Furthermore, we have $\Gamma_0/\ast = \Gamma_0/G_0$, i.e. the $\ast$-equivalence classes are the $G_0$-orbits on the point set of $\Gamma_0$.

By [4] the hyperplane $H$ either contains a unique point in the radical or is non-degenerate. It is clear from Lemma 2.3 that the equivalence classes of $\ast$ and hence the $G_0$-orbits are hyperbolic lines of $\Pi$ not meeting $H$ when $H$ is non-degenerate, and meeting $H$ in its radical when $H$ is degenerate. This allows us to interpret $\Gamma_0/\ast$ as a tangent geometry in the meaning of Section 1 when $\Pi$ is a classical polar space of rank at least 3 defined over a division ring $K$ of characteristic $\neq 2$.

In this case we can assume $\Pi = (P, L)$ to be embedded in a projective space $P$. This means that we have a map of $\Pi$ into the point set of $P$ such that lines of $\Pi$ map onto lines of $P$. We identify the subspaces of $\Pi$ with their images in $P$. Furthermore, we can assume that the points of $\Pi$ generate $P$. Then, by the results of [12] and [19], there is a polarity $\alpha$ such that the points and lines of $\Pi$ are the absolute points respectively lines of $P$ with respect to this polarity. Suppose that $p$ is a point of $P$, then by $p^\perp$ we denote the subspace of $P$ orthogonal to $p$ with respect to $\alpha$.

The geometric hyperplane $H$ of $\Pi$ arises as a section of a hyperplane of $P$ with $\Pi$ (see [4, 12]), just as every hyperbolic line of $\Pi$ arises as a section of a line of $P$ with $\Pi$. Furthermore, if $h$ is a hyperbolic line of $\Pi$ then there is a codimension 2 subspace $X$ of $P$ such that $\langle h \rangle = X^\perp$. (Here $\langle Y \rangle$ denotes the subspace of $P$ generated by the points of $Y$.)

If the geometric hyperplane $H$ of $\Pi$ is non-degenerate then the map $h \mapsto \langle h \rangle \cap \langle H \rangle$ from the set of hyperbolic lines $h$ with $\langle h \rangle^\perp \subseteq \langle H \rangle$ to the non-singular points of $\langle H \rangle$, i.e. the set $\langle H \rangle \setminus H$, clearly induces an isomorphism between $\Gamma_0/\ast$ and one of the connected tangent geometries of the non-singular points in $\langle H \rangle$, as described in the introduction.

Now let us assume that $H$ is degenerate and $p$ is the unique point in the radical of $H$. There is a bijection between the hyperbolic lines on $p$, and hence the projective lines on $p$ that are not in the hyperplane $\langle p^\perp \rangle$ of $P$, and the geometric hyperplanes $q^\perp \cap p^\perp$ of $p^\perp$, where $q$ is a point of $\Pi$ not collinear to $p$. Again it is straightforward to check that this bijection extends to an isomorphism between $\Gamma_0/\ast$ and the tangent geometry of the projective lines on $p$ not in $\langle p^\perp \rangle$ regarded as points of the projective space of lines on $p$ and tangent to the polar space $\text{Res}(p)$. The points of this tangent geometry are the points of the affine space of projective lines on $p$ that are not in $\langle p^\perp \rangle$. We recall that particular instances of this ‘affine realization’ of the quotient $\Gamma_0/\ast$ have already been considered in [14] and [8] for symplectic (resp. unitary) polar spaces.

If the underlying division ring has characteristic 2 the above does not work. In particular, if the geometric hyperplane $H$ of $\Pi$ is non-degenerate we encounter some difficulties in mimicking the above constructions: for then there is a projective embedding of $\Pi$ (the dominant embedding, see [4, 12, 19]) in which $H$ can be obtained as a hyperplane section of the projective space with $\Pi$, but $\Pi$ need not be represented by a non-degenerate polarity. In that case $\Pi$ is represented by a pseudo-quadratic form such that the associated polarity is degenerate. The larger the radical of this polarity, the more trouble we have in mimicking the above.

However, the following shows the way out. As already noted above, the geometric hyperplanes of $\Pi$ are represented by hyperplane sections in the dominant projective
embedding of $\Pi$. This is true in all characteristics. In fact, if $\Pi$ is represented by some non-degenerate polarity in a projective space $\mathcal{P}$, then each point $q$ of $\mathcal{P}$ corresponds to the hyperplane $q^\perp$ in $\mathcal{P}$. This leads us to the following synthetic definition of tangent geometries.

**Definition 2.7.** Let $\Pi$ be a (possibly degenerate) polar space and let $\mathcal{P}$ be the set of non-degenerate geometric hyperplanes of $\Pi$. This set will be the point set of a tangent geometry. As lines we take the sets of elements of $\mathcal{P}$ containing a subspace $H_i \cap H_j$, where $H_i, H_j \in \mathcal{P}$ such that $H_i \cap H_j$ is a degenerate geometric hyperplane of both $H_i$ and $H_j$. Let $\mathcal{L}$ denote the set of all these sets. Clearly, $(\mathcal{P}, \mathcal{L})$ is a space. Finally, let $\mathcal{A}$ be the set of all subsets of $\mathcal{P}$, called planes, that are of the following type. Let $H_i, H_j, H_k \in \mathcal{P}$ such that $H_i \cap H_j$ is degenerate for $i, j \in \{1, 2, 3\}, i \neq j$ and the radical of $H_i \cap H_j \cap H_k$ is a line. Then a plane is the set \{ $H \in \mathcal{P} \mid H \supseteq H_i \cap H_j \cap H_k$ and $H \cap H_i$ is degenerate \}. Now consider the incidence system $(\mathcal{P}, \mathcal{L}, \mathcal{A})$, where incidence is symmetrized inclusion. Every union of connected components of this incidence system, in which there are points and lines and every line is in a plane, is called a tangent geometry of $\Pi$. 

**Proposition 2.8.** Let $\Pi$ be a polar space and $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{A})$ a tangent geometry of $\Pi$. Then $(\mathcal{P}, \mathcal{L})$ is a partial linear space and the planes of $\Gamma$ are singular subspaces of the space $(\mathcal{P}, \mathcal{L})$.

**Proof.** Let $H_1$ and $H_2$ be two collinear point of $\Gamma$ and suppose there are two lines $l_1$ and $l_2$ containing $H_1$ and $H_2$. By definition of the lines, there are elements $G_1, G'_1$ and $G_2, G'_2$ such that $l_i$ consists of all the elements of $\mathcal{P}$ containing $G_i \cap G'_i$, where $i = 1, 2$. Without loss of generality we can assume that $H_1 \neq G_i$, $i = 1, 2$. As $H_1$ and $H_2$ are points of $l$, we have $H_1 \cap H_2 \supseteq G_i \cap G'_i$ for $i = 1, 2$. As the element in $\Gamma$ contain at least a line they have rank at least 2. So, by Lemma 2.2(ii), $G_i \cap G'_i$ is a maximal subspace of $G_i$ contained in $G_i \cap H_1 \cap H_2$, which is a proper subspace of $G_i$. This implies that $G_i \cap G'_i = H_1 \cap H_2$ and hence $l_1 = l_2$. This proves that $(\mathcal{P}, \mathcal{L})$ is a partial linear space.

Now suppose that $\pi$ is a plane of $\Gamma$. Then there are three distinct collinear points $H_1, H_2$ and $H_3$ in $\mathcal{P}$ such that $\pi$ consists of all the elements of $\mathcal{P}$ containing $H_1 \cap H_2 \cap H_3 = l^\perp \cap H_1$ for some line $l$ in $H_1$, and intersect $H_1$ in a degenerate subspace. Let $H$ be an element of $\pi$. We first show that $H$ is collinear with $H_2$ and $H_3$. Let $p$ be the radical of $H \cap H_1$. Then $p^\perp \cap H_1 \supseteq H_1 \cap H_2 \cap H_3 = l^\perp \cap H_1$. Thus, as $H_1$ is non-degenerate, we have $p = p^{\perp \perp} \subseteq l^\perp = l$ in $H_1$ (see [12, 2.3]).

Now suppose that $H$ is not collinear with $H_2$. Then, as $H \cap H_2$ contains $l$, $H$ is a non-degenerate polar space of rank at least 2. If $q$ is the radical of $H_2 \cap H_1$ then by the same argument as above we find that $q$ is a point of $l$. We can assume that $q \neq p$, for otherwise $p^\perp \cap H_2 = p^{\perp \perp} \cap H_1 = p^{\perp \perp} \cap H_2$, which is against our assumption. Since $H$ is not collinear with $H_2$ the intersection $H \cap H_2$ contains a line on $p$ that is not in $q^\perp$. However, this line is inside $H \cap H_1$ and in $H \cap H_2$, and thus in $H_1 \cap H_2 = q^\perp \cap H_2$—a contradiction. Hence $H$ and $H_2$ are collinear, and by a similar argument $H$ is also collinear to $H_3$.

Let $H'$ be another element of $\pi$. Then by the above argument with $H_2$ replaced by $H$ we find that $H'$ and $H$ are collinear. This proves the second part of the proposition. 

By Lemmas 2.3 and 2.4 we immediately have the following.
COROLLARY 2.9. Let \( \Gamma_0 \) be an affine polar space obtained by removing a geometric hyperplane from a non-degenerate polar space of rank at least 3. Then the minimal standard quotient \( \Gamma_0/\ast \) is a tangent geometry of its polar space at infinity.

PROOF. Suppose we remove a geometric hyperplane \( H \) from a polar space \( \Pi \) to obtain the affine polar space \( \Gamma_0 \). Then the map \( p \mapsto H \cap p^\perp \) induces an isomorphism between \( \Gamma_0/\ast \) and a tangent space of \( H \). \( \square \)

We will now prove the converse of the above, provided that the polar space can be embedded in a projective space. This, together with Theorem 1.3, results in a proof of Theorem 1.4.

PROPOSITION 2.10. Let \( \Pi \) be a polar space embedded in a projective space \( P \) such that \( 2 \leq \text{rank}(\Pi) \) and the radical of \( \Pi \) is either empty or contains a unique point. Let \( \Gamma \) be a connected tangent geometry of \( \Pi \) such that the points of \( \Gamma \) arise as hyperplane sections in \( P \). Then \( \Gamma = \Gamma_0/\ast \) for some (up to isomorphism) uniquely determined affine polar space \( \Gamma_0 \) with polar space \( \Pi \) at infinity.

PROOF. As follows from the results of Tits [19], Johnson [12, 13] and Dienst [10], we can assume that \( \Pi \) is isomorphic to a polar space represented by a sesquilinear or a pseudo-quadratic form in \( P \), where \( P = \text{PG}(V) \) for some vector space \( V \). Let \( V' = V \oplus \langle v' \rangle \). First assume that \( \Pi \) is represented by a sesquilinear form \( f \). Let \( H \) be a hyperplane of \( P \) intersecting \( \Pi \) in a point of the tangent geometry \( \Gamma \). If \( \Pi \) is degenerate then let \( v \in V \setminus \{0\} \) be a vector such that \( \langle v \rangle = \text{Rad}(\Pi) \). Otherwise choose \( \langle v \rangle \) outside \( H \) with \( f(v, v) = 0 \). Consider \( V' \) equipped with the form \( f' \), where
\[
\begin{align*}
f'(v', v') &= 0, \\
f'(v', v) &= 1, \\
f'(v', u) &= 0 \quad \text{for all } \langle u \rangle \in H, \\
f_{|v+v'} &= f.
\end{align*}
\]
Then \( f' \) is a non-degenerate sesquilinear form on \( V' \), as follows by an easy computation. Therefore it represents a non-degenerate polar space \( \Pi' \) and \( \Gamma \) is a quotient of the affine polar space obtained by removing \( \Pi \) from \( \Pi' \). (Clearly, \( \Pi \) is a geometric hyperplane of \( \Pi' \).)

The case in which \( \Pi \) is represented by a pseudo-quadratic form can be handled in a similar way, and is left to the reader. This proves the existence of an affine polar space. Uniqueness follows from Corollary 2.6. \( \square \)

REMARK. Not all polar spaces do have tangent geometries. For example, if the radical of the polar space under consideration contains a line, then all geometric hyperplanes meet this line and hence are degenerate. Therefore the definition of tangent geometries is empty in this case. But also the definition of tangent geometries is empty for some non-degenerate polar spaces. If \( \Pi \) is a finite-dimensional non-degenerate symplectic polar spaces in odd characteristic or a non-degenerate non-embeddable polar space of rank 3, then it follows from [4] that all the geometric hyperplanes of \( \Pi \) are degenerate. So, in this case also, the definition of a tangent geometry is empty.

3. PRELIMINARIES

Let \( \Gamma = (P, \mathcal{L}, \mathcal{A}) \) be a geometry satisfying the conditions of Theorem 1.1. In this section we derive some properties of \( \Gamma \) that will be useful in subsequent sections.

PROPOSITION 3.1. Let \( x \) be a point of \( \Gamma \). Then \( \text{Res}(x) \) is a non-degenerate polar space. Furthermore, if \( \Gamma \) satisfies condition (v) of 1.3, then \( \text{Res}(x) \) has rank at least 3.
PROOF. Let \( \pi \) be a plane and \( l \) a line on \( x \) not in \( \pi \). Then by condition (iii) of Theorem 1.1 the line \( l \) is coplanar with one or all lines of \( \pi \) that contain \( x \). Hence we have checked the 'one or all' axiom which implies that \( \text{Res}(x) \) is a polar space. Furthermore, this space is non-degenerate, for, by condition (iv) of 1.1 no line on \( x \) is coplanar with all lines on \( x \).

Now suppose that \( \Gamma \) satisfies condition (v) of Theorem 1.3. Then there is a plane \( \pi \) and a point \( x \) not in \( \pi \) such that \( x \) is coplanar with all lines in \( \pi \). This implies that for all points \( y \) of \( \pi \) the polar space \( \text{Res}(y) \) has rank at least 3. By [12, 3.2], for every plane \( \pi' \) on a point \( y \) of \( \pi \), there is a point \( x \) that is coplanar to all lines in \( \pi' \). But then the polar spaces \( \text{Res}(z) \), where \( z \) is collinear to a point in \( \pi \), have rank at least 3, and by connectedness of \( \Gamma \) the proposition is proved.

The above proposition also tells us that every pair of distinct intersecting coplanar lines is in a unique plane. It readily follows from this and the fact that \((\mathcal{P}, \mathcal{L})\) is a partial linear space that every pair of distinct coplanar lines are in a unique plane.

The above result gives us information about the relations between lines and planes. The next lemma is concerned with points and planes.

**Lemma 3.2.** Let \( x \) be a point and \( \pi \) a plane of \( \Gamma \). If \( x \) is not in \( \pi \) then \( x^\perp \cap \pi \) consists of those points of \( \pi \) that are on the lines of \( \pi \) coplanar with \( x \). Furthermore, either \( x \) is coplanar with all the lines in \( \pi \) or all lines in \( \pi \) coplanar with \( x \) are pairwise parallel.

**Proof.** Let \( x \) be a point not in \( \pi \) such that \( x^\perp \cap \pi \) is non-empty. If \( y \) is a point of \( x^\perp \cap \pi \) then the line through \( x \) and \( y \) is coplanar with a line of \( \pi \) through \( y \). Hence \( y \) is on a line of \( \pi \) coplanar with \( x \). Furthermore, all points of \( \pi \) that are on a line coplanar with \( x \) are collinear to \( x \). The last statement of the lemma is an easy consequence of Proposition 3.1, and we have proved the lemma.

The collinearity graph of \( \Gamma \) is the graph with vertex set \( \mathcal{P} \) and two distinct vertices adjacent iff they are collinear. Let \( \delta \) denote the graph distance in the collinearity graph of \( \Gamma \). If \( x \) and \( y \) are two points of \( \Gamma \) at mutual distance 2, we define \( B(x, y) \) to be the set of lines on \( x \) that contain no point in \( y^\perp \). The following three results can be found in [14] or [4].

**Lemma 3.3.** Let \( x \) and \( y \) be two points of \( \Gamma \) with \( \delta(x, y) = 2 \). Then \( B(x, y) \) is a geometric hyperplane of \( \text{Res}(x) \).

**Lemma 3.4.** The collinearity graph of \( \Gamma \) has diameter at most 3. Moreover, if it has diameter 3 and \( x \) and \( y \) are two points at distance 3, then all point collinear with \( y \) are at distance 2 from \( x \) and, for every point \( z \) collinear to \( y \), the line through \( z \) and \( y \) is the radical of \( B(z, x) \).

**Lemma 3.5.** The reflexive extension of 'being at distance 3' in the collinearity graph of \( \Gamma \) is an equivalence relation on \( \mathcal{P} \).

### 4. The Polar Space at Infinity

Let \( \Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{A}) \) be a geometry satisfying the conditions of Theorem 1.1. In this section we construct a polar space in which we can embed a quotient of \( \Gamma \). This is done along the lines of Section 3 of [4]. Most of the results presented in that section of [4]
carry over to the more general situation considered in this paper. Indeed, the gamma space property assumed by Cohen and Shult in [4] can be replaced by our Lemma 3.2.

Following [4], we define an equivalence relation on the set $\mathcal{L}$ of lines in the following way. Given a line $l$ in $\mathcal{L}$ we set

$$\Delta(l) = \{x \in P \mid x^\perp \cap l = \emptyset \text{ or } x \text{ is coplanar with } l\}.$$ 

Two lines $l$ and $m$ of $\mathcal{L}$ are called parallel, notation $l \parallel m$, iff $\Delta(l) = \Delta(m)$. Parallelism defines an equivalence relation on the set $\mathcal{L}$. For each line $l \in \mathcal{L}$ we denote by $[l]$ the equivalence class of all lines parallel to $l$. This class will be called the parallel class of $l$.

In the following lemma we collect a number of properties of the equivalence relation of being parallel. For the proof of this lemma the reader is referred to Section 3 of [4]. Although a more restrictiv situation is considered in [4], the proofs remain valid in our setting.

**Lemma 4.1.** (i) Let $l$ and $m$ be two distinct coplanar lines of $\Gamma$. Then $l$ is parallel to $m$ iff they do not meet.

(ii) Let $x$ be a point and $l$ a line of $\Gamma$. Then there is a line $l' \in [l]$ such that $x^\perp \cap l'$ is non-empty. In particular, if $x \in \Delta(l)$ then there is a line $l' \in [l]$ that is incident with $x$.

(iii) Let $l$ be a line. Then no two distinct lines of $[l]$ intersect at a point.

(iv) If $l$ is a line then two distinct lines $m$ and $n$ of $[l]$ are either coplanar or $x^\perp \cap n$ is empty for all $x \in m$.

(v) Every plane of $\Gamma$ is uniquely determined by any set of non-collinear points contained in it.

The above shows that parallelism contains the transitive extension of the relation of being parallel in some affine plane. By this remark we obtain the following.

**Corollary 4.2.** Every singular subspace of $\Gamma$ in which every point–line pair is coplanar is an affine space.

It follows from Lemma 4.1 that we can form an incidence system the points of which are the elements of the parallel class $[l]$, where $l$ is some line of $\Gamma$, and the lines of which are the planes of $\Gamma$ containing an element of $[l]$, incidence being symmetrized inclusion. This space will be denoted by $\Delta(l)/l$.

**Lemma 4.3.** Let $x$ be a point and $l$ a line of $\Gamma$ such that $x \notin \Delta(l)$. Let $\pi$ be a plane on $x$. Then either one or all lines of $\pi$ that are incident with $x$ have no point in $\Delta(l)$.

**Proof.** This is Lemma 3.8 of [4].

By the above lemma we have that the set of lines and planes on $x$ not meeting $\Delta(l)$ form a geometric hyperplane of $\text{Res}(x)$. This hyperplane will be denoted as $B(x, [l])$. The complement of $B(x, [l])$ in $\text{Res}(x)$ is denoted by $A(x, [l])$. Clearly $A(x, [l])$ is an affine polar space. As shown in [4], the polar space $B(x, [l])$ is either non-degenerate or contains a unique point in the radical, which is called the deep point of $B(x, [l])$.

**Lemma 4.4.** Let $x$ be a point and $l$ a line such that $x \notin \Delta(l)$. Suppose that $m$ is a line through $x$. Then $m$ is the deep point of $B(x, [l])$ iff $\Delta(l) \cap \Delta(m) = \emptyset$ iff every plane meeting $\Delta(l)$ does not contain any line in $[m]$.

**Proof.** See Lemma 3.10 and Remark 3.11 of [4].
In order to distinguish between points, lines and other subspaces of $\Gamma$ and points, lines and subspaces of $Res(x)$, where $x$ is a point of $\Gamma$, we refer to the latter ones as 'points', 'lines' or 'subspaces'. Furthermore, the collinearity relation in $Res(x)$ will also be denoted by $\perp$. Let $p$ and $q$ be two non-collinear points in a polar space. Then the \textit{hyperbolic line} through $p$ and $q$ is the set $\{p, q\}^{\perp \perp}$.

**Lemma 4.5.** Let $x$ be a point and $l$ a line of $\Gamma$ with $x \notin \Delta(l)$. If there are at least two lines through $x$ meeting $l$ then all lines on $x$ meeting $l$ are 'points' of a hyperbolic 'line' $h$ of $Res(x)$. Furthermore, $h^\perp$ is contained in $B(x, [l])$.

**Proof.** Assume that there are at least two lines on $x$ meeting $l$, and let $m_1$ and $m_2$ be two such lines. Let $n$ be a line on $x$ coplanar with $m_1$ and $m_2$. Suppose that $\pi_i$ is the plane on $m_i$ containing $n$, $i = 1, 2$. Finally, let $y_i$ be the intersection point of $l$ and $m_i$, where $i = 1, 2$. Then $y_2$ is coplanar with $n$, and by Lemma 3.2 and Proposition 3.1 also with a line in $\pi_i$ through $y_i$ parallel to $n$. This implies that $n$ does not meet $\Delta(l)$, and proves the second part of the lemma. Furthermore, it shows that there is a plane on $l$, say $\pi$, meeting $\pi_1$ at a line parallel to $n$. However, by Lemma 1 of [14], there is a unique line at $x$ which is coplanar with all the lines in $\pi$ that are coplanar with $x$. Since the lines in $\pi$ that are coplanar with $x$ are in a parallel class of $\pi$ and parallel to $n$, this line has to be in $\pi_1$ and parallel to $n$ and thus has to be $n$. Therefore, if $m$ is some line on $x$ that meets $l$ in a point $y$, there is a line $m'$ in $\pi$ on $y$ coplanar with $m$ and $n$. By 4.1(v), $n$ is in the plane on $m$ and $m'$ and hence coplanar with $m$. This proves the lemma. \[\square\]

Let $x$ and $l$ be as in the above lemma. If there are at least two lines on $x$ intersecting $l$, then they determine a unique hyperbolic 'line' in $Res(x)$, which will be denoted by $h_l$. Now fix $x$ and $l$ as in Lemma 4.5. Suppose that there are at least two lines through $x$ intersecting $l$. Then all lines of $[l]$ are intersected by at least two of the lines on $x$, as follows from [14]. Furthermore, these lines meeting $\Delta(l)$ account for all the 'points' of $A(x, [l])$. 'Meeting the same line of $[l]$' defines an equivalence relation on the set of 'points' of $A(x, [l])$. Let $\delta_\Delta$ denote the graph distance in the collinearity graph of $\Delta(l)/l$.

**Lemma 4.6.** Let $n$ and $m$ be two lines in $[l]$. Then the following are equivalent:

(i) $n = m$ or $\delta_\Delta(n, m) = 3$;

(ii) $h_n = h_m$.

Furthermore, the diameter of $\Delta(l)/l$ is at most 3.

**Proof.** First observe that whenever two lines on $x$ meeting two distinct lines $n$ and $m$ in $[l]$ are coplanar, then by 4.1(iv) the two lines $n$ and $m$ in $[l]$ are coplanar. This proves that $\Delta(l)/l$ has diameter at most the diameter of $A(x, [l])$, which is less or equal than 3.

Assume that $n$ and $m$ are distinct lines in $[l]$. Let $\delta_\Delta(n, m) = 3$. Choose two different lines $m_1$ and $m_2$ through $x$ intersecting $m$ and let $m_3$ be a line through $x$ that meets $n$. If $k$ is a line on $x$ coplanar with $m_1$ and $m_2$, then there is a unique plane $\pi_k$ on $m$ that intersects the plane $\pi_i$ containing $k$ and $m_i$ in a line, where $i = 1, 2$ (see Lemma 1 of [14]). The line $m_3$ is coplanar with a line on $x$ in $\pi_i$, $i = 1, 2$. If that line is different from $k$ then there is a point $y$ in $\pi_k$ which is collinear with the intersection point of $n$ and $m_3$ and belongs to $\Delta(m) = \Delta(n)$. This would imply that $m$ and $n$ are at distance 2 in $\Delta(l)/l$, against our assumption. Hence $m_3$ is coplanar with $k$. This implies that $m_3$ is an element of the hyperbolic 'line' on $m_1$ and $m_2$. As $m_3$ is an arbitrary line on $x$ meeting $m$, this shows that $h_n = h_m$. 

\[\square\]
Locally polar geometries

Now set $h_m = h_n$ and suppose that $\delta_A(n, m) \leq 2$. Clearly, $m$ and $n$ are not coplanar, so they have distance 2 and there exists a line $k$ in $[l]$ coplanar to both $n$ and $m$. Let $k_x$ be a line on $x$ meeting $k$. Then $k_x$ is coplanar with a unique line in the plane spanned by $n$ and $k$ and with a unique line in the plane on $m$ and $k$. As $k_x$ is not coplanar with an element in $[l]$ it is coplanar with a unique line on $x$ meeting $n$ and a unique line on $x$ meeting $m$. This implies that $h_n \neq h_m$, a contradiction. We have proved the lemma. □

It follows from 4.6 that $\Delta(l)/l$ is a standard quotient of the affine polar space $A(x, [l])$.

**Lemma 4.7.** Let $x$ and $y$ be points and $l$ a line in $\Gamma$ such that $x, y \not\in \Delta(l)$. Then $B(x, [l])$ contains a deep point iff $B(y, [l])$ contains a deep point.

**Proof.** By the above we see that for both $x$ and $y$ the spaces $A(x, [l])$ and $A(y, [l])$ have a standard quotient isomorphic to $\Delta(l)/l$. As follows from Section 2 of [4] and 4.6, the space $A(z, [l])$ is obtained by removing a degenerate hyperplane from $Res(z)$ iff there are two distinct lines $L, M$ in $\Delta(l)/l$ such that we have $\Delta(L) \cap \Delta(M) = \emptyset$ in $\Delta(l)/l$, where $z = x, y$. Therefore, the isomorphism type of $\Delta(l)/l$ is sufficient to tell us whether the affine polar spaces $A(x, [l])$ and $A(y, [l])$ are obtained by removing a degenerate or non-degenerate hyperplane from $Res(x)$ resp. $Res(y)$, the particular choice of the points $x, y \not\in \Delta(l)$ having no effect on this. This proves the lemma. □

**Remark.** If $\Gamma$ satisfies condition (v) of Theorem 1.3, then by Proposition 4.1 the polar space $Res(x)$ has rank at least 3 for all points $x$ of $\Gamma$. In that case we can apply Corollary 2.6 to obtain that both $A(x, [l])$ and $A(y, [l])$ are universal covers of $\Delta(l)/l$ and hence are isomorphic. Futhermore, 2.7 of [4] implies that $Res(x) \cong Res(y)$ in this case.

Let $P_0$ denote the set of parallel classes of lines of $\Gamma$. Then we define the relation $\equiv$ on $P_0$ in the following way. If $[l], [m] \in P_0$, then $[l] \equiv [m]$ iff $\Delta(l) \cap \Delta(m) = \emptyset$. This defines an equivalence relation on $P_0$ as follows from Lemma 3.13 of [4]. The following is 3.14 of [4].

**Lemma 4.8.** Suppose that $\Delta(l) \cap \Delta(m) = \emptyset$ for two lines $l$ and $m$ of $\Gamma$. Then the sets $\Delta(n)$ for $[n]$ running through the $\equiv$-equivalence class of $[l]$ form a partition of $P$.

**Proof.** The proof of [4] remains valid. Instead of referring to 3.9 of [4] we can refer to the slightly weaker version of that result presented in 4.7. □

Let $[l]$ and $[m]$ be two element of $P_0$. Then we write $[l] \sim [m]$ iff there are lines $l' \in [l]$ and $m' \in [m]$ that are coplanar. In the following lemma we give some results concerning this relation. Again, most of these results can be found in [4].

**Lemma 4.9.** Let $l, m$ and $n$ be lines of $\Gamma$. Then:

(i) The subset $\Delta(l)$ is the point set of a subspace of $\Gamma$. Moreover, if $m \subseteq \Delta(l)$ then $[l] \sim [m]$.

(ii) If $l \cap \Delta(m) = \emptyset$ and $\Delta(l) \cap \Delta(m) \neq \emptyset$ then $[l] \sim [m]$.

(iii) If $l \cap \Delta(m) \neq \emptyset$ and $[l] \sim [m]$ then $l \subseteq \Delta(m)$.

(iv) If $[n] \sim [l] \equiv [m]$ then either $[n] = [l]$ or $[n] \sim [m]$.

(v) If $m \cap \Delta(l) \neq \emptyset$ then $[l] \sim [m]$ or $\Delta(m) \cap l \neq \emptyset$.

**Proof.** Statements (i)–(iv) are proved in [4], and (v) is a consequence of (i), (ii) and (iii). □
Now we can state the main result of this section. Let \( \mathcal{P}_\pi \) be the set \( \mathcal{P}_\pi \) if all \( \equiv \)-equivalence classes have size 1 and \( \mathcal{P}_\pi \cup \{\infty\} \) otherwise. Furthermore, let \( \mathcal{L}_\pi \) be the set consisting of all sets \( \{\infty\} \cup X \), where \( X \) is a \( \equiv \)-class of size at least 2 and all sets \( \{[l] \mid l \text{ is a line of } \pi\} \), where \( \pi \) is a plane of \( \Gamma \). Then we have the following theorem.

**Theorem 4.10.** The incidence system \( \Pi = (\mathcal{P}_\pi, \mathcal{L}_\pi) \), where incidence is symmetrized inclusion, is a polar space of rank at least 2 all of the lines of which have at least three points. Furthermore, the following are equivalent:
(i) \( \Pi \) is degenerate;
(ii) \( \Pi \) is degenerate and \( \infty \) is the unique point in the radical of \( \Pi \);
(iii) there is a point \( x \) and a line \( l \) in \( \Gamma \), \( x \notin \Delta(l) \), such that \( B(x, [l]) \) is degenerate;
(iv) for all points \( x \) and lines \( l \) in \( \Gamma \) with \( x \notin \Delta(l) \) the space \( B(x, [l]) \) is degenerate.

**Proof.** This follows from the proof of 4.1 in [4].

The incidence system \( \Pi = (\mathcal{P}_\pi, \mathcal{L}_\pi) \) will be called the polar space at infinity of \( \Gamma \). Collinearity in the polar space at infinity will also be denoted by \( \perp \).

5. The Embedding in the Polar Space at Infinity

In this section we embed the space \( \Gamma \) in its polar space at infinity and prove Theorems 1.1 and 1.2. Let \( \Gamma \) be a geometry satisfying the conditions of Theorem 1.1 and suppose that \( \Pi = (\mathcal{P}_\pi, \mathcal{L}_\pi) \) is its polar space at infinity. Then each point \( x \) of \( \Gamma \) determines a unique subset \( H_x \) of \( \mathcal{P}_\pi \) consisting of all the points \([l]\) of \( \Pi \), where \( l \) is a line of \( \Gamma \) through the point \( x \).

**Lemma 5.1.** Let \( x \) be a point of \( \Gamma \). Then \( H_x \) is a geometric hyperplane of \( \Pi \) isomorphic to Res(\( x \)).

**Proof.** Let \( L \) be a line of \( \Pi \). Then either \( L \) is of the form \( \{\infty\} \cup X \), where \( X \) is a \( \equiv \)-equivalence class, or \( L = \{[l] \mid l \text{ is a line in } \pi\} \), where \( \pi \) is a plane of \( \Gamma \). If \( L = \{[l] \mid l \text{ is a line in } \pi\} \) for some \( \equiv \)-class \( X \), then it follows from 4.7 that \( L \) meets \( H_x \) in a unique point \([l]\). If \( L = \{[l] \mid l \text{ is a line in } \pi\} \) for some plane \( \pi \) then either \( x \in \Delta(l) \) for all lines \( l \) in \( \pi \) and thus \( L \subseteq H_x \), or there is a line \( l \) in \( \pi \) with \( x \notin \Delta(l) \). Thus \( x \perp \pi \) is non-empty and by Lemma 3.2 there is a unique parallel class of lines of \( \pi \) in \( H_x \). Hence \( H_x \) is a geometric hyperplane of \( \Pi \). It remains to prove that \( H_x \) is isomorphic to Res(\( x \)).

Let \( l \) and \( m \) be two lines on \( x \) such that \([l]\) is collinear to \([m]\) in \( \Pi \). Then, by Lemma 4.9(iii), \( l \) and \( m \) are coplanar. On the other hand, if \( l \) and \( m \) are coplanar, then the points \([l]\) and \([m]\) are collinear in \( \Pi \) by definition. This proves the isomorphism.

The next lemmas allow us to define a quotient of \( \Gamma \) over the reflexive closure of the relation of 'having distance 3' in the collinearity graph of \( \Gamma \) (see Lemma 3.5).

**Lemma 5.2.** Let \( x \) and \( y \) be points of \( \Gamma \). Then \( H_x = H_y \) iff \( x = y \) or \( \delta(x, y) = 3 \).

**Proof.** This follows from Lemma 4.1.

**Lemma 5.3.** Let \( l \) be a line of \( \Gamma \). Then \( H_x \cap H_y = H_z \cap [l] \) for any pair of distinct points \( x \) and \( y \) on \( l \). Furthermore, if \( H \) is a non-degenerate hyperplane of \( \Pi \) and \( x, y \) are any two collinear points of \( \Gamma \) such that \( H_x \cap H_y \subseteq H \), then there is a point \( z \) on the line through \( x \) and \( y \) such that \( H = H_z \).
Let $x$ and $y$ be two points of the line $l$ of $I$. Then, clearly $[l]^\perp \cap H_x \subseteq H_x \cap H_y \supseteq [l]^\perp \cap H_y$. Since $[l]^\perp \cap H_x$, $[l]^\perp \cap H_y$, and $H_x \cap H_y$ are geometric hyperplanes of $H_x$ and $H_y$, respectively, the first part of the lemma follows from Lemma 2.2.

Now let $H$ be a non-degenerate hyperplane of $\Pi$ containing $[l]^\perp \cap H_x$, and suppose that $[m]$ is a point of $H$ not collinear to $[l]$. Then $\Delta(m) \cap l \neq \emptyset$, as follows from 4.9. Let $z$ be a point in $\Delta(m) \cap l$, then $H_z$ contains $[m]$ and $[l]^\perp \cap H_z$ by the above. However, $[l]^\perp \cap H_z$ is a geometric hyperplane of both $H$ and $H_z$. Thus, as both $H$ and $H_z$ have rank at least 2, we have $H_z = H$ by Lemma 2.2(ii).

Let $l$ be a line of $I$. By $J_l$, we denote the subspace $H_x \cap H_y$ of $\Pi$, where $x$ and $y$ are two distinct points of $l$. By the above lemma this is independent of the choice of $x$ and $y$.

**Lemma 5.4.** Let $l$ and $m$ be distinct lines. Then the following are equivalent:

(i) $J_l = J_m$;
(ii) for every point $p \in l$ there is a point $q \in m$ with $\delta(p, q) = 3$;
(iii) the lines $l$ and $m$ are parallel and $\delta_\Delta(l, m) = 3$.

**Proof.** The equivalence of (i) and (ii) follows from Lemmas 5.2 and 5.3. It remains to prove that (iii) is equivalent to (i) and (ii).

Suppose that (i) and (ii) hold. Then $[l]$ is the radical of $J_l$ and $[m]$ is the radical of $J_m$. But since $J_l = J_m$, we have $[l] = [m]$. Now consider $\Delta(l) \cap l$. By (ii) we have $\delta_\Delta(l, m) = 3$ and (iii) follows from 4.6.

Now assume that (iii) holds. Let $[n]$ be a point of $J_l$. A point $[n]$ of $\Pi$ is in $J_l$ iff there is an $n' \in [n]$ intersecting $l$ and coplanar with $l$. Let $n'$ be that line in $[n]$. Then $n' \subseteq \Delta(m)$. Fix a point $x$ coplanar with $n$ but not in $\Delta(l)$. Then there is a unique line $n''$ on $x$ that is in $[n]$ and coplanar with all the lines through $x$ meeting $l$. By Lemma 4.6, the lines through $x$ meeting $m$ are coplanar with $n''$. Therefore, as follows from Lemma 1 of [14], there is a line $n'''$ in $[n]$ that is coplanar with $m$. Hence $[n] \in J_m$ and $J_l \subseteq J_m$, and (i) follows by symmetry of the argument. This proves the lemma.

Let $H$ be the set of all geometric hyperplanes $H_x$ of $\Pi$, where $x$ is a point of $I$ and $\mathcal{H}$ is the set of all subhyperplanes $H_x \cap H_y$, where $x$ and $y$ are two distinct but collinear point of $I$. We identify the elements $J_l$ with the subsets $\{H_x \mid x \in l\}$ of $\Pi$. Furthermore, we denote by $\mathcal{H}$ the set of all sets $\{H_x \mid x \in \pi\}$, where $\pi \in \mathcal{A}$.

Now we can consider the geometry $\tilde{\Gamma} = (\mathcal{H}, \mathcal{F}, \mathcal{A})$, where incidence is symmetrized inclusion. This geometry is a tangent geometry of $\Pi$, as easily follows from Lemma 5.3. Furthermore, $\tilde{\Gamma}$ is a quotient geometry of $\Gamma$ and it is irreducible and isomorphic to $\Gamma$ iff $\Gamma$ is irreducible. In fact, if $\star$ denotes the relation of 'having distance 3' in the collinearity graph of $\Gamma$, then we have that $\tilde{\Gamma} \cong \Gamma/\star$. The above result together with Lemma 5.3 shows that two hyperplanes $H_x$ and $H_y$ are collinear in $\tilde{\Gamma}$ iff they intersect in a degenerate subspace of each other. This proves Theorem 1.2 and hence 1.1.

**6. Projective Embeddings**

In this final section, we prove Theorem 1.3. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{A})$ be a geometry satisfying the hypotheses of Theorem 1.3. Then by Theorem 1.2 we can identify the points of $\Gamma$ with geometric hyperplanes of its polar space $\Pi$ at infinity.

The Veldkamp space of a polar space $\mathcal{F}$, denoted by $V(\mathcal{F})$, is the incidence structure the point set of which is the set $\mathcal{H}(\mathcal{F})$ of all geometric hyperplanes of $\mathcal{F}$, and the lines of which are the sets $\{H \in \mathcal{H} \mid H \supseteq H_1 \cap H_2\}$, where $H_1$ and $H_2$ are two distinct
geometric hyperplanes of \( \mathcal{F} \), incidence being symmetrized inclusion. The following result is proved in [1].

**Theorem 6.1 ([1])**. *Let \( \mathcal{F} \) be a non-degenerate polar space of rank at least 3, all the singular subspaces of which are Desarguesian projective spaces. Then the Veldkamp space \( \mathcal{V}(\mathcal{F}) \) of \( \mathcal{F} \) is a projective space.*

We will use the above theorem to prove Theorem 1.3 in the case that the polar space at infinity of \( \Gamma \) is non-degenerate.

**Corollary 6.2.** *Let \( \Pi \) be a non-degenerate polar space. Then the Veldkamp space \( \mathcal{V}(\Pi) \) of \( \Pi \) is a projective space.*

**Proof.** By construction, we have that \( \Pi \) is a polar space of rank at least 3. By condition (v) of 1.3 and Corollary 4.2 every singular subspace of \( \Gamma \), in which every point line pair is coplanar, is an affine space of dimension at least 3. This implies that the singular subspaces of \( \Pi \) are Desarguesian. So we can apply Theorem 6.1.

By \( \mathcal{V}(\Gamma) \) we denote the subspace of \( \mathcal{V}(\Pi) \) generated by the geometric hyperplanes \( H_x \) of \( \Pi \), where \( x \) is a point of \( \Gamma \). For every point \( p \) of \( \Pi \), we denote by \( \mathcal{V}(p) \) the set of all points of \( \mathcal{V}(\Gamma) \) containing \( p \).

**Lemma 6.3.** *Suppose that \( \Pi \) is non-degenerate and let \( p \) be a point of \( \Pi \). Then \( \mathcal{V}(p) \) is the point set of a hyperplane of \( \mathcal{V}(\Gamma) \).*

**Proof.** First of all, notice that \( \mathcal{V}(p) \) forms a proper subspace of \( \mathcal{V}(\Gamma) \). So we only have to prove that this is a maximal subspace.

Since \( \Gamma \) is connected, \( \mathcal{V}(\Gamma) \) is generated by the points \( q^+ \), where \( q \) is a point of \( \Pi \), and one point \( H_x \), where \( x \) is a point of \( \Gamma \). Clearly, we can choose \( x \) in such a way that \( H_x \) contains \( p \). Thus it suffices to show that \( \mathcal{V}(p) \) intersects the subspace of \( \mathcal{V}(\Gamma) \) generated by the points \( q^+ \) in a hyperplane. Indeed, this follows from the fact that \( \Pi \) is generated by all points of \( \Pi \) collinear to \( p \), i.e. the hyperplane \( p^+ \), together with one point not in \( p^+ \).

Let \( \Pi \) be non-degenerate. Suppose that \( p \) and \( q \) are two distinct but collinear points of \( \Pi \) and let \( W \) be a hyperplane containing \( \mathcal{V}(p) \cap \mathcal{V}(q) \). Then let \( H \) be a point of \( W \) not in \( \mathcal{V}(p) \cap \mathcal{V}(q) \). As \( H \) is a geometric hyperplane of \( \Pi \) there is a point \( r \in H \) on the line of \( \Pi \) through \( p \) and \( q \). Hence \( H \) and \( \mathcal{V}(p) \cap \mathcal{V}(q) \) are contained in \( \mathcal{V}(r) \). This shows that the map \( p \mapsto \mathcal{V}(p) \), where \( \mathcal{V}(p) \) is identified with the hyperplane it spans, maps lines of \( \Pi \) onto lines of the dual space \( \mathcal{V}(\Gamma)^* \) of \( \mathcal{V}(\Gamma) \), and thus defines an embedding of \( \Pi \) into the space \( \mathcal{V}(\Gamma)^* \).

Let \( \mathcal{P} \) be the subspace of \( \mathcal{V}(\Gamma)^* \) generated by the elements \( V(p) \), where \( p \) is a point of \( \Pi \). Every point \( x \) of \( \Gamma \) determines a unique subspace \( H_x^* = \{ p \in \mathcal{P} \mid H_x \in p \} \) of \( \mathcal{P} \). If \( p \in P_x \) and \( x \in \mathcal{P} \), then \( V(p) \in H_x^* \) if and only if \( p \in H_x^* \). This shows that \( H_x^* \) is an hyperplane of \( \mathcal{P} \) and, using Lemma 5.3, that the map \( x \mapsto H_x^* \) induces an isomorphism between \( \Gamma \) and one of the projective tangent geometries of \( \Pi \) embedded in \( \mathcal{P} \). This finishes the proof of Theorem 1.3 in the case in which the polar space at infinity is non-degenerate.

**Remark.** In the above proof it becomes clear why we have to switch from points to hyperplanes to catch the right tangent geometries. It is possible that \( \mathcal{V}(\Gamma) \) contains \( \langle p^+ \mid p \in P_x \rangle \) as a hyperplane. This would imply that every point \( H_x \) in \( \mathcal{V}(\Gamma) \), \( x \in \mathcal{P} \), is on a tangent line with every point \( p^+ \), \( p \in P_x \), and not only with the points \( p^+ \) of \( \mathcal{V}(\Gamma) \) where \( p \in H_x \). By switching to the dual space, this problem has been avoided.
In the remainder of this section we consider the case in which II is degenerate. Since no analogue of Theorem 6.1 exists for the degenerate case, we will show that the subspace \( V(\Gamma) \) of the Veldkamp space \( V(\Pi) \) is a projective space in which all points of \( \Gamma \) can be identified with points. As before, we then switch to the dual space to obtain the points of \( \Gamma \) as hyperplane sections of a projective space. Here we follow the lines of the proof of Section 3 of [9].

Let \( x \) be a point of \( \Gamma \). As before, we denote by \( H_\lambda \) the unique geometric hyperplane of \( \Pi \) consisting of all the points \([l]\) of \( \Pi \), where \( l \) is a line of \( \Gamma \) containing \( x \). Now let \( x \) and \( y \) be two distinct points in \( \Gamma \). Then we define \( H^{xy} \) to be the union of all lines of \( \Pi \) through \( \propto \) meeting \( H_\lambda \cap H_\eta \).

**Lemma 6.4.** Let \( x \) and \( y \) be two distinct points of \( \Gamma \). Then \( H^{xy} \) is a geometric hyperplane of \( \Pi \).

**Proof.** Let \( L \) be a line of \( \Pi \) not containing \( \propto \), and suppose that \([l]\) and \([m]\) are two distinct points on \( L \). Denote by \([l_1]\) (resp. \([m_1]\)) the intersection points of the lines through \( \propto \) and \([l]\) (resp. \([m]\)) and \( H_\lambda \). Then \([l_1]\) and \([m_1]\) are collinear. Let \( L_\lambda \) denote the line in \( \Pi \) containing \([l_1]\) and \([m_1]\). Now suppose that \([n]\) is a point on \( L \). Consider the intersection point \([n_1]\) of \( H_\lambda \) with the line through \([n]\) and \( \propto \). For every point \([k]\) in \( H_\lambda \) we have that whenever \([k]\) \( \propto \) \([l_1]\), \([m_1]\) then \([k]\) \( \propto \) \([l]\), \([m]\) and hence \([k]\) \( \propto \) \([n]\), \([n_1]\). Thus \([n_1]\) \( \in \) \( L_\lambda \) = \( L_\lambda \) (see [3] or [12, Lemma 3.1]). Hence the map \([n] \mapsto [n_1]\) maps \( L \) into \( L_\lambda \).

Since \( L \) does not contain \( \propto \) there is a point \( z \) such that \( H_\lambda \) contains \( L \). Let \( z \) be such a point in \( \Gamma \) with \( H_\lambda \) containing \( L \). Then we can define a map from \( L \) to \( H_\lambda \) as above, which maps \( L_\lambda \) into \( L_\lambda \). This map is the inverse of the above map. This shows that the map \([n_1] \mapsto [n]\) is onto.

The line \( L_\lambda \) either meets \( H_\lambda \cap H_\eta \) in one point or is completely contained in \( H_\lambda \cap H_\eta \). This implies that \( L \) meets \( H^{xy} \) in one point or is contained in \( H^{xy} \).

Finally, if \( L \) is a line on \( \propto \) then clearly either \( L \) is in \( H^{xy} \) or \( \propto \) is the unique point on \( L \) in \( H^{xy} \). This proves the lemma. \( \square \)

By \( \mathcal{H} \) we denote the set of geometric hyperplanes of \( \Pi \) of the form \( H_\lambda \) and \( H^{xy} \), where \( x \) and \( y \) are distinct points of \( \Pi \).

**Lemma 6.5.** Let \( x \) and \( y \) be distinct points of \( \Gamma \) and \([l]\) \( \in \) \( H_\lambda \setminus (H_\lambda \cap H_\eta) \). Then there is a point \([m]\) in \( H_\lambda \) not collinear to \([l]\) such that \([l]\) \( \perp \) \([m]\) \( \cap \) \( H_\lambda \setminus H_\eta \). \( [l]\) \( \not\subset \) \([m]\). Hence \([m]\) is the point of \( \Pi \) we are looking for.

**Proof.** First consider the case that \( x \) and \( y \) are collinear by a line \( m \). Then \( H_\lambda \cap H_\eta = [m] \setminus H_\lambda \) and \([l]\) \( \not\subset \) \([m]\). Hence \([m]\) is the point of \( \Pi \) we are looking for.

Now assume that \( x \) and \( y \) are non-collinear. Then, as \([l]\) \( \not\in \) \( H_\lambda \), there is a line \( m' \) on \( y \) meeting \( \Delta(l) \). Let \( m \) be the deep point of \( B(x, [m']) \). Then \([m]\) \( \not\subset \) \([l]\) and, by Lemma 1 of [14], we have for every line \( k \) on \( x \) that if \([k]\) \( \perp \) \([l]\), \([m]\), then \([k]\) \( \perp \) \([m']\) and \([k]\) \( \in \) \( H_\lambda \).

\( \square \)

**Lemma 6.6.** Let \( p \) be point of \( \Pi \) and \( x \) and \( y \) two distinct points of \( \Gamma \). Then there is an element \( H \in \mathcal{H} \) such that \( H \) contains \( H_\lambda \cap H_\eta \) and \( p \).

**Proof.** Let \( p \) be a point of \( \Pi \). Clearly, we can assume that \( p \not\in H_\lambda \cup H_\eta \cup H^{xy} \). So let \( p = [l] \) for some line \( l \) of \( \Gamma \). Let \([l]\) be the intersection point of the line through \( \propto \) and \([l]\) with \( H_\lambda \). By the above lemma there exists a line \( m \), in \( \Gamma \) containing \( x \) such that
Let \([n]\) be a point of \((H_v \cap H_x) \setminus \{m_x\} \cup \{l_x\} \cap H_x\). Such a point \([n]\) exists, for otherwise \(H_x \cap H_v \subseteq \{l_x\} \cup \{m_x\} \cap H_x\) and thus \(H^{xv} \subseteq \{l_x\} \cup \{m_x\}\), which contradicts the fact that \(H^{xv}\) is a geometric hyperplane. By Lemma 2.2 we have that \(H_x \cap H_v\) is generated by \([n]\) and its geometric hyperplane \([l_x] \cup \{m_x\} \cap H_x\). Since \([l_x] \cup \{m_x\}\) we have \([l_x] \cup \{m_x\}\) and hence \(m_x \cap \Delta(l_x)\) is non-empty, by Lemma 4.9(ii). Hence we can assume without loss of generality that \(l_x\) meets \(m_x\). If \([n] \perp [l_x]\) then \(H_x \cap H_v \subseteq H_x \cap \{l_x\} \cup \{m_x\}\). As both sides of the inclusion relation are hyperplanes of \(H_x\), we have equality, which implies that \([l_x] \cap H^{xv}\) a contradiction. Thus \([n] \perp [l_x]\) and we have \(l_x \cap \Delta(n) \neq \emptyset\), by Lemma 4.9(ii). Let \(z\) be a point on \(l_x\) in \(\Delta(n)\). Then \(H_x\) contains \([n]\), \([l_x]\) and \(H_x \cap H_v \supseteq H_x \cap \{m_x\} \cup [l_x] \cup [m_x]\). Hence \(H_x \cap H_v \supseteq H_x \cap H_x\), which has rank at least 2. This proves that \(H^{xv} = H^{xv}\). 

**Lemma 6.7.** Let \(x, y\) and \(z\) be different points in \(\Gamma\). Then there is a point \(u\) in \(\Gamma\) such that \(H^{xv} = H^{xz}\).

**Proof.** As \(\Gamma\) is connected we may assume, without loss of generality, that \(x\) and \(z\) are collinear. Let \(l_x\) be the line through \(x\) and \(z\). Then by \(l_x\), we denote the unique line of \(\Delta(n)\) that \(l_x\) is on the line of \(\Pi\) through \(\infty\) and \([l_x]\). If \(H_x \cap H_v \subseteq H_x\) then, clearly, \(H^{xz} = H^{xv}\). Thus we can assume that there is a point \([m]\) in \(H_x \cap H_v \setminus (H_x \cap H_v \cap H_x)\). Let \(m_x\) be the unique line on \(z\) such that \([m_x]\) is on the line of \(\Pi\) through \([m]\) and \(\infty\). Then, by 4.9(v) there is a point \(u\) in \(l_x \cap \Delta(m_x)\). We have \(H_u \cap H_v = [l_x] \cup H_v \supseteq [l_x] \cup H_x \cap H_v = H_x \cap H_v \cap H_x \cap H_x\). Hence \(H^{xz}\) contains \([m_x]\) and thus \([m]\) and \(H_x \cap H_v \cap H_x\). By Lemma 2.2 we also have that \(H_x \cap H_v \cap H_x\), together with \([m]\) generates \(H_x \cap H_v\), which has rank at least 2. This proves that \(H^{xz} = H^{xv}\).

**Lemma 6.8.** Let \(p\) be a point of \(\Pi\), and \(H_1\) and \(H_2\) two distinct elements of \(\mathcal{H}\). Then there is an element \(H \in \mathcal{H}\) containing \(H_1 \cap H_2\) and \(p\).

**Proof.** See Lemma 7 of [9].

**Proposition 6.9.** The space \(V(\Gamma)\) is a projective space.

**Proof.** By Lemma 1 of [9] and Lemma 6.8 we obtain that the space \(V(\Gamma)\) has the set \(\mathcal{H}\) as point set. Thus the proposition follows by the same arguments as used in the proof of Proposition 1 of [9].

As follows from Lemma 6.8, each point \(p\) of \(\Pi\) determines the hyperplane of the space \(V(\Gamma)\) consisting of all the points of \(V(\Gamma)\) that contain \(p\). This hyperplane will be denoted by \(V(p)\). As before, the map \(p \mapsto V(p)\) defines an embedding of \(\Pi\) into the dual space of \(V(\Gamma)\). Let \(P\) be the subspace of \(V(\Gamma)^*\) generated by the elements \(V(p)\), where \(p\) is a point of \(\Pi\).

By the same arguments as in the case that \(\Pi\) is non-degenerate, we find that every point \(x\) of \(\Gamma\) determines a unique hyperplane \(H^x = \{p \in P \mid H_x \subseteq p\}\) in \(P\). Using Lemma 5.3, it is now straightforward to check that the map \(x \mapsto H^x\) induces an isomorphism between \(\Gamma\) and one of the projective tangent geometries of \(\Pi\) inside \(P\). This proves Theorem 1.3 in the case in which \(\Pi\) is degenerate.

**References**


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