Enumeration of Maps Regardless of Genus.
Geometric Approach.

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Abstract
We use the conceptual idea of “maps on orbifolds” and the theory of the non-Euclidian crystallographic groups (NEC groups) to enumerate rooted and unrooted maps (both sensed and unsensed) on surfaces regardless of genus. As a consequence we deduce a formula for the number of chiral pairs of maps. The enumeration principle used in this paper is due to A. D. Mednykh (2006), it counts the number of conjugacy classes of subgroups in NEC groups which are in one-two-one correspondence with unrooted (sensed or unsensed) maps.

Keywords: sensed or unsensed map; rooted map; chiral pairs of maps; chiral twins of maps; maps on orbifolds; maps on bordered surfaces; maps on closed surfaces; reflexible maps

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Introduction

A sensed (respectively unsensed) map is an equivalence class of maps on a closed orientable surface, where the equivalence relation is given by sense-preserving (respectively, sense-preserving or -reversing) map homeomorphisms. Wormald [Wor1], [Wor2] and Walsh [Wa05] have calculated sensed and unsensed 1-, 2- and 3-connected planar maps. A formula for the number of sensed planar maps with a given number of edges was obtained by Liskovets [Li85]. More recently, Mednykh and Nedela [MeN06] enumerated sensed maps of a given genus.

In the present paper we employ a geometric approach to enumerate sensed and unsensed maps based on the enumeration of rooted maps on cyclic orbifolds and on the determination of the unrooted coefficients in terms of torsion free epimorphisms from orbifold fundamental groups onto cyclic groups. This method further extends the one used in [MeN06] and [MN06] to count sensed unrooted maps and hypermaps on closed orientable surfaces of given genus. We also deduce formulas that give the number of “reflexible” maps and the number of chiral pairs (or twins) of maps. A sensed map \( M \) may, or may not, be isomorphic to its mirror image. In the first case \( M \) is reflexible, while in the second case it gives rise to a chiral pair of maps. Enumerating twins in a given family of maps is notoriously a hard problem. To our knowledge there are no published results on enumerations of twins regardless of genus. The present paper provides such an enumeration formula.

We believe that an asymptotic analysis of the formulas derived in this paper is possible and that such analyses would resolve an widely circulated belief, but not yet proven, that almost all maps are chiral.

Although we only enumerate rooted and unrooted maps on surfaces regardless of genus, we point out that the method used here is quite general in the sense that it allows, for instance, the enumeration of sensed and unsensed maps with a given distribution of vertices, edges and faces, provided the corresponding problem is already solved for rooted maps on closed and bordered surfaces. It is worth to mention that our enumeration results can be translated into results on enumeration of free subgroups and their conjugacy classes in certain universal Fuchsian groups.

1 Preliminaries

We start by listing some statements that will be extensively used in the subsequent sections.

The signature \( \sigma(\alpha) = (1^{s_1}2^{s_2}...n^{s_n}) \) of a permutation \( \alpha \) of degree \( n \) determines the cycle structure of \( \alpha \); the term \( k^{s_k} \) means that \( \alpha \) has \( s_k \) cycles of length \( k \) (\( k = 1, 2, \ldots, n \)). A formula yielding the number of permutations of given signature, which is well-known (see for example [St97], Prop. 1.3.2, p. 18), is reproduced in the following lemma.

Lemma 1.1 The number of permutations of degree \( n \) with signature \( \sigma(\alpha) = (1^{s_1}2^{s_2}...n^{s_n}) \)
is given by

\[ \nu(\alpha) = \frac{n!}{1^{s_1} s_1! 2^{s_2} s_2! \cdots n^{s_n} s_n!} = \frac{n!}{\prod_{k=1}^{n} k^{s_k} s_k!}. \]

Let \( G = \langle x_1, x_2, \ldots, x_r \rangle \) be a group generated by \( x_1, x_2, \ldots, x_r \) and let \( f : G \to S_n \) be a homomorphism satisfying some property \( \mathcal{P} \). We say that \( \mathcal{P} \) is \textit{invariant under conjugation} if for any \( \sigma \in S_n \) the homomorphism \( f^\sigma : G \to S_n, g \mapsto f^\sigma(g) = \sigma \circ f(g) \circ \sigma^{-1} \) also satisfies \( \mathcal{P} \). Typical examples of invariant properties \( \mathcal{P} \) under conjugation are:

- \( f \) satisfies \( \mathcal{P} \) if the images \( x_1 = f(x_1), x_2 = f(x_2), \ldots, x_n = f(x_n) \) act fixed point freely on the set \( \{1, 2, \ldots, n\} \);
- \( f \) satisfies \( \mathcal{P} \) if for any \( i = 1, 2, \ldots, n \) the element \( x_i = f(x_i) \) has fixed cycle structure \( \sigma(x_i) = (1^{s_1} 2^{s_2} \cdots n^{s_n}) \).

Let \( G \) be a finitely generated group and let \( \mathcal{H}om^\mathcal{P}(G, S_n) \) be the set of homomorphisms satisfying a property \( \mathcal{P} \) invariant under conjugation. Each element \( f \in \mathcal{H}om^\mathcal{P}(G, S_n) \) defines an action of \( G \) on the set \( \{1, 2, \ldots, n\} \). Denote by \( \mathcal{T}ran^\mathcal{P}(G, S_n) \) the subset of \( \mathcal{H}om^\mathcal{P}(G, S_n) \) composed of the transitive actions of \( G \) on \( \{1, 2, \ldots, n\} \). The elements of \( \mathcal{T}ran^\mathcal{P}(G, S_n) \) will be called \textit{transitive homomorphisms}.

Let \( \mathcal{P} \) be a set of subgroups of a finitely generated group \( \Gamma \), closed under conjugation. Denote by \( \mathcal{E}pi^\mathcal{P}(K, \mathbb{Z}_\ell) \) the number of epimorphisms \( K \to \mathbb{Z}_\ell \) with kernel in \( \mathcal{P} \).

By [Me06] and [MN06] we have the following counting lemma.

**Lemma 1.2** The number \( B_n^\mathcal{P} \) of homomorphisms \( f : G \to S_n \) satisfying a property \( \mathcal{P} \) invariant under conjugation, and the number \( T_n^\mathcal{P} \) of transitive homomorphisms with the same property are related as follows:

1) \[ B_n^\mathcal{P} = \sum_{i+j=n} \binom{n-1}{j} T_i^\mathcal{P} B_j^\mathcal{P} \]

2) \[ \sum_{k=1}^{\infty} \frac{T_k^\mathcal{P}}{k! z^k} = \log \sum_{k=0}^{\infty} \frac{B_k^\mathcal{P}}{k! z^k}. \]

**Remark.** This has first appeared in the paper by Hurwitz [Hu891] where he had computed the number of non-equivalent coverings over the sphere having simple branch points of order two. A similar result has been used later by Hall [Ha49] to calculate the number of subgroups of given index in a free group. In a more general form it has appeared (though not stated in a form of generating-functions) in [De65] and [Wo77].

Let \( \mathcal{P} \) be a set of subgroups of a finitely generated group \( \Gamma \), closed under conjugation. Denote by \( \mathcal{E}pi_\mathcal{P}(K, \mathbb{Z}_\ell) \) the number of epimorphisms \( K \to \mathbb{Z}_\ell \) with kernel in \( \mathcal{P} \).

By [Me06] and [MN06] we have the following counting lemma.
Theorem 1.3 Let $\Gamma$ be a finitely generated group. Let $P$ be a set of subgroups of $\Gamma$ closed under conjugation. Then the number of conjugacy classes of subgroups in $P$ of index $n$ in $\Gamma$ is given by the formula

$$N^P_\Gamma(n) = \frac{1}{n} \sum_{\ell|n} \sum_{K < \Gamma \ [\Gamma:K]=\ell} \text{Epi}_P(K, \mathbb{Z}_\ell).$$

As we shall see later, conjugacy classes of subgroups of a certain universal group are closely related with isomorphism classes of maps. The need to find $\text{Epi}_P(K, \mathbb{Z}_\ell)$ is clear from Theorem 1.3. In this direction the following lemma, by Jones [Jo95], is useful.

Lemma 1.4 Let $K$ be a finitely generated group. Then the number $\text{Hom}(K, \mathbb{Z}_\ell)$ of homomorphisms and the number $\text{Epi}(K, \mathbb{Z}_\ell)$ of epimorphisms $K \to \mathbb{Z}_\ell$ are related by

\begin{enumerate}
\item $\text{Hom}(K, \mathbb{Z}_\ell) = \sum_{d|\ell} \text{Epi}(K, \mathbb{Z}_d)$;
\item $\text{Epi}(K, \mathbb{Z}_\ell) = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) \text{Hom}(K, \mathbb{Z}_d)$.
\end{enumerate}

where $\mu$ is the arithmetic Möbius function.

In what follows we shall employ a particular generalization of Lemma 1.4. Let $G$ be a group and $\omega : G \to \mathbb{Z}_2 = \{1, -1\}$ be a homomorphism. Following [SSk92] we shall call a pair $(G, \omega)$ a group with sign structure. An element $x \in G$ will be called positive if $\omega(x) = 1$ and negative if $\omega(x) = -1$. Equivalently, since the image of a homomorphism is determined by its kernel, a group with sign structure is a couple $(G, G^+)$, where $G^+$ is the subgroup of positive elements. Let $G = (G, \omega)$ and $A = (A, \eta)$ be two groups with sign structure. A homomorphism $\psi : G \to A$ is said to be orientation-preserving if $\psi(G^+) \subseteq A^+$ and $\psi(G^-) \subseteq A^-$. An epimorphism $\psi : G \to A$ is orientation-preserving if and only if its kernel is a subgroup of $G^+$. Note that the commutator $[a, b] = aba^{-1}b^{-1}$ of two elements in a group with signed structure is always positive. Thus the derived group $G' = [G, G] < G^+$. It follows that there is an induced signature on the abelianization $H_1(G) = G/G'$ with group positive elements $G^+/G'$. Denote by $\text{Hom}^+(G, A)$ and $\text{Epi}^+(G, A)$ the respective numbers of orientation-preserving homomorphisms and epimorphisms $G \to A$. As a consequence we have,

Lemma 1.5 Let $G = (G, \omega)$ and $A = (A, \eta)$ be groups with sign structure and let $A$ be abelian. Then $\text{Hom}^+(G, A) = \text{Hom}^+(H_1(G), A)$.

Note that a finite cyclic group with a nontrivial signed structure is of even order, and the set of positive elements is determined by the unique subgroup of index two. The following lemma from [KLS04] (see also [KMN07]) will be useful.

Lemma 1.6 Let $G$ be a group with sign structure. Then

$$\text{Epi}^+(G, \mathbb{Z}_{2\ell}) = \sum_{d|\ell, \frac{\ell}{d} \text{ odd}} \mu\left(\frac{\ell}{d}\right) \text{Hom}^+(G, \mathbb{Z}_{2d}).$$
We shall also use extensively the following observation. Let
\[ G = \langle x_1^+, x_2^+, \ldots, x_k^+, y_1^-, \ldots, y_m^- \rangle, \]
be a group with sign structure generated by positive elements \( x_1^+, x_2^+, \ldots, x_k^+ \) and negative elements \( y_1^-, \ldots, y_m^- \). Then a homomorphism \( \psi : G \to A \) is orientation-preserving if and only if the elements \( \psi(x_1^+), \psi(x_2^+), \ldots, \psi(x_k^+) \) are positive and \( \psi(y_1^-), \psi(y_2^-), \ldots, \psi(y_m^-) \) are negative.

## 2 Maps on orbifolds

A **topological map** is a 2-cell decomposition of a surface. Maps can be viewed as cell-embeddings of graphs into surfaces. Two categories of maps will be distinguished depending on whether orientation-reversing morphisms (coverings) between maps are considered or not. Our aim is to study maps on closed orientable surfaces.

### 2.1 Sensed maps

By a (combinatorial) **sensed map** we mean a triple \((D; R, L)\) composed by a finite set \(D\) and two permutations \(R\) and \(L\), with \(L\) satisfying \(L^2 = 1\), generating a transitive subgroup of the symmetric group \(S_D\). The elements of \(D\) are called *darts* and the respective orbits of \(R, L\) and \(RL\) are called *vertices, edges* and *faces*. Edges of size one are called *semiedges* (topologically these correspond to free-edges). The genus \(g\) of a sensed map \(\mathcal{M} = (D; R, L)\) is given by \(2 - 2g = V + E + F - |D|\) where \(V\) is the number of vertices, \(E\) is the number of edges and \(F\) is the number of faces. If \(\mathcal{M}\) has no semiedges (i.e. if \(L\) is fixed point free) then \(|D| = 2E\) and then \(2 - 2g = V - E + F\). Sensed maps describe topological maps on orientable surfaces with a chosen global orientation. Hence they are determined up to orientation-preserving self-homeomorphisms of the surface preserving vertices, edges and faces. This gives rise the following definition: two sensed maps are **isomorphic**, and write \((D_1; R_1, L_1) \cong (D_2; R_2, L_2)\), if there is a bijection \(D_1 \to D_2\) such that \(\psi R_1 = R_2 \psi\) and \(\psi L_1 = L_2 \psi\). In particular, if \(D_1 = D_2 = D\), then \((D; R^\psi, L^\psi)\), where \(\psi \in S_D\) ranges through all permutations of degree \(n\), determines an isomorphism class of maps based on \(D\).

A map \(\mathcal{M} = (D; R, L)\) may, or may not, be isomorphic to its **mirror image** \(\mathcal{M}^{-1} = (D; R^{-1}, L)\). If \(\mathcal{M} \cong \mathcal{M}^{-1}\) we say that \(\mathcal{M}\) is **reflexible**, otherwise the maps \(\mathcal{M}\) and \(\mathcal{M}^{-1}\) are **chiral** and the pair will be called a **chiral twin**. A **sensed rooted map** is a 4-tuple \((D, x_0; R, L)\), where \(x_0 \in D\) is a root, and \((D; R, L)\) is a sensed map. Finally, a map where all the darts are distinguished (or labelled) will be called a **labeled map**. Note that the number of labeled maps on \(n\) darts is equal to \((n - 1)!\) times the number of rooted maps. Relaxing the condition on the transitivity of \((R, L)\) we get the concept of sensed **premap**, **rooted premap** and **labeled premap**. Clearly, each premap \((D; R, L)\) is a disjoint union of maps, those corresponding to the orbits of \((R, L)\) on \(D\).

To each sensed map \(\mathcal{M} = (D; R, L)\) there is an associated **closed** orientable surface (that is, a compact orientable surface without boundary) which can be constructed by
attaching a 2-cell to each orbit of the permutation \( RL \). Hence \( M \) can be considered as a topological map. In turn, any topological map on a closed orientable surface can be realized as a combinatorial sensed map. Topological maps correspond to cellular embeddings of graphs and since graphs were generally assumed to be without semiedges, we will interpret “sensed maps on closed surfaces” as maps without semiedges. The theory of sensed maps [JS78] was built around a close relationship between maps and subgroups of a certain universal group. Denote by \( \Delta^+ = \Delta^+(\infty, \infty, 2) = \langle \alpha, \beta | \beta^2 = 1 \rangle \cong \mathbb{Z} \ast \mathbb{Z}_2 \). Given a sensed map \((D; R, L)\) the assignment \( \alpha \mapsto R \) and \( \beta \mapsto L \) extends to an epimorphism \( \Phi : \Delta^+ \to \langle R, L \rangle \). It follows that \( \Delta^+ \) acts on \( D \) by \( z \cdot x = \Phi(z)x \) for \( z \in \Delta^+ \) and \( x \in D \). The stabilizer \( K \leq \Delta^+ \) of a point \( x \in D \), has index \([\Delta^+: K] = |D|\). Vice versa, each subgroup \( K \leq \Delta^+ \) of finite index determines a rooted sensed map \( M = (D; R, L, x_0) \), where \( D \) is the set of left cosets \( xK, x \in \Delta^+ \), and the action of \( R \) and \( L \) is defined by left multiplication \( R(xK) = \alpha xK, \ L(xK) = \beta xK \) and \( x_0 = K \) is the trivial coset. Moreover, \( M \) has no semiedges if and only if \( K \) is torsion-free. We summarize the above considerations in the following proposition, see [JS78].

**Proposition 2.1** The following statements hold true.

(1) Rooted sensed maps on \( n \) darts are in 1-1 correspondence with subgroups of \( \Delta^+ \) of index \( n \).

(2) Isomorphism classes of sensed maps on \( n \) darts are in 1-1 correspondence with conjugacy classes of subgroups of \( \Delta^+ \) of index \( n \).

(3) The above maps are free of semiedges if and only if the respective subgroups of \( \Delta^+ \) are torsion-free.

The goal is to understand the structure of the subgroups \( K \leq \Delta^+ \) of finite index that give rise to maps on close orientable surfaces. This is the case when \( K \) is isomorphic to the fundamental group of the punctured surface \( S - B \), where \( B \) is the the set of points corresponding to vertices and face-centers. In this case the structure of \( K \) is determined by the underlying surface \( S \). If \([\Delta^+: K] = 2e = n\), where \( e \) is the number of edges, then \( K \) is a free group of rank \( e + 1 \). Sensed maps on closed orientable surfaces are (by definition) semidge-free sensed maps and so they are in 1-1 correspondence with torsion-free subgroups of \( \Delta^+ \).

**Proposition 2.2** In the following two cases the refered objects are in 1-1 correspondence:

(1) Rooted sensed maps on closed orientable surfaces with \( e \) edges, and free subgroups of \( \Delta^+ \) of rank \( e + 1 \);

(2) Isomorphism classes of sensed maps on closed orientable surfaces with \( e \) edges, and conjugacy classes of free subgroups of rank \( e + 1 \) in \( \Delta^+ \).
In this paper we consider maps on orbifolds. This is a new and fruitful idea which was already used in our previous articles ([MeN06], [MN06]). By an oriented orbifold \( O \) we mean an oriented surface \( S \) with a distinguished discrete set of points \( B \) assigned by integers \( m_1, m_2, \ldots, m_i, \ldots \) such that \( m_i \geq 2 \) or \( m_i = \infty \), for \( i = 1, 2, \ldots \). Elements of \( B \) will be called branch points and the respective numbers \( m_1, m_2, \ldots, m_i, \ldots \) will be called branch indices. If \( S \) is a compact connected orientable surface of genus \( g \) then \( B \) is finite of cardinality \( |B| = r \) and \( O \) is determined by its signature \([g; m_1, m_2, \ldots, m_r]\).

Hence we write \( O = O[g; m_1, m_2, \ldots, m_r] \). The fundamental group \( \pi_1(O) \) of \( O \) is an \( F \)-group defined as follows

\[
\pi_1(O) = F[g; m_1, m_2, \ldots, m_r] = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, e_1, \ldots, e_r \prod_{i=1}^g [a_i, b_i] \prod_{j=1}^r e_j = 1, e_1^{m_1} = \ldots e_r^{m_r} = 1 \rangle, \tag{2.1}
\]

with a convention that \( e_i^{m_i} \) vanishes if \( m_i = \infty \) for some \( i \) (see [JS87]).

A map on an orbifold \( O \) is a map on the underlying surface \( S_g \) of genus \( g \) satisfying the following three properties:

(P1) if \( x \in B \) then \( x \) is either an internal point of a face, or a vertex, or an end-point of a semiedge (free end) which is not a vertex,

(P2) each face contains at most one branch point,

(P3) each free end of a semiedge is a branch point and the branch index of this point is two.

Maps on orbifolds arise naturally when we take quotients of sensed maps by a group \( G \) of automorphisms. Such coverings are called regular. Then the numbers \( m_1, \ldots, m_r \) are orders of stabilizers of faces, vertices and edges in the action of \( G \). Note that these stabilizers are by definition cyclic. Further information on sensed maps on orbifolds can be found in [MeN06] and [MN06].

2.2 Unsensed maps on closed and bordered surfaces

By a (unsensed) map we mean a quadruple \((F; l, r, t)\) where \( (r, t, l) \) is a transitive subgroup of the symmetric group \( S_F \) and \( r^2 = l^2 = t^2 = (tl)^2 = 1 \). The elements of \( F \) are called flags, the respective orbits of \( (r, t), (t, l) \) and \( (r, l) \) are vertices, edges and faces. Since \( (tl)^2 = 1 \) the edges of our maps can be of size four, two or one. Let \( x \) be a flag incident to an edge \( e \). We say that \( e \) is a

- complete edge if the orbit \( \{x, l(x), t(x), tl(x)\} \) of \( (t, l) \) has size 4,
- boundary edge if \( l(x) = tl(x) \neq x = t(x) \),
- internal semiedge if \( l(x) = t(x) \neq x = tl(x) \),
- halfedge if \( l(x) = x \neq t(x) = tl(x) \).
- boundary semiedge if \( l(x) = x = t(x) = tl(x) \).

![Diagram of edges in a map](image)

Figure 1: Types of edges in a map.

A diagonal is an orbit of \( r \). A diagonal is called internal or boundary, if it is of size two or one, respectively.

A rooted map \((F,x_0;l,r,t)\), is a map with one distinguished flag \(x_0 \in F\) called a root. A Labeled map is a map with all flags distinguished (or labeled). As above, relaxing the condition on the transitivity of \((l,r,t)\) we get the concept of premap, rooted premap and labeled premap. Clearly each premap \((F;l,r,t)\) is the disjoint union of the maps corresponding to the orbits of \((l,r,t)\) on \(D\).

Two maps are isomorphic, \((F_1;l_1,r_1,t_1) \cong (F_2;l_2,r_2,t_2)\), if there is a bijection \(F_1 \to F_2\) such that \(\psi l_1 = l_2 \psi\), \(\psi r_1 = r_2 \psi\) and \(\psi t_1 = t_2 \psi\). In particular if \(F_1 = F_2 = F\) then \((F;l^\psi,r^\psi,t^\psi)\), where \(\psi \in S_F\) ranges through all permutations in \(S_F\), determines, as before, an isomorphism class of maps based on \(F\). An isomorphism between rooted maps takes root into root.

As explained in [BS85] maps describe topological maps on closed surfaces (orientable or not) possibly with non-empty border. A map \((F;l,r,t)\) is on a closed surface (or briefly a closed map) if the involutions \(r,t,tl\) are fixed point free. It follows that closed maps have all edges complete. A map \((F;l,r,t)\) is on a compact surface, possibly with non-empty border, (or, just a compact map) if it has no semiedges (either boundary or internal). This is the case when \(tl\) is fixed point free. Closed maps are compact maps. Note that the underlying surface of a map may or may not be orientable, depending on whether the set of flag transposes can be split into two disjoint blocks such that each one of the three involutions \(r,t,tl\) transposes the two blocks. The Euler characteristic of \(S\) can be computed in the usual way. Note that the faces of such maps are of two types: internal faces, homeomorphic to \(\mathbb{R}^2\), and halved faces, homeomorphic to \(\mathbb{R}_+^2\).

Maps can be described in terms of subgroups of a certain universal group ([JS87]). Let

\[ \Delta = \Delta(\infty, \infty, 2) = \langle \lambda, \rho, \tau | \lambda^2 = \rho^2 = \tau^2 = (\lambda \tau)^2 = 1 \rangle. \]

The group \(\Delta\) can be realized as a discrete group of isometries of the hyperbolic plane \(\mathbb{H}^2\). It is generated by reflections in the sides of a hyperbolic triangle with internal angles \(0, 0\) and \(\frac{\pi}{2}\) (see Fig. 2). Given a map \((F;l,r,t)\), the assignment \(\lambda \mapsto l\), \(\rho \mapsto r\) and \(\tau \mapsto t\) extends to an epimorphism \(\Phi : \Delta \to \langle l,r,t \rangle\). It follows that \(\Delta\) acts on \(F\) by \(z \cdot x = \Phi(z)x\), where \(z \in \Delta\) and \(x \in F\). The stabilizer \(K \leq \Delta\) of a flag \(x \in F\), has index \([\Delta : K] = |F|\). Conversely, each subgroup \(K \leq \Delta\) of finite index determines a
rooted map \( \mathcal{M} = (F, x_0; l, r, t) \) by setting \( F \) to be the set of left cosets \( xK, x \in \Delta \), with action \( r(xK) = r xK, l(xK) = lxK, t(xK) = \tau xK \) and \( x_0 = K \). Moreover, \( \mathcal{M} \) is a closed map (orientable or not) if and only if \( K \) is torsion-free. As usually, the subgroup inclusion \( H < K \) correspond to a map covering \( \mathcal{H} \rightarrow \mathcal{E} \). In particular, taking \( K = \Delta \) we get \( \mathcal{H} \rightarrow \mathcal{E} \), where \( \mathcal{E} \) is the trivial (one-flag) map. The supporting orbifold of \( \mathcal{E} \) is the hyperbolic triangle shown in Fig. 2. We summarize the above considerations in a proposition.

**Proposition 2.3** The following statements hold true.

(1) Rooted maps with \( n \) flags are in 1-1 correspondence with subgroups of \( \Delta \) of index \( n \).

(2) Isomorphism classes of (unsensed) maps on \( n \) flags are in 1-1 correspondence with conjugacy classes of subgroups of \( \Delta \) of index \( n \).

(3) An (unsensed) map is closed if and only if the respective subgroup of \( \Delta \) is torsion-free.

Similarly as before we have,

**Proposition 2.4** The objects refered in each item are in 1-1 correspondence:

(1) Rooted maps on closed surfaces (rooted “closed maps”) with \( e \) edges, and free subgroups of \( \Delta \) of rank \( e + 1 \);

(2) Isomorphism classes of closed maps with \( e \) edges, and conjugacy classes of free subgroups of rank \( e + 1 \) in \( \Delta \).

Clearly, the mapping \( \alpha \mapsto \rho \tau, \beta \mapsto \tau \lambda \) extends to a monomorphism \( \Delta^+ \rightarrow \Delta \). Hence \( \Delta^+ \) can be identified with the subgroup \( \langle \rho \tau, \tau \lambda \rangle \) of index two in \( \Delta \). We say that a (unsensed) map \( \mathcal{M} \) is orientable if the respective subgroup \( K \leq \Delta \) is a subgroup of \( \Delta^+ \). Otherwise \( \mathcal{M} \) is non-orientable. The underlying surface of a non-orientable map may have boundary and may be either orientable or not. If a non-orientable map has no boundary (i.e. no boundary edges, no halfedges and no boundary semiedges) then the underlying surface is also non-orientable.

By a Klein surface we mean an orbifold \( \mathcal{S} = \mathbb{H}^2/K \) determined by a non-Euclidean crystallographic (NEC) group \( K \), that is, a discrete group of isometries of the hyperbolic plane \( \mathbb{H}^2 \) inducing a compact quotient space. Such orbifold \( \mathcal{S} \) as a topological space is indeed a surface and it may have boundary. If \( K \) has signature

\[
(g, \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\})
\]

(2.1)

(notation from [Buj90]) then \( \mathcal{S} \) has genus \( g \), has \( k \) boundary components and is orientable if \( \pm = + \) and non-orientable if \( \pm = - \). The integers \( m_1, \ldots, m_r \) are the “proper periods” of \( K \) and represent the branching over \( r \) interior points of \( \mathcal{S} \) under the natural projection
\[ \mathbb{H}^2 \to \mathbb{H}^2/K. \] The “period cycle” \((n_{i1}, \ldots, n_{i_s})\), \(i = 1, \ldots, k\), represent the branching over the \(i^{th}\) hole (the integers \(n_{ij}\) are the “link periods”). Let \(b = s_1 + \cdots + s_k\). Then \(K\) is generated by

\[
\begin{align*}
\text{r} & \text{ elliptic elements } (+) : x_1, \ldots, x_r \\
\text{b} + k & \text{ reflections } (-) : c_{i,0}, \ldots, c_{i,s_i} , \ i = 1, \ldots, k \\
\text{k} & \text{ orientation-preserving elements } (+) : e_1, \ldots, e_k \\
2g & \text{ hyperbolic elements } (+) : a_1, b_1, \ldots, a_g, b_g \text{ if } \pm = + \\
\text{g} & \text{ glide reflections } (-) : a_1, \ldots, a_g \text{ if } \pm = -
\end{align*}
\]

subject to the following relations

\[
\begin{align*}
x_i^{n_1} &= \cdots = x_i^{n_r} = 1 \\
c_{i,j-1}^2 &= c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{ij}} = 1 , \ i = 1, \ldots, k , \ j = 1, \ldots, s_i \\
e_i c_{i,0} e_i^{-1} &= c_{i,s_i}, \ i = 1, \ldots, k \\
\prod_{i=1}^r x_i \prod_{j=1}^k e_j \prod_{\ell=1}^s [a_\ell, b_\ell] &= 1 , \ \text{ if } \pm = + \\
\prod_{i=1}^r x_i \prod_{j=1}^k e_j \prod_{\ell=1}^s a_\ell^2 &= 1 , \ \text{ if } \pm = -
\end{align*}
\]

The hyperbolic area of a fundamental region for \(K\) is given by

\[
\Phi(K) = 2\pi \left( -\chi + \sum_{i=1}^r (1 - \frac{1}{m_i}) + \frac{1}{2} \sum_{i,j} (1 - \frac{1}{n_{ij}}) \right)
\]

where \(\chi = 2 - 2g - k\) or \(2 - g - k\), according as \(\pm = +\) or \(-\), is the (topological) characteristic of \(\mathcal{S}\).

If \(\mathcal{S} = \mathbb{H}^2/K\) is a \(m\)-fold cover of another Klein surface \(\mathcal{S}' = \mathbb{H}^2/K'\) (that is, if \(K < K'\) and \([K' : K] = m\)) then their hyperbolic areas are related by the Riemann-Hurwitz equation:

\[
\Phi(K) = m \Phi(K').
\]

Similarly as in the case of sensed maps, a subgroup \(M < \Delta\) associated with a map \(\mathcal{M}\) gives rise to a Klein surface \(\mathbb{H}^2/M\). Such subgroup can then be interpreted geometrically as the fundamental group of an orbifold arising from the supporting surface by deleting points representing vertices and centers of faces. The topology on this “multi-punctured” surface (possibly with non-empty border) which makes this surface compact is the \((V + F)\)-compactification topology, where \(V\) and \(F\) are the number of vertices and faces respectively. Moreover, to each map \(\mathcal{M}\) having \(b\) internal semiedges, \(V_0\) internal vertices and \(F_0\) internal faces, there is an associated orbifold \(\mathbb{H}^2/M\). If \(\mathcal{M}\) has no boundary semiedges, then the orbifold \(O = \mathbb{H}^2/M\) has signature \((g, +; [2^k, \infty^{F_0 + V_0}]; \{(\infty^s) : 0 \leq i \leq k}\), or \((p, -; [2^k, \infty^{F_0 + V_0}]; \{(\infty^s) : 0 \leq i \leq k}\); here \(k\) is the number of boundary components, and \(g\) and \(p\) are the respective orientable and non-orientable genus of the supporting surface. Otherwise, the link periods in the period cycles corresponding to boundary semiedges is 2. This relationship between the associated group \(M < \Delta\) and the orbifold \(\mathbb{H}^2/M\), will play a central role in this paper.
We illustrate our definitions with the following examples:

**Example 1.** The elementary map $\mathcal{E}$ has only one flag (Fig. 2.) One can write $\mathcal{E} = (F; l, r, t)$, where $F = \{1\}$, $l = r = t = (1)$. The associated group $\Delta$ is generated by reflections $\lambda$, $\rho$ and $\tau$ insides of a hyperbolic triangle with angles $0, 0$ and $\frac{\pi}{2}$. The signature of the group $\Delta$ is $(0^+, \emptyset; \{(2, \infty, \infty)\})$ and $\Phi(\Delta) = \frac{\pi}{2}$. The elementary map $\mathcal{E}$ is not a compact map. It has a boundary semiedge.

![Figure 2: The elementary map $\mathcal{E}$ with associated group $\Delta$.](image)

**Example 2.** Let $F = \{1, 2, 3, 4, 5, 6\}$, $l = (16)(24)(35)$, $r = (12)(34)(56)$ and $t = (16)(23)(45)$. Then $(F; l, r, t)$ determines a non compact map on the projective plane with one branch point of order two (Fig. 3). The associated NEC group $K$ has signature $(1^-, [2^1, \infty^{1+1}]; \emptyset)$.

![Figure 3: A map on the projective plane with one branch point of order two.](image)

**Example 3.** Consider $F = \{1, 2, \ldots, 16\}$ and set

$$l = (1)(2)(3\ 4)(5\ 8)(6\ 7)(9\ 10)(11\ 12)(13\ 14)(15\ 16),$$

$$r = (1)(2\ 3)(4\ 5)(6\ 7)(8\ 9\ 16)(10\ 11)(12\ 13)(14\ 15)$$

and

$$t = (1\ 2)(3\ 10)(4\ 9)(5\ 6)(7\ 8)(11)(12)(13)(14)(15)(16).$$

Then $(F; l, r, t)$ is a compact map on the disc (characteristic $\chi = 1$) with 0 internal vertices, 1 internal face, 2 complete edges, 0 internal semiedges, 3 boundary edges, 1 halfedge and 1 boundary component (one chain of 8 linked boundary reflections). Then the associated group $K$ has signature $(0, +; [\infty^{0+1}]; \{\infty^8\})$. See Fig. 4.
3 Enumeration of rooted maps

In this section we enumerate rooted maps on closed and compact surfaces regardless of genus. The results will be used in subsequent sections.

Proposition 3.1 The number of rooted sensed maps on closed orientable surfaces $R^+(e)$ with $e$ edges is given by the following equation

$$
\sum_{e \geq 1} \frac{R^+(e)}{e} 2^{e-1} u^e = \log \left( \sum_{e \geq 0} \frac{(2e)!}{e!} u^e \right).
$$

Proof: Denote by $T_{2k}$ the family of sensed labeled maps $(D; R, L)$ without semiedges based on a fixed set of darts $D = D_{2k}$, $|D_{2k}| = 2k$. Similarly, denote by $B_{2k}$ the family of sensed labeled premaps $(D; R, L)$ without semiedges based on $D$. Set $T_{2k} = |T_{2k}|$ and $B_{2k} = |B_{2k}|$. Choose a dart $x_0 \in D$ and denote by $B_{2e}^{(i)}$, $i = 0, 1, \ldots, e - 1$, the set of sensed labeled premaps such that the orbit of $\langle R, L \rangle$ containing $x_0$ is of size $2(e - i)$. Clearly, $B_{2e}$ is a disjoint union $B_{2e} = \bigcup_{i=0}^{e-1} B_{2e}^{(i)}$. The orbit $[x_0]$ (of size $2(e - i)$) in a premap in $B_{2e}^{(i)}$ can be chosen by $\binom{2e-1}{2(e-i)-1}$ ways. For a given set $W \subseteq D$ containing $x_0$ such that $|W| = 2(e - i)$, there are $T_{2(e-i)} B_{2i}$ ways to build a labeled premap with $[x_0] = W$. Thus we have

$$
B_{2e} = \sum_{i=0}^{e-1} |B_{2e}^{(i)}| = \sum_{i=0}^{e-1} \binom{2e-1}{2(e-i)-1} T_{2(e-i)} B_{2i}. \tag{3.2}
$$

Remark. This coincides with the first statement of Lemma 1.2 for the particular case when $T_k = B_k = 0$ for $k$ odd.

Multiplying the equation (3.2) by $\frac{z^{2e-1}}{(2e-1)!}$ we get

$$
\frac{B_{2e}}{(2e-1)!} z^{2e-1} = \sum_{i=0}^{e-1} \frac{T_{2(e-i)}}{(2(e-i)-1)!} z^{2(e-i)-1} \cdot \frac{B_{2i}}{(2i)!} z^{2i}. \tag{3.3}
$$
Denote by
\[ B(z) = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k} \quad \text{and} \quad T(z) = \sum_{k=1}^{\infty} \frac{T_{2k}}{(2k)!} z^{2k}, \]
where the coefficient \( B_0 \) is defined to be \( B_0 = 1 \).

Then the equation (3.3) can be expressed in a form of a differential equation \( B'(z) = T'(z)B(z) \) with \( B(0) = 1 \) and \( T(0) = 0 \). It follows that \( T(z) = \log B(z) \). Now we calculate \( B_{2e} \) directly. Let \( D = D_{2e} \). Since \( R \) is any permutation of \( D \), there are \( (2e)! \) choices for \( R \).

The number of permutations with prescribed cycle structure (see Lemma 1.1) determining the number of fixed-point-free involutions acting on \( 2e \) elements is given by \( \frac{(2e)!}{2e!} \); this counts the number of choices for \( L \). Hence \( B_{2e} = (2e)! \frac{(2e)!}{2e!} \). Substituting in \( T(z) = \log B(z) \) gives
\[ \sum_{k=1}^{\infty} \frac{T_{2k}}{(2k)!} z^{2k} = \log \sum_{k=0}^{\infty} \frac{(2k)!}{2^k k!} z^{2k}. \]

Changing variable \( u = \frac{z^2}{2} \) we get
\[ \sum_{k=1}^{\infty} \frac{T_{2k}}{(2k)!} 2^k u^k = \log \sum_{k=0}^{\infty} \frac{(2k)!}{k!} u^k. \quad (3.4) \]

The final step is established by observing that the coefficient \( T_{2k} \) counts the number of labeled maps with \( k \) edges, thus we have \( T_{2k} = (2k-1)! R^+(k) \). Using the substitution \( T_{2k} = (2k-1)! R^+(k) \) the equality (3.4) transforms into the required form. \( \square \)

**Proposition 3.2** The number \( R(e) \) of rooted (orientable or not) maps on closed surfaces with \( e \) edges is given by the following equation
\[ \sum_{e \geq 1} \frac{R(e)}{e} 4^{2e-1} u^e = \log \left( \sum_{e \geq 0} \frac{(4e)!}{(2e)! e!} u^e \right). \]

Proof: The proof is done in a similar manner as above. Note that if \( (F; l, r, t) \) is an ordinary closed map with \( e \) edges then we have \( |F| = 4e \). In what follows “map” means “closed map” and similarly for “premap”. We first derive a relation between the number \( T_{4e} \) of labeled ordinary maps based on a set \( F \) having \( 4e \) flags, and the number \( B_{4e} \) of ordinary premaps on \( F \). Similarly as above (compare with equation (3.2)), the coefficients are related by
\[ B_{4e} = \sum_{i=0}^{e-1} \left( \frac{4e-1}{4(e-i)-1} \right) T_{4(e-i)} B_{4i}. \quad (3.5) \]

Multiplying both sides by \( \frac{z^{4e-1}}{(4e-1)!} \) we derive
\[ \frac{B_{4e}}{(4e-1)!} z^{4e-1} = \sum_{i=0}^{e-1} \frac{T_{4(e-i)}}{(4(e-i)-1)!} z^{4(e-i)-1} \frac{B_{4i}}{(4i)!} z^{4i}. \quad (3.6) \]
Now we calculate \( B_{4e} \), directly. Up to this point, there was no essential difference with the proof of Proposition 3.1. Now we employ the condition that \( l, r, t \) and \( tl \) are fixed point free on \( F \). There are \( \frac{(4e)!}{2^{4e}(2e)!} \) fixed point free involutions on \( F \) with \( |F| = 4e \).

The choices of \( r \) and \( t \) can be done independently, hence we have \( \frac{(2e)!}{2^{e} e!} \) couples of \( r, t \). Assume \( r \) and \( t \) is given. Since \((tl)^2 = 1\) and each edge is complete, the involution \( l \) freely permutes the orbits of \( t \) as blocks. Since the orbits of \( t \) are all of length two, \( t \) has exactly \( 2e \) orbits. It follows that we have \( \frac{(2e)!}{2^{e} e!} \) choices for the action of \( l \) on the orbits of \( t \). Once the action of \( l \) is determined on blocks of \( t \), the edges of our premap are determined. If \( \{x, t(x), y, t(y)\} \) is an edge then either \( l(x) = y \) and \( l(t(x)) = t(y) \), or \( l(x) = t(y) \) and \( l(t(x)) = y \). Since there are no further constraints the choice for each edge is independent, so the number of choices for \( l \) (provided \( t \) and \( r \) are prescribed) is \( 2^{e} \frac{(2e)!}{2^{e} e!} = \frac{(2e)!}{e!} \).

It follows that
\[
B_{4e} = \frac{(4e)!}{2^{4e}((2e)!)^2} \cdot \frac{(2e)!}{e!} = \frac{(4e)!^2}{2^{4e} e!}.
\]

Denoting
\[
B(z) = \sum_{k=0}^{\infty} \frac{B_{4k}}{(4k)!} z^{4k} \quad \text{and} \quad T(z) = \sum_{k=1}^{\infty} \frac{T_{4k}}{(4k)!} z^{4k},
\]
where the coefficient \( B_0 \) is defined to be \( B_0 = 1 \), the equation (3.6) transforms into a differential equation \( B'(z) = T'(z)B(z) \), with \( B(0) = 1 \) and \( T(0) = 0 \). It follows that \( T(z) = \log B(z) \) which gives
\[
\sum_{k=1}^{\infty} \frac{T_{4k}}{(4k)!} z^{4k} = \log \sum_{k=0}^{\infty} \frac{(4k)!}{2^{4k} \cdot k!} z^{4k}.
\]

Finally, we use the substitutions \( z^4 = 2^4 u \) and \( T_{4k} = (4k - 1)! R(k) \) to derive the required equality. \( \Box \)

**Remark.** Observe that the number \( R^+(e) \) of rooted semiedge-free sensed maps with \( e \) edges coincides with the number of rooted orientable closed maps with \( e \) edges. Then the number \( R^-(e) \) of rooted non-orientable closed maps with \( e \) edges is equal to \( R^-(e) = R(e) - R^+(e) \), where the numbers \( R^+(e) \) and \( R(e) \) are given in Propositions 3.1 and 3.2, respectively.

The number \( R^+(e) \) was determined earlier by Jackson and Visentin [JV90] and by Arquès and Béraud [AB00].

Rooted orientable maps have no boundary, yet they may have semiedges. The following proposition counts the number of them with \( m \) darts and \( q \) complete edges.
Proposition 3.3 The number of rooted orientable maps $R^+(m, q)$ with $m$ darts and $q$ complete edges (and hence with $m - 2q$ semiedges) is given by the following equation

$$\sum_{m \geq 1} \sum_{q \geq 1} \frac{R^+(m, q)}{m} x^m y^q = \log(\sum_{m \geq 0} \sum_{q \geq 0} \frac{m!}{(m - 2q)! 2q!} x^m y^q).$$

Remark. Note that $R^+(2e, e) = R^+(e)$ is the number of rooted orientable closed maps with $e$ edges. We put also $R^+(m, \frac{m}{2}) = 0$ for odd $m$.

Proposition 3.4 The number $R(m, q)$ of rooted boundary-free maps with $m$ darts and $q$ complete edges (hence with $m - 2q$ semiedges) is given by the following equation

$$\sum_{m \geq 1} \sum_{q \geq 1} \frac{R(m, q)}{2m} x^{2m} y^q = \log(\sum_{m \geq 0} \sum_{q \geq 0} \frac{(2m)!}{2^{2m} m! (m - 2q)! q!} x^{2m} y^q).$$

Moreover, $R(m, q)$ can be determined by the following recursive relation

$$R(m, q) = 2m Q(m, q) - \sum_{k=1}^{m-1} \sum_{s=0}^q R(m - k, q - s) Q(k, s),$$

where

$$Q(m, q) = \frac{(2m)!}{2^{2m} m! (m - 2q)! q!}.$$

Proof: Let $\Delta(\infty, \infty, 2) = \langle \lambda, \rho, \tau \mid \lambda^2 = \rho^2 = \tau^2 = (\lambda \tau)^2 = 1 \rangle$. Labeled boundary-free maps with $m$ darts and $q$ complete edges are in correspondence with transitive homomorphisms $h : \Delta(\infty, \infty, 2) \rightarrow S_{2m}$ such that the involutions $l = h(\lambda)$, $r = h(\rho)$, $t = h(\tau)$ are fixed point free and $\sigma(lt) = (1^{2(m-2q)} 2q)$. The base set is the set $F$ of flags and we have $|F| = 2m$. Denote by $B_{m, q}$ and $T_{m, q}$ the respective numbers of homomorphisms and transitive homomorphisms $\Delta(\infty, \infty, 2) \rightarrow S_{2m}$ satisfying the above properties. Repeating the arguments in the proof of Proposition 3.1 we get

$$B_{m, q} = \sum_{m_1 + m_2 = m \atop q_1 + q_2 = q} \binom{2m - 1}{2m_2} T_{m_1, q_1} B_{m_2, q_2}, \tag{3.8}$$

where $B_{m, q} = T_{m, q} = 0$ if $m < 2q$, $B_{0, 0} = 1$ and $T_{0, 0} = 0$. Multiplying (3.8) by $\frac{z^{2m-1} w^q}{(2m - 1)!}$ leads to

$$\frac{B_{m, q}}{(2m - 1)!} z^{2m-1} w^q = \sum_{m_1 + m_2 = m \atop q_1 + q_2 = q} \frac{T_{m_1, q_1}}{(2m_1 - 1)!} z^{2m_1-1} w^{q_1} \frac{B_{m_2, q_2}}{(2m_2)!} z^{2m_2} w^{q_2}. \tag{3.9}$$
Summing left- and right-hand side of this equation for all $m$ and $q$ we get

$$\sum_{m,q} \frac{B_{m,q}}{(2m - 1)!} z^{2m-1} w^q = \left( \sum_{m,q} \frac{T_{m,q}}{(2m - 1)!} z^{2m-1} w^q \right) \left( \sum_{m,q} \frac{B_{m,q}}{(2m)!} z^{2m} w^q \right). \quad (3.10)$$

Denote by

$$T(z, w) = \sum_{m,q} \frac{T_{m,q}}{(2m)!} z^{2m} w^q$$

and by

$$B(z, w) = \sum_{m,q} \frac{B_{m,q}}{(2m)!} z^{2m} w^q.$$

Then (3.10) can be rewritten as a differential equation

$$\frac{\partial B(z, w)}{\partial z} = \frac{\partial T(z, w)}{\partial z} B(z, w)$$

with constraints $T(0, w) = 0$ and $B(0, w) = 1$, and solution

$$T(z, w) = \log B(z, w). \quad (3.11)$$

We now compute the coefficients $B_{m,q}$ directly. First we count the number of choices for the two fixed point free involutions $r$ and $t$. Since there are no other restrictions, by Lemma 1.1 there are $\nu(t)\nu(r) = \frac{(2m)!^2}{2^{2m}m!^2}$ choices for $r$ and $t$. Fix one pair $(r, t)$ from that choices. Now we are going to count the number of choices for $l$. Each $l$ (by determining a map) splits the $m$ orbits of $t$ into 2 sets: the first, consisting of $2q$ orbits, forms the $q$ complete edges, while the second set, consisting of $m - 2q$ orbits, forms the semiedges. Let $Q$ be set of orbits of $t$ forming complete edges. There are $\binom{m}{2q}$ choices for $Q$. The action of $l$ on the flags incident to semiedges coincides with the action of $t$. Hence $l$ is determined by its action on the flags incident to complete edges, that is, on the union of the $2q$ orbits in $Q$.

Since $(lt)^2 = 1$, each $l$ determines a perfect matching of elements (orbits) in $Q$. By Lemma 1.1 there are $\frac{(2q)!}{2q q!}$ such perfect matchings. Each perfect match just counted determines a complete edge and for each of the $q$ complete edges there are exactly two ways to define $l$, so we have $\binom{m}{2q} \frac{(2q)!}{2q q!} 2^q$ possibilities for $l$.

Summarizing,

$$B_{m,q} = \frac{(2m)!^2}{2^{2m}m!^2} \binom{m}{2q} \frac{(2q)!}{2q q!} 2^q = \frac{(2m)!^2}{2^{2m}m!(m - 2q)q!}. \quad (3.12)$$

Inserting (3.12) into (3.11) and taking into the account that $T_{m,q} = R(m, q)(2m - 1)!$ we obtain the required result. □

Remarks.
1. The number $R^-(m, q)$ of rooted non-orientable boundary-free maps with $n$ darts and $q$ complete edges (thus with $m - 2q$ semiedges) is given by the formula $R^-(m, q) = R(m, q) - R^+(m, q)$.

2. The proof of Proposition 3.3 is a replica of the above proof with the following adjustments $h : \Delta^+ (\infty, \infty, 2) \rightarrow S_m, \sigma(L) = (1^{m-2q} 2^q)$ and $T_{m,q} = (m-1)!R^+(m, q)$.

Denoted by $R(m, q, s)$ the number of rooted compact maps with $2m$ flags, $q$ complete edges ($q \leq \lceil m/2 \rceil$) and $s$ internal diagonals ($s \leq m$). In this case the bordered surface has $b = 3m - 2q - 2s \geq 0$ boundary reflections (halfedges, boundary edges and boundary semiedges), which correspond to half the number of fixed points of $t$ plus half the number of fixed points of $l$ and plus the number of fixed points of $r$.

**Proposition 3.5** The number $R(m, q, s)$ of rooted maps on compact surfaces with $2m$ flags, $q$ complete edges, $s$ internal diagonals (thus with $3m - 2q - 2s$ boundary reflections) is given by the following equation

$$
\sum_{m \geq 1, q, s \geq 0} R(m, q, s) \frac{(2m)!}{2^q+s (m-2q)! (2m-2s)! q! s!} w^{2m} x^q y^s = \log \left( \sum_{m, q, s \geq 0} (2m)! \frac{2^q+s (m-2q)! (2m-2s)! q! s!} {w^{2m} x^q y^s} \right).
$$

Moreover, $R(m, q, s)$ can be determined by the following recursive relation

$$
R(m, q, s) = 2m Q(m, q, s) - \sum_{k=1}^{m-1} \sum_{i=0}^{q} \sum_{j=0}^{s} R(m-k, q-i, s-j) Q(k, i, j),
$$

where

$$
Q(m, q, s) = \frac{(2m)!}{2^q+s (m-2q)! (2m-2s)! q! s!}.
$$

Proof: As above we use Lemma 1.2. Labeled compact maps are in 1-1 correspondence with transitive homomorphisms $h : \Delta(\infty, \infty, 2) \rightarrow S_{2m}$. The characteristic property $P$ of our homomorphisms $h : \lambda \rightarrow l, \rho \rightarrow r, \tau \rightarrow t$ reads as follows:

1) $\sigma(lt) = (2^m)$,

2) $\sigma(t) = (2^{2p_2} 2^{q+p_1}), \sigma(l) = (2^{p_1} 2^{2q+p_2})$, for some $p_1 \geq 0, p_2 \geq 0, p_1 + p_2 = m - 2q$,

3) $\sigma(r) = (2^{m-2s} 2^s)$.

The base set is the set $F$ of flags and we have $|F| = 2m$. Denote by $B_{m,q,s}$ and $T_{m,q,s}$ the respective numbers homomorphisms and transitive homomorphisms $\Delta(\infty, \infty, 2) \rightarrow S_{2m}$ satisfying the above properties.
As in the proof of Proposition 3.4 (by Lemma 1.2),

\[ B_{m, q, s} = \sum_{m_1 + m_2 = m \atop q_1 + q_2 = q \atop s_1 + s_2 = s} \frac{(2m - 1)}{2m_2} T_{m_1, q_1, s_1} B_{m_2, q_2, s_2} \]  
(3.13)

where \( B_{0, q, s} = 1 \) and \( T_{0, q, s} = 0 \). Multiplying (3.13) by \( \frac{w^{2m_1 - 1} x^q y^s}{(2m_1 - 1)!} \) we get

\[ \frac{B_{m, q, s}}{(2m - 1)!} w^{2m_1 - 1} x^q y^s = \sum_{m_1 + m_2 = m \atop q_1 + q_2 = q \atop s_1 + s_2 = s} \frac{T_{m_1, q_1, s_1}}{(2m_1 - 1)!} w^{2m_1 - 1} x^{q_1} y^{s_1} \frac{B_{m_2, q_2, s_2}}{(2m_2)!} w^{2m_2} x^{q_2} y^{s_2}. \]
(3.14)

Summing left- and right-hand side of this equation,

\[ \sum_{m, q, s} \frac{B_{m, q, s}}{(2m - 1)!} w^{2m_1 - 1} x^q y^s = \left( \sum_{m, q, s} \frac{T_{m, q_1, s_1}}{(2m - 1)!} w^{2m_1 - 1} x^{q_1} y^{s_1} \right) \left( \sum_{m, q, s} \frac{B_{m, q_2, s_2}}{(2m)!} w^{2m} x^{q_2} y^{s_2} \right). \]
(3.15)

Consider the following generating functions

\[ T(w, x, y) = \sum_{m \geq 1 \atop q, s \geq 0} \frac{T_{m, q, s}}{(2m)!} w^{2m} x^q y^s \quad \text{and} \quad B(w, x, y) = \sum_{m, q, s \geq 0} \frac{B_{m, q, s}}{(2m)!} w^{2m} x^q y^s. \]

Then (3.15) rewrites as a differential equation

\[ \frac{\partial B(w, x, y)}{\partial w} = \frac{\partial T(w, x, y)}{\partial w} B(w, x, y) \]

with constraints \( T(0, x, y) = 0 \) and \( B(0, x, y) = 1 \), and solution

\[ T(w, x, y) = \log B(w, x, y). \]
(3.16)

We now compute the coefficients \( B_{m,q,s} \). The incomplete edges make the boundary reflections. Denote by \( P_1 \) (resp. \( P_2 \)) the set of incomplete edges of the form \( \{x, t(x)\} \) (resp. \( \{x, l(x)\} \)). Recall that \( p_i = |P_i| \) (i = 1, 2), and since \( P_1 \cap P_2 = \emptyset \), we have \( m - 2q = p_1 + p_2 \). We first compute the number \( B_{m,q,s,p_1} \) of premaps for a given \( p_1 \). By Lemma 1.1 the number of choices for \( t \) and \( r \) are given by:

\[ \nu(t) = \frac{(2m)!}{(2p_2)! 2^{2q+p_1}(2q+p_1)!} \quad \text{and} \quad \nu(r) = \frac{(2m)!}{(2m-2s)! 2^s s!} \]

Fix a pair \( r \) and \( t \). The permutation \( t \) has \( 2q + p_1 \) orbits. To count the numbers of choices for \( l \) let, as before, \( Q \) be the set of orbits of \( t \) forming complete edges. The number of choices for \( Q \) is \( \binom{2q + p_1}{2q} \). Each choice of \( l \) induces a perfect matching of elements of \( Q \). The number of such matchings coincides with the number of fixed point free
involutions acting on \(Q\). By Lemma 1.1 this number is \(\frac{(2q)!}{2^q q!}\). Now, for each complete edge there are two possibilities for the action of \(l\), hence there are \(\binom{2q+p_1}{2q}\) \(\frac{(2q)!}{2^q q!}\) ways to determine the action of \(l\) on the flags incident to complete edges. It remains to determine the action of \(l\) on the \(2p_2\) fixed points of \(t\). Since \(l\) acts freely on the set of flags fixed by \(t\), by Lemma 1.2 there are \(\frac{(2p_2)!}{2^{p_2} p_2!}\) such possibilities. Summarizing we have

\[
B_{m,q,s,p_1} = \frac{(2m)!}{(2m-2s)!2^s s! (2p_2)!2^{2q+p_1}(2q+p_1)!} \frac{(2q+p_1)!}{(2q)!(p_1)!} \frac{(2q)!}{2^q q!} \frac{(2p_2)!}{2^{p_2} p_2!} = \frac{((2m)!)^2}{2^{m+s}(2m-2s)!q!s!p_1!p_2!},
\]

and thus

\[
B_{m,q,s} = \sum_{p_1+p_2=m-2q} B_{m,q,s,p_1} = \frac{((2m)!)^2}{2^{m+s}(2m-2s)!q!s!(m-2q)!} \sum_{p_1+p_2=m-2q} \frac{(m-2q)!}{p_1!p_2!} = 2^{2q+s}(2m-2s)!q!s!(m-2q)!.
\]

Inserting the above expression into (3.16) and taking into the account that \(T_{m,q,s} = R(m,q,s)(2m-1)!\) it leads to the required result.

To check the initial conditions \(R(1, 0, 0) = R(1, 0, 1) = 2\), the case \(m = 1\) leads to (necessarily regular) maps on two darts, we note that there are seven such (regular) maps [BJ00]. Three of them (the last three in Figure 5) have branch points and so are not compact maps, although two of them are on the disc and the third on the sphere. The first two of the 4 compact maps have no internal diagonals while the remaining two have one internal diagonal.

![Figure 5: The seven maps with 2 darts.](image)

**Remark.** In the case of empty border we have \(m = 2e, q = e, s = 2e\) and \(R(2e,e,2e) = R(2e,e)\), so this number can be determined by both propositions 3.4 and 3.5.
4 Enumeration of unrooted maps

4.1 Unrooted sensed maps on closed orientable surfaces

We introduce the following functions, the Jordan function
\[ \phi_p(\ell) = \sum_{d | \ell} \mu\left(\frac{\ell}{d}\right) d^p \]
and the even Jordan function
\[ \phi_{even}^p(\ell) = \sum_{d | \ell} d \mu\left(\frac{\ell}{d}\right) d^p. \]

**Theorem 4.1** The number of (unrooted) sensed maps on closed orientable surfaces \( U(n) \) with \( n \) edges is given by the following formula
\[
U(n) = \frac{1}{2n} \sum_{\ell | 2n} \left( \sum_{0 \leq q < \frac{n}{2}} R^+(m, q) \phi_{even}^{q+1}(\ell) + R^+(m, \frac{m}{2}) \phi_{even}^{q+1}(\ell) \right),
\]
where \( R^+(m, q) \) is the number of rooted orientable maps with \( m \) darts and \( q \) complete edges (or, equivalently, with \( m - 2q \) semiedges). Moreover, \( R^+(m, q) \) can be determined by the following recursive formula
\[
R^+(1, 0) = 1, \quad R^+(1, q) = 0 \quad \text{for} \quad q \neq 0,
\]
\[
R^+(m, q) = m b(m, q) - \sum_{s=0}^{q-1} R^+(m - k, q - s) b(k, s),
\]
where \( b(m, q) = \frac{m!}{(m - 2q)! 2^q q!} \).

**Proof:** We employ Theorem 1.3. Since we are going to enumerate sensed maps, the universal group is \( \Gamma = \Delta^+ \). Since we want to compute sensed maps without semiedges (unbranched maps), the property \( \mathcal{P} \) reads as follows: the kernels \( H < \Delta^+ \) are torsion free. Each \( K < \Delta^+ \) of finite index \( m \) gives rise to exactly one rooted map (possibly with semiedges) with \( m \) darts. Let \( q \) be the number of complete edges and \( V, F \) be the number of vertices and faces, respectively. The genus of the supporting surface is related to the parameters \( V, F \) and \( q \) by the Riemann-Hurwitz equation
\[
2g - 2 + V + F = q.
\]
Given \( m, q \) and \( g \), the group \( K \) is the fundamental group of an orbifold supporting the map with signature \([g; 2^{m-2q}, \infty^{V+F}] = [g; 2^{m-2q}, \infty^{q-2g+2}]\). Note that the term \( m - 2q \) counts the number of semiedges. Hence \( K \) has presentation
\[
\pi_1[g; 2^{m-2q}, \infty^{q-2g+2}] = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_{q-2g+2}, x_1, \ldots, x_{m-2q} \rangle.
\]

\[ \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{q-2g+2} [x_j] \prod_{k=1}^{m-2q} x_j = 1, x_j^2 = 1, k = 1, \ldots, m - 2q \].
Claim 1. The abelianization $H_1(K) \cong \mathbb{Z}^{q+1} \oplus \mathbb{Z}_2^{m-2q}$.

Considering the abelianization, the term $\prod_{i=1}^g[a_i, b_i]$ in the nontrivial relator vanishes. Since every map on an orientable surface has at least one vertex and at least one face, $V + F = q - 2g + 2 > 0$ (in fact it must be $\geq 2$). Using the relator containing $c_1$, one can express $c_1$ in terms of the other generators, and reduce the presentation by deleting $c_1$ together with the related relator. Hence $H_1(K) \cong \mathbb{Z}^{2g-2+V+F+1} \oplus \mathbb{Z}_2^{m-2q} = \mathbb{Z}^{q+1} \oplus \mathbb{Z}_2^{m-2q}$.

It follows that the abelianization of $K$ does not depend on $g$, and so we may decompose the set of all subgroups $K < \Delta^+$ of finite index into classes $K(m, q)$ with the same abelianization determined by the parameters $m$ and $q$. Then the number of rooted semi-edge-free sensed maps $R^+(m, q)$ determines the cardinality $|K(m, q)| = R^+(m, q)$.

Denote by $\text{Epi}_o(K, \mathbb{Z}_d)$ the number of (finite) order preserving epimorphisms $K \to \mathbb{Z}_d$.

Claim 2. $\text{Epi}_o(K, \mathbb{Z}_d) = \text{Epi}_o(H_1(K), \mathbb{Z}_d)$.

It is well-known that every element of finite order in a Fuchsian group (2.1) is conjugate to a power of a generator. Thus when counting order-preserving epimorphism onto a cyclic group we may replace $K$ with its abelianization.

Denote by $N_{\Delta^+}(m)$ the number of conjugacy classes of torsion free subgroups of index $m$ in $\Delta^+$. By Proposition 2.1 we have $U(n) = N_{\Delta^+}(2n)$. By Theorem 1.3, Claim 1 and Claim 2 we get the number of sensed maps with $2n$ darts

\begin{equation}
N_{\Delta^+}(2n) = \frac{1}{2n} \sum_{\ell \mid 2n} \left( \sum_{\ell \mid m, m \mid \ell, \ell \mid \ell} \text{Epi}_o(K, \mathbb{Z}_d) \right) \frac{1}{2n} \sum_{\ell \mid 2n} \left( \sum_{\ell \mid m, m \mid \ell, \ell \mid \ell} \text{Epi}_o(\mathbb{Z}^{q+1} \oplus \mathbb{Z}_2^{m-2q}, \mathbb{Z}_d) R^+(m, q). \right)
\end{equation}

The numbers $R^+(m, q)$ are determined in Proposition 3.3. We now determine $\text{Epi}_o(\mathbb{Z}^{q+1} \oplus \mathbb{Z}_2^{m-2q}, \mathbb{Z}_d)$.

Case $m - 2q > 0$. Let us first compute the number of order preserving homomorphisms $\text{Hom}_o(\mathbb{Z}^{q+1} \oplus \mathbb{Z}_2^{m-2q}, \mathbb{Z}_d)$. We have

$$\text{Hom}_o(\mathbb{Z}^{q+1} \oplus \mathbb{Z}_2^{m-2q}, \mathbb{Z}_d) = \delta_{d, \text{even}} d^{q+1},$$

where $\delta_{d, \text{even}} = 0$ if $d$ is odd and 1 otherwise. By Lemma 1.4 we have

$$\text{Epi}_o(\mathbb{Z}^{q+1} \oplus \mathbb{Z}_2^{m-2q}, \mathbb{Z}_d) = \sum_{d \mid \ell} \mu\left(\frac{\ell}{d}\right) \text{Hom}_o(\mathbb{Z}^{q+1} \oplus \mathbb{Z}_2^{m-2q}, \mathbb{Z}_d) = \sum_{d \mid \ell} \mu\left(\frac{\ell}{d}\right) \delta_{d, \text{even}} d^{q+1} = \varphi_{q+1}^{\text{even}}(\ell).$$
Case \( m = 2q \). In this case we want to compute
\[
\mathrm{Epi}_o(\mathbb{Z}^{q+1}, \mathbb{Z}_\ell),
\]
and consequently there is no restriction on the order \( \ell \). In the same way as above we get
\[
\mathrm{Epi}_o(\mathbb{Z}^{q+1}, \mathbb{Z}_\ell) = \varphi_{q+1}(\ell).
\]
Inserting (4.18) and (4.19) into (4.17) we obtain the required result. \( \square \)

### 4.2 Unrooted orientable reflexible maps

We first introduce some notation. Note that \( \Delta^+ < \Delta \) induces a sign structure on \( \Delta \). To simplify notation we write \( K < \Delta^+ < \Delta \) instead of \( K < \Delta \) and \( K \not< \Delta^+ \). Denote by \( N_{\Delta^+}(n) \) the number of conjugacy classes of torsion free subgroups of \( \Delta^+ \) of index \( n \) (note that \( n \) must be even). Further, denote by \( N_{\Delta^+}^+(2n) \) the number of conjugacy classes of torsion free subgroups \( K < \Delta^+ < \Delta \) of index \( 2n \) in \( \Delta \). Set
\[
I(n) = \frac{1}{n} \sum_{\ell | 2n} \sum_{m | \ell \Delta} \mathrm{Epi}^+_o(K^-, \mathbb{Z}_\ell),
\]
where \( \mathrm{Epi}^+_o(K^-, \mathbb{Z}_\ell) \) is the number of orientation and order preserving epimorphisms of \( K^- \) onto \( \mathbb{Z}_\ell \) with torsion free kernels.

**Lemma 4.2** With the above notation \( N_{\Delta^+}^+(2n) = \frac{N_{\Delta^+}(n) + I(n)}{2} \).

Proof: We apply Theorem 1.3. The property \( \mathcal{P} \) reads as follows: \( G \) is a torsion-free subgroup of \( \Delta \) such that \( G < \Delta^+ < \Delta \), \( [\Delta : G] = 2n \)
\[
N_{\Delta^+}^+(2n) = \frac{1}{2n} \sum_{\ell | 2n} \sum_{m | \ell \Delta} \mathrm{Epi}^+_o(K, \mathbb{Z}_\ell) =
\]
\[
\frac{1}{2n} \sum_{\ell | 2n} \sum_{m | \ell \Delta} \mathrm{Epi}^+_o(K^+, \mathbb{Z}_\ell) + \frac{1}{2n} \sum_{\ell | 2n} \sum_{m | \ell \Delta} \mathrm{Epi}^+_o(K^-, \mathbb{Z}_\ell) =
\]
\[
\frac{1}{2n} \sum_{\ell | 2n} \sum_{m | \ell \Delta} \mathrm{Epi}^+_o(K, \mathbb{Z}_\ell) + \frac{1}{2n} \sum_{\ell | 2n} \sum_{m | \ell \Delta} \mathrm{Epi}^+_o(K^-, \mathbb{Z}_\ell),
\]
where in the first double summand \( m = 2\tilde{m} \) is even and in the second one \( \ell = 2\tilde{\ell} \) is even (shown next). The first term is precisely \( \frac{1}{2} N_{\Delta^+}(n) \) and the second one is \( \frac{1}{2} I(n) \).

To complete the proof we need to see that indeed \( \ell \) is even. Let \( \psi : K^- \to \mathbb{Z}_\ell \) be an epimorphism. Since \( \ker(\psi) \leq K^- \cap \Delta^+ < K^- \) and \( [K^- : K^- \cap \Delta^+] = 2 \), it follows that \( \mathbb{Z}_\ell \cong K^- / \ker(\psi) \) is even. \( \square \)
REMARK. By Proposition 2.3(3) the number of $N^+_\Delta(2n)$ coincides with the number $Z(n)$ of unrooted unsensed maps on closed orientable surfaces having $n$ edges. We will use this observation later to calculate $Z(n)$.

Consider the following coset decomposition $\Delta = \Delta^+ \cup \rho \Delta^+$. Let $K$ be a subgroup of $\Delta^+$. Denote by $[K]_\Delta^+$ and $[K]_\Delta$ the conjugacy classes of $K$ in $\Delta^+$ and $\Delta$, respectively.

There are two kinds of subgroups $K$ in $\Delta^+$: the reflexible ones with the property $[K]_\Delta^+ = [K^\rho]_\Delta^+ = [K]_\Delta$ and the twins ones with $[K]_\Delta^+ \neq [K^\rho]_\Delta^+$ and $[K]_\Delta = [K]_\Delta^+ \cup [K^\rho]_\Delta^+$.

Denote by $A_\Delta(2n)$ and $T_\Delta(2n)$ the respective numbers of conjugacy classes of torsion free subgroups $K$ of index $2n$ in $\Delta$. We note that the numbers $A_\Delta(2n)$ and $T_\Delta(2n)$ coincide with the number $A(n)$ of reflexible (or achiral) maps and the number $T(n)$ of twin maps (or chiral pairs) with $n$ edges on closed orientable surfaces, respectively.

By definition $A_\Delta(2n) = N^+_\Delta(2n) - T_\Delta(2n)$ and $T_\Delta(2n) = \frac{1}{2}(N^+_\Delta(n) - A_\Delta(2n))$.

Then by Lemma 4.2

$$A_\Delta(2n) - T_\Delta(2n) = \frac{1}{2}N^+_\Delta(n) - \frac{1}{2}I(n) - \frac{1}{2}(N^+_\Delta(n) - A_\Delta(2n)) = \frac{1}{2}I(n) + \frac{1}{2}A_\Delta(2n).$$

We have proved:

**Proposition 4.3** With the above notation we have

(i) $A_\Delta(2n) = I(n)$,

(ii) $T_\Delta(2n) = \frac{N^+_\Delta(n) - I(n)}{2}$.

The function $N^+_\Delta(n)$ (= $U(\frac{n}{2})$) gives the number of semiedge-free sensed maps with $\frac{n}{2}$ edges and this was already determined in Proposition 4.1. We shall now concentrate on $I(n)$. To do this we need to recognize the structure of the orbifolds $O = \mathbb{H}^2/K$ given by (possibly non-torsion-free) subgroups $K^- < \Delta$ such that there exists an order and orientation preserving epimorphism $\psi : K \to \mathbb{Z}_{2\ell}$ with torsion free kernel in $\Delta$. In this case, $H = \text{Ker}(\psi) < K \cap \Delta^+ \lhd \Delta^+$ is a torsion-free subgroup of $\Delta$ and $S = \mathbb{H}^2/H \to O = \mathbb{H}^2/K$ is a cyclic $\mathbb{Z}_{2\ell}$-covering of the non-orientable orbifold $O$ by the orientable surface $S$. Since $\psi(K \cap \Delta^+)$ has index 2 in $\mathbb{Z}_{2\ell}$, the covering group $\mathbb{Z}_{2\ell}$ is provided by the natural sign structure: orientation-preserving elements are positive and orientation-reversing ones are negative. The signatures of the admissible orbifolds $O$ split into the following 3 types:

**I** $(p, -; [2^b, \infty^{V+F}]; \emptyset)$, with $b \geq 0$, $V \geq 1$ and $F \geq 1$.

**II** $(g, +; [\infty^{V_0+F_0}]; \{\infty^i : 1 \leq i \leq k\})$, with $V_0 \geq 0$, $F_0 \geq 0$ and $k \geq 1$.

**III** $(p, -; [\infty^{V_0+F_0}]; \{\infty^i : 1 \leq i \leq k\})$, with $V_0 \geq 0$, $F_0 \geq 0$ and $k \geq 1$. 

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The notation for the signature of NEC groups used here is, as said earlier, standard and comes from the book [Buj90]. The interpretation of the parameters in the map given by $K^-$ follows: $b$ is the number of (internal) semiedges, $V$ and $F$ are the numbers of vertices and faces, respectively, and $V_0$ and $F_0$ are the numbers of internal vertices and faces, respectively. Since the map associated with $K^-$ is a cyclic quotient of the map on an orientable surface given by $H$, there are no boundary semiedges, and internal semiedges may appear only if the boundary is empty. Roughly speaking, the orbifold $O$ contains either branch points or boundary components. Translating to group theory, there is at most one involution in $\mathbb{Z}_2$ and $\psi$ cannot send at the same time a negative and a positive element onto this involution.

We introduce the following odd Jordan function

$$\varphi^\text{odd}_k(\ell) = \sum_{d|\ell, \frac{\ell}{d} \text{ odd}} \mu\left(\frac{\ell}{d}\right) d^k$$

and will analyze each case separately.

**Lemma 4.4** Let $K^- < \Delta$ be a NEC group of type (I), with $b \geq 0$, $V \geq 1$ and $F \geq 1$. Then the index $[\Delta : K^-] = 2m$ is an even number and

(i) $\text{Epi}^+_o(K^-, \mathbb{Z}_2\ell) = 0$, if $\ell$ is odd and $b > 0$,

(ii) $\text{Epi}^+_o(K^-, \mathbb{Z}_2\ell) = \sum_{d|\ell, \frac{\ell}{d} \text{ odd}} \mu\left(\frac{\ell}{d}\right) d^k = \varphi^\text{odd}_{q+1}(\ell)$, if $\ell$ is even, or

(iii) $\text{Epi}^+_o(K^-, \mathbb{Z}_2\ell) = \varphi^\text{odd}_{q+1}(\ell)$, if $b = 0$,

where $q = \frac{m-b}{2}$ is the number of complete edges of the map associated with $K^-$. 

**Proof:** By the Riemann-Hurwitz formula [Buj90] we have

$$p - 2 + k + \sum_{i=1}^{b} (1 - \frac{1}{2}) + \sum_{i=1}^{V+F} (1 - \frac{1}{\infty}) = [\Delta : K^-] \cdot \frac{1}{4}.$$

Hence, $[\Delta : K^-] = 2m$, where $m = 2(p - 2 + V + F) + b$ is an integer.

Let the group $\mathbb{Z}_2\ell = \langle \gamma^- \rangle$ be endowed with a nontrivial sign structure. First note that if $\ell$ is odd then the unique involution $(\gamma^-)^\ell$ of $\mathbb{Z}_2\ell$ is negative. If $b > 0$ then $\text{Epi}^+_o(K^-, \mathbb{Z}_2\ell) = 0$, since the image of any positive involutions in $K$ under an orientation- and order-preserving epimorphism cannot be negative.

Now let $\ell$ be even. By Lemma 1.6 we have

$$\text{Epi}^+_o(K^-, \mathbb{Z}_2\ell) = \sum_{d|\ell, \frac{\ell}{d} \text{ odd}} \mu\left(\frac{\ell}{d}\right) |\text{Hom}^+_o(K^-, \mathbb{Z}_{2d})|.$$
We now compute the number of orientation- and order-preserving homomorphisms $K^- = \pi_1(O) \to \mathbb{Z}_{2d}$ for $d|\ell$. By definition

$$\pi_1(0) = \langle x_1, \ldots, x_{V+F}, y_1, \ldots, y_b, d_1, \ldots, d_p \rangle$$

satisfying the relations

$$x_1 \cdots x_{V+F}y_1 \cdots y_b d_1^2 \cdots d_p^2 = 1, \quad y_i^2 = 1, \quad i = 1, \ldots, b.$$ 

Since $V+F > 0$ we can reduce the presentation by expressing $x_1$ from the non-trivial relator. This way we get

$$\pi_1(0) = \langle x_2^+, \ldots, x_{V+F}^+, y_1^+, \ldots, y_b^+, d_1^-, \ldots, d_p^- | (y_i^+)^2 = 1, \quad i = 1, \ldots, b, \rangle$$

where the signs ± indicate whether a generator is positive or negative.

By Lemma 1.5 we can replace $\pi_1(O)$ by its abelianization

$$H_1(0) = (\mathbb{Z}^+)^{V+F-1} \oplus (\mathbb{Z}_2^+)^b \oplus (\mathbb{Z}^-)^p.$$ 

Hence

$$\text{Hom}^+_o(K^-, \mathbb{Z}_{2d}) = \text{Hom}^+_o(H_1(O), \mathbb{Z}_{2d}) = \text{Hom}^+_o(\mathbb{Z}^+, \mathbb{Z}_{2d})^{V+F-1} \text{Hom}^+_o(\mathbb{Z}^+, \mathbb{Z}_{2d})^b \text{Hom}^+_o(\mathbb{Z}^-, \mathbb{Z}_{2d})^p.$$ 

Recall that $\mathbb{Z}_{2d} \cong \langle \gamma^2 \rangle$ is a group with the unique nontrivial sign structure. It follows that $\text{Hom}^+_o(\mathbb{Z}^+, \mathbb{Z}_{2d}) = d$, $\text{Hom}^+_o(\mathbb{Z}^-, \mathbb{Z}_{2d}) = d$ and $\text{Hom}^+_o(\mathbb{Z}_2^+, \mathbb{Z}_{2d}) = \delta_{d, \text{even}}$. Hence

$$\text{Epi}^+_o(K^-, \mathbb{Z}_{2d}) = \sum_{d|\ell, \frac{d}{2} \text{ odd}} \mu\left(\frac{\ell}{d}\right) \left| \text{Hom}^+_o(K^-, \mathbb{Z}_{2d}) \right| = \sum_{d|\ell, \frac{d}{2} \text{ odd}} \mu\left(\frac{\ell}{d}\right) d^{V+F-p-1} \delta_{d, \text{even}}.$$ 

By the Euler-Poincaré equation we have $V + F + p - 1 = q + 1$, where $q$ is the number of complete edges. Thus, taking into account that $\ell$ is even

$$\text{Epi}^+_o(K^-, \mathbb{Z}_{2d}) = \sum_{d|\ell, \frac{d}{2} \text{ odd}} \mu\left(\frac{\ell}{d}\right) d^{q+1} \delta_{d, \text{even}} = \sum_{d|\ell, \frac{d}{2} \text{ odd}} \mu\left(\frac{\ell}{d}\right) d^{q+1} = \varphi^{\text{odd}}_{q+1}(\ell).$$ 

Applying the same arguments to the case $b = 0$ we obtain

$$\text{Hom}^+_o(K^-, \mathbb{Z}_{2d}) = \text{Hom}^+_o(\mathbb{Z}^+, \mathbb{Z}_{2d})^{V+F-1} \text{Hom}^+_o(\mathbb{Z}^-, \mathbb{Z}_{2d})^p = d^{q+1}$$ 

and

$$\text{Epi}^+_o(K^-, \mathbb{Z}_{2d}) = \sum_{d|\ell, \frac{d}{2} \text{ odd}} \mu\left(\frac{\ell}{d}\right) d^{q+1} = \varphi^{\text{odd}}_{q+1}(\ell).$$ 

□
Lemma 4.5 Let $K^- < \Delta$ be a NEC group of type (II), or of type (III), with $V_0 \geq 0$, $F_0 \geq 0$ and $k \geq 1$. Then the index $[\Delta : K^-] = 2m$ is an even number and

(i) $\text{Epi}_o^+(K^-, \mathbb{Z}_2^\ell) = 0$, if $\ell$ is even, or

(ii) $\text{Epi}_o^+(K^-, \mathbb{Z}_2^\ell) = \varphi_{-m+q+s+1}(\ell)$, if $\ell$ is odd,

where $s_1 + s_2 + \cdots + s_k = 3m - 2q - 2s$, $q$ is the number of complete edges, $2m$ is the number of flags and $s$ is the number of internal diagonals of the map associated with $K^-$. 

Remark. Formula (ii) extends to the case $k = 0$ as well, since in this case $s = m$ and (ii) becomes (iii) of Lemma 4.4.

Proof: The parity of the index $[\Delta : K^-] = 2m$ follows from the Riemann-Hurwitz formula. Denote by $H$ the kernel of an orientation- and order-preserving epimorphism $K^- \to \mathbb{Z}_2^\ell$. Since the supporting surface of the orbifold $O = \mathbb{H}^2/K$ has a nonempty boundary the unique involution $(\gamma^-)^e$ acts as a reflection on the orientable surface $S = \mathbb{H}^2/H$. Hence, $\ell$ is odd. We only provide a proof for the orientable case. The proof is similar for the non-orientable case. Denote by $b = s_1 + s_2 + \cdots + s_k$. The orbifold fundamental group is generated by $V_0 + F_0 + 2g + k$ generators of infinite order and $b + k$ generators $c_{i,j}$ of order two, where $1 \leq i \leq k$ and $0 \leq j \leq s_i$. The relations are

$$x_1 x_2 \cdots x_{V_0 + F_0} e_1 e_2 \cdots e_k [a_1, b_1] \cdots [a_g, b_g] = 1, e_i^{-1} c_{i,0} e_i = c_{i,s_i} \text{ for } i = 1, 2, \ldots, k, c_{i,j} = 1 \text{ for } 1 \leq i \leq k, 0 \leq j \leq s_i. \text{ Since } k \geq 1 \text{ we may express } e_i \text{ from the non-trivial relator and reduce the presentation. Considering the abelianization, the set of relators } e_i^{-1} c_{i,0} e_i = c_{i,s_i} \text{ allow us to reduce further the presentation of } H_1(O) \text{ by eliminating } k \text{ involutions, namely the images of } c_{i,0}. \text{ Hence } H_1(O) = (\mathbb{Z}^+)^{V_0 + F_0 + 2g + k - 1} \oplus (\mathbb{Z}_2)^b. \text{ Denote } h = V_0 + F_0 + 2g + k - 1 \text{ and } H_1 = H_1(O). \text{ The induced signature on } H_1 \text{ forces all the generators of infinite order to be positive and all generators of order two to be negative.}

As above we first count $\text{Hom}_o^+(\mathbb{Z}^+) \oplus (\mathbb{Z}_2)^b, \mathbb{Z}_2^d) = d^h \cdot \delta_{d, odd}$. Since, if $\ell$ is odd and $d|\ell$, then $d$ is also odd, we have

$$\text{Epi}_o^+(K^-, \mathbb{Z}_2^\ell) = \sum_{d | \ell, \frac{\ell}{d} \text{ odd}} \mu\left(\frac{\ell}{d}\right) |\text{Hom}_o^+(K^-, \mathbb{Z}_2^d)| = \sum_{d | \ell, \frac{\ell}{d} \text{ odd}} \mu\left(\frac{\ell}{d}\right) d^h \delta_{d, odd}.$$ 

Hence

$$\text{Epi}_o^+(K^-, \mathbb{Z}_2^\ell) = \sum_{d | \ell, \frac{\ell}{d} \text{ odd}} \mu\left(\frac{\ell}{d}\right) d^h = \varphi_{h}^{\text{odd}}(\ell).$$

Now we compute the parameter $h = V_0 + F_0 + 2g + k - 1$. According to [BS85] the topological characteristic of a reflexible compact map $\mathcal{K} = (F; l, r, t)$ with $q$ complete edges is

$$2 - 2g - k = \chi(\mathcal{K}) = V_0 - q + F_0 + \frac{1}{2}(V_b - E_b + F_b),$$

where $V_b$, $E_b$ and $F_b$ are the number of vertices, edges and faces on the boundary respectively. Now
\( V_b = \frac{1}{2}(\# \text{ fixed points of } r + \# \text{ fixed points of } t) \)

\( E_b = \frac{1}{2}(\# \text{ fixed points of } t + \# \text{ fixed points of } l) \)

\( F_b = \frac{1}{2}(\# \text{ fixed points of } l + \# \text{ fixed points of } r) \)

so \( V_b - E_b + F_b = \# \text{ fixed points of } r = 2m - 2s \), where \( s \) is the number of internal diagonals. Hence we have \( V_0 - q + F_0 + m - s = 2 - 2g - k \) from which we derive \( h = q + s - m + 1 \).

**Theorem 4.6** The number of (unrooted) reflexible maps \( A(e) \) with \( e \) edges is given by the following formula

\[
A(e) = \frac{1}{2^e} \sum_{\ell|m} E(\ell, m),
\]

where

\[
E(\ell, m) = \begin{cases}
\sum_{q=0}^{[\frac{m}{2}]} \sum_{s=0}^{m} R(m, q, s) \varphi_{-m+q+s+1}(\ell) - R^+(m, \frac{m}{2}) \varphi_{\frac{m}{2}+1}(\ell), & \text{if } \ell \text{ is odd}, \\
\sum_{q=0}^{[\frac{m}{2}]} R^-(m, q) \varphi_{q+1}(\ell), & \text{if } \ell \text{ is even},
\end{cases}
\]

where the functions \( R^-(m, q), R^+(m, q) \) and \( R(m, q, s) \) are determined by the Propositions 3.3, 3.4 and 3.5, respectively.

**Proof:** By Proposition 4.3 we have

\[
A(e) = A_\Delta(4e) = I(2e) = \frac{1}{2^e} \sum_{\ell|m} \sum_{\ell\in\Delta} \text{Epi}_o^+(K^-, \mathbb{Z}_{2\ell}).
\]

Since by Lemma 4.4 and Lemma 4.5 the index \([\Delta : K^-] = \tilde{m} \) is even, we write \( \tilde{m} = 2m \) and thus

\[
A(e) = \frac{1}{2^e} \sum_{\ell|m} \sum_{\ell\in\Delta} \text{Epi}_o^+(K^-, \mathbb{Z}_{2\ell}).
\]

It remains to prove \( \sum_{K^-<\Delta} \text{Epi}_o^+(K^-, \mathbb{Z}_{2\ell}) = E(\ell, m) \). We count the number of such epimorphisms.

First assume that \( \ell \) is odd. Since \( \text{Epi}_o^+(K^-, \mathbb{Z}_{2\ell}) = 0 \) if \( b > 0 \), we want to count only subgroups \( K^- < \Delta \) which are NEC groups with signatures of type (I) with \( b = 0 \) and of type (III). Now \( R(m, q, s) \) is the number of rooted compact maps (hence \( b = 0 \)) with \( 2m \) flags, \( q \) complete edges and \( s \) internal diagonals. Denote by \( R^+(m, q, s) \) the number
of those rooted compact maps with $2m$ flags, $q$ complete edges and $s$ internal diagonals which have empty border ($k = 0$) and are orientable. Then

$$R^+(m, q, s) = \begin{cases} R(m, \frac{m}{2}, m) = R^+(m, \frac{m}{2}), & \text{if } m \text{ is even, } q = \frac{m}{2}, s = m, \\ 0, & \text{if otherwise.} \end{cases} \quad (4.20)$$

Then $R(m, q, s) - R^+(m, q, s)$ counts the number of non-orientable rooted compact maps with $2m$ flags, $q$ complete edges and $s$ internal diagonals. Thus the sum

$$\sum_{q=0}^{[m/2]} \sum_{s=0}^{m} (R(m, q, s) - R^+(m, q, s))$$

counts the number of non-orientable rooted compact maps with $2m$ flags, that is, the number of subgroups $K^- < \Delta$ which are NEC groups with signatures of type (I) with $b = 0$ and of type (III). Now for each $K^- < \Delta$ counted above, if $K$ has signature of type (I), that is if $K$ is a NEC non-orientable group without border, it gives rise to $\varphi_{m+q+s+1}(\ell)$ orientation- and order-preserving epimorphisms (Lemma 4.4), while if it has signature (III), that is if $K$ is a non-orientable NEC group with non-empty border ($k > 1$), it gives rise to $\varphi_{-m+q+s+1}(\ell)$ orientation- and order-preserving epimorphisms (Lemma 4.5). However, the last expression is also valid for $k = 0$ giving the same value as (iii) of Lemma 4.4. Therefore since any $K^- < \Delta$ counted above gives rise to $\varphi_{-m+q+s+1}(\ell)$ orientation- and order-preserving epimorphisms we must have (taking into account (4.20))

$$\sum_{K^- < 2m \Delta} \text{Epi}^+(K^-, \mathbb{Z}_{2\ell}) = \sum_{q=0}^{[m/2]} \sum_{s=0}^{m} (R(m, q, s) - R^+(m, q, s)) \varphi_{-m+q+s+1}(\ell)$$

$$= \sum_{q=0}^{[m/2]} \sum_{s=0}^{m} R(m, q, s) \varphi_{-m+q+s+1}(\ell) - R^+(m, \frac{m}{2}) \varphi_{\frac{m}{2}+1}(\ell).$$

Now let $\ell$ be even. It follows from Lemma 4.5 that it is sufficient to take into account groups $K^-$ of type (I) for $b \geq 0$. There are $R^-(m, q)$ such groups, thus by Lemma 4.4(ii) and (iii) we obtain

$$\sum_{K^- < 2m \Delta} \text{Epi}^+(K^-, \mathbb{Z}_{2\ell}) = \sum_{q=0}^{[m/2]} R^-(m, q) \varphi_{q+1}(\ell), \quad \text{if } \ell \text{ is even.}$$

□

Employing now Lemma 4.2 and Proposition 4.3 we get the following two corollaries:

**Corollary 4.7** The number of (unrooted) unsensed maps $Z(e)$ with $e$ edges is given by

$$Z(e) = \frac{1}{2} (U(e) + A(e)),$$

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where the numbers $U(e)$ and $A(e)$ are determined by the Theorems 4.1 and 4.6.

Corollary 4.8 The number of twin maps (or chiral pairs) with $e$ edges is given by

$$T(e) = \frac{1}{2}(U(e) - A(e)), \quad \text{where the numbers } U(e) \text{ and } A(e) \text{ are determined by the Theorems 4.1 and 4.6.}$$

5 Final Remarks

During preparation of this paper we have discovered that R. Robinson has investigated the same problem (enumeration of maps regardless of genus), see the lecture notes [Rb05]. The tables generated following his recursive formulas fits perfectly with the ones produced by Mathematica 5.1 based on our formulas, see the attached tables. However, the method of Robinson is different from the approach undertaken in this article. Moreover, as indicated in [Rb05] his method cannot be applied for enumeration of maps with given genus (compare with [MeN06], where sensed unrooted maps of given genus are enumerated).

The last two authors are grateful to the Combinatorial and Computational Mathematics Center at Pohang University of Science and Technology as well as to Department of Mathematics of University Aveiro for creating excellent conditions on the work on the problem investigated in this paper.
6 Tables for sensed unrooted maps and twins

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Table 5. The number $U(n)$ of sensed unrooted maps with $n$ edges
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Table 6. The number $A(n)$ of reflexible unrooted maps with $n$ edges
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Table 7. The number $T(n)$ of twins with $n$ edges

References


