Half-arc-transitive graphs and chiral hypermaps

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Abstract

A subgroup $G$ of automorphisms of a graph $X$ is said to be $\frac{1}{2}$-arc-transitive if it is vertex- and edge- but not arc-transitive. The graph $X$ is said to be $\frac{1}{2}$-arc-transitive if $\text{Aut} X$ is $\frac{1}{2}$-arc-transitive. The interplay of two different concepts, maps and hypermaps on one side and $\frac{1}{2}$-arc-transitive group actions on graphs on the other, is investigated. The correspondence between regular maps and $\frac{1}{2}$-arc-transitive group actions on graphs of valency 4 given via the well known concept of medial graphs (European J. Combin. 19 (1998) 345) is generalised. Any orientably regular hypermap $H$ gives rise to a uniquely determined medial map whose underlying graph $Y$ admits a $\frac{1}{2}$-arc-transitive group action of the automorphism group $G$ of the original hypermap $H$. Moreover, the vertex stabiliser of the action of $G$ on $Y$ is cyclic. On the other hand, given graph $X$ and $G \leq \text{Aut} X$ acting $\frac{1}{2}$-arc-transitively with a cyclic vertex stabiliser, we can construct an orientably regular hypermap $H$ with $G$ being the orientation preserving automorphism group. In particularly, if the graph $X$ is $\frac{1}{2}$-arc-transitive, the corresponding hypermap is necessarily chiral, that is, not isomorphic to its mirror image. Note that the associated $\frac{1}{2}$-arc-transitive group action on the medial graph induced by a map always has a stabiliser of order two, while when it is induced by a (pure) hypermap the stabiliser can be cyclic of arbitrarily large order. Hence moving from maps to hypermaps increases our chance of getting different types of $\frac{1}{2}$-arc-transitive group action. Indeed, in last section we have applied general results to construct $\frac{1}{2}$-arc-transitive graphs with cycle stabilisers of arbitrarily large orders.

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1. Introduction

Throughout this paper we allow graphs to have multiple edges and loops unless otherwise specified. Furthermore, all graphs and groups are assumed to be finite. For group-theoretic terms not defined here we refer the reader to [30].
An automorphism of a graph $X$ is a permutation of its arcs preserving the incidence relation and commuting with the arc-reversing involution. If $X$ is simple then the induced action on the vertex set is faithful and thus our automorphisms can be alternatively described as permutations of vertices of $X$ preserving the adjacency relation. A subgroup $G \leq \text{Aut} X$ of the automorphism group is said to be vertex-transitive, edge-transitive and arc-transitive provided that it acts transitively on the sets of vertices, edges and arcs of $X$, respectively. Moreover, $G$ is said to be half-transitive if it is vertex and edge but not arc-transitive. The graph $X$ is said to be vertex-transitive, edge-transitive and arc-transitive if $\text{Aut} X$ is vertex-transitive, edge-transitive and arc-transitive, respectively. We shall say that the graph $X$ is $(G, \frac{1}{2})$-transitive if the group $G \leq \text{Aut} X$ is half-transitive. The graph $X$ is half-transitive if it is $(\text{Aut} X, \frac{1}{2})$-transitive.

There will be instances in our discussion where information on the vertex stabiliser of a subgroup $G \leq \text{Aut} X$ acting transitively on the vertex set $V(X)$ of $X$ is relevant. Let $v \in V(X)$ and let $H = G_v$ be the vertex stabiliser of $v$. We shall say that $X$ is $(G, 1, H)$-transitive and $(G, \frac{1}{2}, H)$-arc-transitive, respectively, provided that the group $G$ acts arc-transitively, and half-arc-transitively on $X$ and $H \cong G_v$ for some $v \in V(X)$.

In 1966 Tutte [28] proved that a vertex-transitive and edge-transitive graph of odd valency is necessarily arc-transitive. Regarding even valency, in 1970 Bouwer [5] constructed a $2k$-valent $\frac{1}{2}$-arc-transitive graph for every $k \geq 2$. The smallest graph in his family has 54 vertices and valency 4. Some years later, Holt [15] found one with 27 vertices. In [1], Alspach et al. proved that no smaller $\frac{1}{2}$-arc-transitive graph exists and exhibited an infinite family of $\frac{1}{2}$-arc-transitive graphs of valency 4. Further results on $\frac{1}{2}$-arc-transitive graphs can be found in [13, 20, 21, 23, 24].

By a topological map we mean a cellular decomposition of a closed surface. A common way of constructing maps is by embedding a graph into a surface. It is well known that a map $\mathcal{M}$ given by an embedding of a graph $X$ into an orientable surface can be completely described by means of its rotation system, that is by listing, for every vertex $v$ of $\mathcal{M}$, the cyclic permutation of the incident outgoing arcs of $v$ induced by the rotation of the supporting surface. We can therefore identify $\mathcal{M}$ with the pair $(X; R)$, where $R$ is the above rotation system for $\mathcal{M}$. An automorphism of $\mathcal{M}$ is an automorphism of its underlying graph commuting with the rotation system of $\mathcal{M}$. The map $\mathcal{M}$ is called orientably regular if its automorphism group $\text{Aut} \mathcal{M}$ acts transitively (and therefore regularly) on the set of arcs of $X$. Regular maps are extensively studied in various branches of mathematics, including combinatorics, Riemann surfaces and group theory [6, 14, 16–18].

We say that a map is 3-coloured if there is a colouring of its faces, say black (0), grey (1) and white (2), such that every vertex of the map is incident with three faces coloured by mutually different colours. A (topological) hypermap $\mathcal{H}$ is a 3-coloured map. It follows that all the 2-cells of a 3-coloured map are of even size. The coloured faces of the map will be called the hypercells of $\mathcal{H}$ while the black, grey and white coloured faces (now referred as 0-, 1-, and 2-faces of the coloured map) will be called hypervertices, hyperedges and hyperfaces of $\mathcal{H}$, respectively. The size of a hypercell is half the size of the corresponding face of the 3-coloured map. The vertices of the underlying 3-coloured map will be called flags of the hypermap. Note that every map $\mathcal{M}$ gives rise to a hypermap $\mathcal{H}(\mathcal{M})$ where...
the hyperedges\(^1\) are of size 2. The flags of \(\mathcal{H}(\mathcal{M})\) are triples \((x, v, f)\), where \(x = vu\) is a dart, \(v\) and \(u\) are the initial and the terminal vertex of \(x\), and \(f\) is one of the two faces \(f\) and \(g\) incident with \(x\). The hypermap \(\mathcal{H}(\mathcal{M})\) is formed by joining the flag \((x, v, f)\) to flags \((x^{-1}, u, f), (x, v, g)\) and \((y, v, f)\), where \(y\) is the other dart with initial vertex \(v\) and incident with the face \(f\). The above construction of \(\mathcal{H}(\mathcal{M})\) basically coincides with the combinatorial description of topological maps by means of three involutions developed earlier (see [19, 29]).

An automorphism of a hypermap \(\mathcal{H}\) is a permutation of flags of \(\mathcal{H}\) which extends to a self-homeomorphism of the surface, mapping hypervertices onto hypervertices, hyperfaces onto hyperfaces and hyperedges onto hyperedges and preserving the incidence relation between them. If the surface is orientable there is an orientation preserving subgroup of \(\text{Aut}\mathcal{H}\) of index two. This group will be denoted by \(\text{Aut}^+\mathcal{H}\). It is well known that \(|\text{Aut}^+\mathcal{H}| \leq |F|/2\) and \(|\text{Aut}\mathcal{H}| \leq |F|\), where \(F\) is the set of flags of \(\mathcal{H}\). The hypermap \(\mathcal{H}\) will be called orientably regular if \(|\text{Aut}^+| = |F|/2\) and regular if \(|\text{Aut}\mathcal{H}| = |F|\). For more information on hypermaps and their automorphisms the reader is referred to [9–12].

The notion of medial graphs has arisen in the study of polyhedra, that is the 3-connected spherical maps. Given a map \(\mathcal{M}\) on a surface, its medial map is constructed as follows. First subdivide each edge of \(\mathcal{M}\) by one new vertex. If \(e\) and \(f\) are consecutive edges in a boundary walk of a face of \(\mathcal{M}\), join the corresponding new vertices by an edge. Finally delete all the old vertices and incident edges from the surface. The 4-valent map that we have constructed is the medial map for \(\mathcal{M}\). As a consequence of the definition we get that \(\mathcal{M}\) and its dual \(\mathcal{M}^*\) share the same medial map. Another important feature of the medial map is that its faces split into two disjoint families: vertex-faces containing vertices of the original map \(\mathcal{M}\) and face-faces containing the centres of faces of \(\mathcal{M}\). In other words, the dual map of the medial map is necessarily bipartite. It is not difficult to see that to be 4-valent and to have a bipartite dual are on the other hand sufficient conditions for a map to be a medial map of some map. It is proved in [22] that if the map automorphism group \(G = \text{Aut}^+\mathcal{M}\) acts regularly on arcs of \(\mathcal{M}\), then the medial graph is \((G, \frac{1}{2}, Z_2)\)-arc-transitive. On the other hand, every \((G, \frac{1}{2}, Z_2)\)-arc-transitive graph \(X\) defines an orientably regular map with the automorphism group \(G = \text{Aut}^+\mathcal{M}\) and such that \(X\) is the medial graph for \(\mathcal{M}\).

In the present paper we first generalise the concept of a medial map for hypermaps. Second, we show that results proved in [22] present particular cases of more general statements on hypermaps and their medials. To be more precise, given a hypermap \(\mathcal{H}\), its medial map \(\text{Med}(\mathcal{H})\) is constructed as follows (see Fig. 2).

Take the topological map associated with \(\mathcal{H}\). Choose one vertex inside each grey face representing a hyperedge of \(\mathcal{H}\). Form \(\text{Med}(\mathcal{H})\) by contracting grey faces to chosen vertices lying in the middle of them (see also Fig. 1). It is straightforward to see that the valency of a vertex in \(\text{Med}(\mathcal{H})\) is equal to twice the size of the corresponding hyperedge. Note that the underlying graph of the medial map may have multiple edges even if the original 3-coloured map representing \(\mathcal{H}\) is free of them.

\(^1\)The edges of \(\mathcal{M}\) (as the faces and vertices) can be arbitrarily associated with any of the three cell colours. Without losing generality we assign colour 1 to edges (colour 2 to faces and colour 0 to vertices) and in this way view \(\mathcal{M}\) as a hypermap with hyperedges of size 2.
In Section 2 we prove that if \( \mathcal{H} \) is an orientably regular hypermap where the hyperedges are of size \( k \geq 2 \), then the medial graph \( X = \text{Med}(\mathcal{H}) \) is \((G, \frac{1}{2}, Z_k)\)-arc-transitive for \( G = \text{Aut}^+\mathcal{H} \). On the other hand, any \((G, \frac{1}{2}, Z_k)\)-arc-transitive graph \( X \) gives rise to an orientably regular hypermap \( \mathcal{H} \) satisfying \( \text{Aut}^+\mathcal{H} = G \) and \( \text{Med}(\mathcal{H}) \cong X \).

In Section 3 we inspect how the existence of some additional self-homeomorphisms of the surface associated with the hypermap \( \mathcal{H} \) is reflected in the medial graph \( Y = \text{Med}(\mathcal{M}) \) and in its automorphism group (Theorem 3.1). In particular, we prove that an orientably regular hypermap for which the medial graph is \( \frac{1}{2} \)-arc-transitive is necessarily chiral. A hypermap is chiral if it is not isomorphic to its mirror image. A more precise definition is given in Section 3.

In Section 4 we develop a method allowing us to construct infinite families of hypermaps with \( \frac{1}{2} \)-arc-transitive medials. As an application we construct a family of \( \frac{1}{2} \)-arc-transitive graphs with cyclic stabilisers of arbitrarily large order (see Corollary 4.5).

2. Orientably regular hypermaps and graphs admitting \( \frac{1}{2} \)-transitive group actions

The aim of this section is to introduce algebraic counterparts of topological notions such as map, hypermap and hypervertex, mentioned in the introduction. While the topological definitions of maps and hypermaps are important for getting general ideas of the objects we are dealing with, their algebraic counterparts are relevant as tools for proving statements. Notions such as planar width are difficult to explain without having a topological point of view.
Let $\mathcal{M} = (X; R)$ be a map on an orientable surface, where $X$ is the underlying graph and $R$ is its rotation system (the permutation of darts of $X$ permuting cyclically, and consistently with a chosen global orientation, the darts based at the same vertices). We set $\mathcal{M} = (D; R, L)$, where $D = D(X)$ denotes the set of arcs of $X$ and $L$ is the arc reversing involution interchanging oppositely directed arcs arising from the same edge. The vertices and edges of the underlying graph correspond to orbits of $R$ and $L$, respectively. An edge is incident with a vertex if and only if the corresponding orbits intersect. Finally, the boundaries of faces of $\mathcal{M}$ are determined by the orbits of $RL$. A map automorphism of a map $\mathcal{M} = (D; R, L)$ is a permutation $\psi$ of $D$ such that $\psi R = R \psi$ and $\psi L = L \psi$. The fact that the underlying graph of a $\mathcal{M}$ is connected is reflected by the transitivity of the action of $(R, L)$ on the set darts. (We refer the reader to [17] for more detailed information.)

A hypermap $\mathcal{H}$ is a 4-tuple $\mathcal{H} = (F; r_0, r_1, r_2)$, where $F$ is the set of flags and $r_0, r_1$ and $r_2$ are fixed-point free involutory permutations of $F$ such that $\langle r_0, r_1, r_2 \rangle$ acts transitively on $F$. The hyperedges are the orbits of $(r_0, r_2)$, hypervertices are the orbits of $(r_1, r_2)$ and hyperfaces are the orbits of $(r_1, r_0)$. Given the topological hypermap $\mathcal{H}$ the three involutory permutations acting on the flags of the associated hypermap $\mathcal{H}$ are derived from the induced 3-edge colouring of the underlying 3-valent graph. That is, an edge is coloured by the unique colour $i \in \{0, 1, 2\}$ not shared by the two incident faces. An $i$-coloured edge is usually called an $i$-edge.

An automorphism of the hypermap $\mathcal{H}$ is a permutation $\psi$ of $F$ satisfying $r_0 \psi = \psi r_0$, $r_1 \psi = \psi r_1$ and $r_2 \psi = \psi r_2$. It is known that the underlying surface of a hypermap is orientable if and only if the even-word subgroup $G^+ = \langle r_1 r_2, r_2 r_0 \rangle = \langle R, L \rangle \leq G$ is the general monodromy group $\text{GMon}(\mathcal{H}) = \langle r_0, r_1, r_2 \rangle$ acts with two orbits $F^+$ and $F^-$. Thus we have an assignment $\alpha : F \to \{-1, 1\}$ defined by the rule $\alpha(x) = 1$ if and only if $x \in F^+$. In what follows we always consider only orientable maps and hypermaps. An automorphism $\psi$ of $\mathcal{H}$ is orientation preserving if $\alpha \psi x = \alpha x$ for every flag $x$ in $F$. The hypermap $\mathcal{H}$ is said to be orientably regular if the group of orientation preserving automorphisms $\text{Aut}^+ \mathcal{H}$ acts transitively (regularly) on $F^+$ and $\mathcal{H}$ is regular if $\text{Aut} \mathcal{H}$ acts transitively (regularly) on $F$. It can be easily seen that a regular hypermap on an orientable surface is orientably regular as well. On the other hand, there are examples of hypermaps which are orientably regular but not regular. We call such hypermaps chiral. In other words a chiral hypermap is an orientable regular hypermap which is not isomorphic to its mirror image. To verify whether a given orientable regular hypermap $(F; r_0, r_1, r_2)$ is chiral it is sufficient to see that the assignment $R \mapsto R^{-1}$, $L \mapsto L^{-1}$ does not extend to a group automorphism $G^+ \to G^-$, where $G^+ = \langle r_1 r_2, r_2 r_0 \rangle = (R, L)$.

Given an orientable hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ we associate a map $\text{Med}(\mathcal{H}) = (D; L, R)$ by setting $D = F$, $L = r_1$ and $Rx = r_0 x$ if $\alpha x = 1$ and $Rx = r_2 x$ if $\alpha x = -1$. Since $\alpha r_2 x = \alpha r_0 x = -\alpha x$, the permutation $R$ is well defined. The underlying graph of $\text{Med}(\mathcal{H})$ will be called the medial graph of $\mathcal{H}$. Depending on the context, $\text{Med}(\mathcal{H})$ will denote either the medial map or the corresponding medial graph. Let us also note that the medial map $\text{Med}(\mathcal{H}) = (D; R, L)$ is, up to mirror symmetry, uniquely determined.

In what follows we shall assume that the surface is endowed with the preferred global orientation and by $\text{Med}(\mathcal{H})$ we mean the medial map whose rotation $R$ is consistent with the chosen global rotation.
The following result is fundamental.

**Proposition 2.1.** Let $H$ be an orientably regular hypermap. Then the group of orientation preserving automorphisms $\text{Aut}^+ H$ acts $\frac{k}{2}$-transitively on the medial graph $\text{Med}(H)$ and the vertex stabiliser is isomorphic to $Z_k$, where $k \geq 2$ is the size of the hyperedges in $H$, that is, $\text{Med}(H)$ has valency $2k$.

**Proof.** Firstly, we prove that every automorphism of the hypermap $H = (F; r_0, r_1, r_2)$ is an automorphism of $\text{Med}(H) = (D; R, L)$. To do this, it is sufficient to see that

$$\psi Lx = \psi r_1 x = r_1 \psi x = L\psi x,$$

for every dart $x \in D = F$ and for every $\psi \in \text{Aut}^+ H$. Since $r_1 \psi x = \alpha x$ we have

$$\psi Rx = \psi r_0 x = r_0 \psi x = R\psi x,$$

if $x \in F^+$, and similarly

$$\psi Rx = \psi r_2 x = r_2 \psi x = R\psi x,$$

if $x \in F^-$. Thus every $\psi \in \text{Aut}^+ H$ is a map automorphism of $\text{Med}(H)$. Since the permutation $L = r_1$ interchanges $F^+$ and $F^-$, it follows that precisely one of two oppositely directed darts $x$ and $Lx$ belongs to $F^+$ and the other one to $F^-$. Since $\text{Aut}^+ H$ acts regularly on $F^+$, it acts transitively on edges and vertices of $\text{Med}(H)$. On the other hand, the orbits of the action of $\text{Aut}^+ H$ on $D$ are $F^+$ and $F^-$. Thus the action is not transitive on darts of the medial graph. Consequently, $\text{Aut}^+ H$ acts half-transitively on $\text{Med}(H)$. To see that the vertex stabiliser is $Z_k$ it is sufficient to observe that $\psi$ stabilises a vertex of $\text{Med}(H)$ if and only if $\psi$ stabilises an orbit of the action of $R$ which is of the form

$$O = \{ x, r_0x, r_2r_0x, r_0r_2r_0x, \ldots, r_2(r_2r_0)^{k-1}x \}$$

for some $x \in F^+$. Since there are exactly $k$ positive darts in $O$ and a vertex stabiliser of an (orientable) map is always cyclic, we are done. $\square$

The converse of this result is proved in Proposition 2.2 below.

**Proposition 2.2.** Let $Y$ be a $(G, \frac{k}{2}, Z_k)$-transitive graph of valency $2k$ for some subgroup of automorphisms $G \leq \text{Aut} Y$. Then there is an orientably regular hypermap $H$ with the orientation preserving automorphism group $G$ such that $Y$ is the medial graph of $H$.

**Proof.** Let us denote by $D$ the set of darts of $Y$ and by $L$ the dart-reversing involution. We define the hypermap $H = (F; r_0, r_1, r_2)$ as follows. We set $F = D$ and $r_i = L$. It follows from the assumption that the group $G$ acts with two orbits on $F = D$ which we denote by $F^+$ and $F^-$. Moreover, the action of $G$ is regular on $F^+$. Thus we can think of darts of $F^+$ as elements of $G$ and we denote by $\psi_g$ the unique automorphism in $G$ mapping a fixed flag $x_0 = 1$ onto flag $g$. Let us denote by $v_0$ the initial vertex of $x_0$. To define $r_0$, first choose one of the $k$ flags in $F^- \cap G_{v_0}$ to be $r_0(x_0) = r_0(1)$. Now set $r_0(g) = \psi_g r_0 \psi_g^{-1}(g)$. In this way $r_0$ is defined for flags in $F^+$. If $x \in F^-$ then there is exactly one flag $g \in G$ such that $x = \psi_g r_0 \psi_g^{-1}(g)$ and in this case we set $r_0(x) = g$. To define $r_2$ we use a generator $\gamma$ of the stabiliser $\gamma = G_{v_0} \cong Z_k$. For every vertex $u \in Y$ let $\gamma_u = \psi_u \gamma \psi_u^{-1}$, where $\psi_u$ is an
automorphism in $G$ mapping $v_0$ to $u$. Clearly, $\gamma_u$ generates $G_u$ and it is independent of the choice of $\psi_u$. Let us note that $\psi_u = \psi_g$, where $g = \psi_u(1)$. Now we set $r_2(x) = r_0\gamma_u^{-1}(x)$ if $x \in F^+$, $r_2(x) = r_0\gamma_u(x)$ if $x \in F^-$, where $u$ is the initial vertex of $x$. It is a consequence of their definitions that all $r_0$, $r_1$ and $r_2$ are fixed-point free permutations. It is easy to see that $r_0$ and $r_1$ are involutory.

We show now that every automorphism in $G$ commutes with each of the permutations $r_0$, $r_1$ and $r_2$. Obviously any $\phi \in G$ commutes with $r_1 = L$. Further we have

$$
\phi r_0x = \phi \psi_x r_0 \psi_x^{-1}(x) = \phi \psi_x r_0 \psi_x^{-1} \phi^{-1} \phi(x) = \phi \psi_x r_0 (\phi \psi_x^{-1} \phi (x)) = \psi \phi \phi^{-1} \phi (x) = r_0 \phi x,
$$

for $x \in F^+$ and $\phi \in G$. If $y \in F^-$ then $y = r_0 x$ for some $x \in F^+$. Thus $\phi r_0 y = r_0 \phi y$ is equivalent to $\phi x = r_0 \phi r_0 x$ which was already verified for any $x \in F^+$. Using the definition of $r_2$ and the regularity of the action of $G$ on $F^+$ we get

$$
\phi r_2x = \phi r_0 \gamma_u^{-1}(x) = r_0 \phi \gamma_u^{-1}(x) = r_0 \phi \gamma_u^{-1} \phi^{-1} \phi (x) = r_0 \gamma_u^{-1} \phi(x) = r_2 \phi x.
$$

Similar calculation can be done for $x \in F^-$. Now we are ready to check that $r_2$ is involutory as well. Since $r_2$ maps $F^+$ to $F^-$ it is sufficient to check it for a flag $g = x \in F^+$. Indeed, for any $g \in G$ we have

$$
\phi (r_2^2(g)) = r_2(r_0 \gamma_u^{-1} \psi_g^{-1} \psi_g(1)) = r_2(\gamma_u^{-1} \psi_g^{-1}(r_0(1)))
= r_0 \gamma_u^{-1} \psi_g^{-1} \psi_g \gamma_u^{-1} \gamma_u(1) = r_0 \gamma_u^{-1} \gamma_u(1) = g.
$$

Thus the group $G$ acts as a group of automorphisms of a hypermap $H = (D^r; r_0, r_1, r_2)$, where $D^r \subseteq D = F$. The regularity of the action of $G$ on $F^+$ implies that $D^r = D$ and proves that $H$ is orientably regular.

Let us denote by $X$ the medial graph of $H$. We claim that the identity mapping $\text{Id}$ on $F$ establishes an isomorphism $Y \rightarrow X$. Indeed, since $L = r_1$ and by the definition of $X$ the involution $L$ is the arc-reversing involution of $X$ as well. Let $R$ be the rotation of $\text{Med}(H)$. To see that the incidence relation associating an arc with its initial vertex is preserved by $\text{Id}$, it is sufficient to see that $x$ and $Rx$ have the same initial vertices in $Y$. By its definition $Rx = r_0(x)$ or $Rx = r_2x$ depending on whether $x \in F^+$ or $F^-$. Now the above definitions of permutations $r_0$ and $r_2$ imply the statement. □

Observe that given a hypermap $H$ its medial map $\text{Med}(H)$ is uniquely determined up to mirror image. Consequently, the underlying $(G, \frac{1}{2}, Z_2)$-arc-transitive graph is uniquely determined by the hypermap $H$. On the other hand, in general, there are many different hypermaps arising from the same pair $(X, G)$ where $X$ is a $(G, \frac{1}{2}, Z_2)$-transitive graph of valency $2k$. The reason is that the medial embedding of $X$ is not uniquely determined by the action of $G$. One can prove that if $M = (D; R, L)$ and $M = (D; R', L)$ are two such embeddings of $X$, then $R' = R^e$, where $e$ is a coprime to the valency $2k$ of $X$.

Let us also note that given an orientably regular hypermap $H = (F; r_0, r_1, r_2)$, by interchanging the roles of the three involutions we can get virtually six hypermaps. Since the hypermap $H^* = (F; r_2, r_1, r_0)$ gives rise to the same medial map as $H$ (up to mirror image), out of these six hypermaps one can get at most three different medial graphs admitting a $\frac{1}{2}$-arc-transitive group action of $G \approx \text{Aut } H$. 

3. External symmetries of hypermaps and their reflection in the medial graphs

Let $\mathcal{H} = (F; r_0, r_1, r_2)$ be an orientable hypermap. An automorphism of $\mathcal{H}$ interchanging $F^+$ and $F^-$ will be called a mirror symmetry. A hypermap admitting a mirror symmetry will be called mirror symmetric. A regular hypermap on an orientable surface is mirror symmetric. A mirror symmetry that fixes a dart of $\mathcal{H}$ is uniquely determined.

As was already mentioned, given a hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ we can permute (re-colour) the three involutions thus obtaining virtually six hypermaps, the duals of $\mathcal{H}$. However, it may happen that some of them, sometimes all of them, are mutually isomorphic. Since a recolouring of faces of the corresponding topological hypermap does not change the underlying surface, all the duals are sitting on the same surface. For our purposes it is important to see whether the mapping realising a particular isomorphism between two duals preserves or changes the global orientation of the underlying surface. This gives rise to the following definition. Let $\sigma$ be a permutation of the set $\{0, 1, 2\}$. The hypermap $K = (F; r_0, r_1, r_2, r_3)$ will be called a $\sigma$-self-dual to $\mathcal{H}$. Hypermap $\mathcal{H}$ will be called positively $\sigma$-self-dual if there exists a hypermap isomorphism $\phi : \mathcal{H} \to \mathcal{H}' = (F; r_2, r_1, r_0)$ mapping $F^+$ onto $F^-$, and we say that $\mathcal{H}$ is negatively $\sigma$-self-dual if there exists a hypermap isomorphism $\mathcal{H} \to \mathcal{H}'$ taking $F^+$ onto $F^-$. The corresponding hypermap isomorphisms will be called a positive self-duality and a negative self-duality, respectively. The geometric difference between the two self-dualities consists in the fact that a positive self-duality preserves the global orientation of the surface, whereas a negative self-duality reverses it. If $\mathcal{H}$ is both positively $\sigma$-self-dual and negatively $\sigma$-self-dual, then we say that $\mathcal{H}$ is $\sigma$-self-dual. It is easy to observe that $\mathcal{H}$ is $\sigma$-self-dual if and only if it is mirror symmetric and positively $\sigma$-self-dual (or negatively $\sigma$-self-dual). Since we want to study medials of hypermaps we have to restrict our consideration to $(02)$-duality, since it follows from the definition of the medial map that the $(02)$-dual of a hypermap $\mathcal{H}$ has the same medial map $\text{Med}(\mathcal{H})$ as $\mathcal{H}$ does and, in general, this is not true for the duals of other sorts. Thus in what follows, $\sigma$ will always be omitted and we shall simply use the terms dual, positive self-duality and negative self-duality of a hypermap meaning $(02)$-dual, positive $(02)$-self-duality and negative $(02)$-self-duality, respectively.

Now we turn our attention to the medial map of an orientably regular hypermap $\mathcal{H}$. Let us denote by $D_m$ the dihedral group of order $2m$. As was already proved, $\text{Aut}^+\mathcal{H}$ acts $\frac{1}{2}$-arc-transitively on the medial graph $\text{Med}(\mathcal{H})$. Generally, the action of the automorphism group $G$ of a (hyper)map is semiregular on flags and, consequently, the vertex stabiliser $G_v$ acts faithfully on the set of darts incident on $v$ provided that the valency $m$ of $v$ is at least 3. Moreover, $G_v$ is always a subgroup of $D_{2m}$. In particular, it follows that $Z_k \leq G_v \leq D_{2k}$, where $G$ is the automorphism group of the medial map $\text{Med}(\mathcal{H})$ of valency $2k$ (for some $k \geq 2$). Hence $G_v$ is isomorphic to one of the following five groups: $Z_k$, $Z_k \times Z_2$, $Z_{2k}$, $D_k$ and $D_{2k}$. To be more precise we state the following theorem. We say that a hypermap $\mathcal{H}$ is $k$-edge-uniform if every hyperedge of $\mathcal{H}$ has size $k$.

**Theorem 3.1.** Let, for some $k \geq 2$, $\mathcal{H}$ be a $k$-edge-uniform hypermap on an orientable surface, let $G = \text{Aut}^+\mathcal{H}$ and let $\mathcal{M} = \text{Med}(\mathcal{H})$ be the medial map of $\mathcal{H}$ with the underlying graph $Y$. Then the following equivalences hold.
1. $\mathcal{H}$ is orientably regular if and only if $Y$ is $(G, \frac{1}{2}, Z_k)$-transitive;
2. $\mathcal{H}$ is regular if and only if $Y$ is $(G, \frac{1}{2}, Z_k)$-transitive and $(G_2, 1, Z_k \times Z_2)$-transitive for some group $G_2$ such that $G \leq G_2 \leq \text{Aut} (\text{Med} (\mathcal{H}))$;
3. $\mathcal{H}$ is orientably regular and positively self-dual if and only if $Y$ is $(G, \frac{1}{2}, Z_k)$-transitive and $(G_3, 1, Z_{2k})$-transitive for some group $G_3$ such that $G \leq G_3 \leq \text{Aut} (\text{Med} (\mathcal{H}))$;
4. $\mathcal{H}$ is orientably regular and negatively self-dual if and only if $Y$ is $(G, \frac{1}{2}, Z_k)$-transitive and $(G_4, \frac{1}{2}, D_k)$-transitive for some group $G_4$ such that $G \leq G_4 \leq \text{Aut} (\text{Med} (\mathcal{H}))$;
5. $\mathcal{H}$ is regular and self-dual if and only if $Y$ is $(G, \frac{1}{2}, Z_k)$-transitive and $(G_5, 1, D_{2k})$-transitive for some group $G_5$, $G \leq G_5 \leq \text{Aut} (\text{Med} (\mathcal{H}))$.

**Proof.** First of all, the equivalence (1) follows from Propositions 2.1 and 2.2. Here we prove the equivalence (4); the proofs of the remaining equivalences may be obtained in a similar way and we omit them.

Let us start with the proof of the first part of (4). Assume that $\mathcal{H}$ is orientably regular and negatively self-dual. Part (1) of the theorem implies that $Y$ is $(G, \frac{1}{2}, Z_k)$-transitive. This action preserves the black–white bipartition $\{B, W\}$ of the family of all cycles bounding faces of the medial map of $\mathcal{H}$. Moreover it induces a decomposition $D = D^+ \cup D^-$ of arcs of the medial graph $Y$ which is compatible with the action of $G$. Recall that every negative-self-duality of $\mathcal{H} = (F; r_0, r_1, r_2)$ can be viewed as an automorphism of the medial graph $Y$. Since $G$ acts transitively on the vertices of $Y$ and a composition of an orientation preserving automorphism of $\mathcal{H}$ with a negative self-duality is again a negative self-duality, we may assume that there exists a negative self-duality $\sigma$ fixing some vertex $v$ of $Y$. By composing $\sigma$ with an appropriate element in $G$, we get a negative-self-duality which fixes a dart $x_0$ of $Y$. Thus we may assume that $\sigma x_0 = x_0$. Since $\sigma$ maps $F^+ = D^+$ onto itself and $\sigma^2$ commutes with $r_0$, $r_1$ and $r_2$, we conclude that $\sigma^2 \in G$, which means that $G \leq G_4 = (G, \sigma)$ is an index two subgroup of $G_4$. To see the structure of the vertex stabiliser in the action of $G_4$ let us consider the stabiliser $G_4(v) = \langle \gamma, \sigma \rangle$, where $\gamma \in G$ is the generator of the cyclic subgroup of order $k$ such that $\gamma(x) = r_0 r_2 x$ for every $x \in D_v = \{x_0, r_0 r_2 x_0, \ldots, (r_0 r_2)^{k-1} x_0\}$. We already have the relation $\gamma^k = \sigma^2 = 1$. Now we show that $\gamma \gamma \sigma = \gamma^{-1}$. Let $x \in D_v$ and note that $G_4(v)$ acts faithfully on $D_v$. We have

$$\sigma \gamma \sigma x = \sigma \gamma (r_0 r_2)x_0 = \sigma \gamma (r_2 r_0)x_0 = \sigma (r_2 r_0)^{i-1} x_0 = (r_0 r_2)^{i-1} x_0 = \gamma^{-1} x.$$ 

Hence the vertex stabiliser $G_4(v)$ is a dihedral group of order $2k$. By its definition, a negative self-duality preserves the set $F^+ = D^+$. Hence the action of $G_4$ is half-transitive.

To prove the converse, let us choose a vertex $v$ of $Y$ and let $\sigma$ be an involutory element of the stabiliser $G_4(v)$ not belonging to $G$. We may assume that $\sigma$ fixes a dart $z$ incident to $v$. We want to prove that $\sigma$ is a hypermap isomorphism $(F; r_0, r_1, r_2) \rightarrow (F; r_2, r_1, r_0)$. Since $\sigma$ is a graph automorphism of $Y$ it commutes with $L = r_1$ (recall the definition of the medial map $\text{Med}(\mathcal{H}) = (D; R, L)$). Let us consider the action of $\sigma$ on the set $D_v = \{z, Rz, R^2 z, \ldots, R^{2k-1} z\}$. Since $\sigma \in \text{Aut}(\text{Med}(\mathcal{H}))$ and not in $G_v$, it is a uniquely determined involutory automorphism of the medial map sending $R^i (z) \rightarrow R^{-i} (z)$. Thus $\sigma$ is a mirror symmetry of the medial map; i.e. we conclude that $R^{-i} \sigma = \sigma R$. Since $\sigma$ maps
$F^+$ onto $F^+$, the definition of $R$ implies $r_2 \sigma = \sigma r_0$ and $r_0 \sigma = \sigma r_2$. Thus $\sigma$ is a negative self-duality and so $\mathcal{H}$ is negatively self-dual. □

It follows from Theorem 3.1, case (3), that the medial map $\text{Med}(\mathcal{H})$ of a hypermap $\mathcal{H}$ is orientably regular if and only if $\mathcal{H}$ is orientably regular and positively self-dual (a particular case of this statement can be found in [2, 22]). This gives rise to a method producing positively self-dual orientably regular $k$-edge-uniform hypermaps from $2k$-valent orientably regular maps with bipartite dual. This method is used in the construction of infinite families of regular positively self-dual orientably regular maps in [2].

Recall that a map is chiral if it is orientably regular and mirror asymmetric. The following corollary is an immediate consequence of Theorem 3.1.

**Corollary 3.2.** Let $Y$ be a $2k$-valent $\frac{1}{2}$-arc-transitive graph. If its vertex stabiliser is $Z_k$, then $Y$ is a medial graph of a chiral hypermap which is neither positively nor negatively self-dual.

This corollary enables us to construct families of chiral maps from known examples of $\frac{1}{2}$-transitive graphs with cyclic stabilisers. The problem of whether one can derive $\frac{1}{2}$-arc-transitive graphs from chiral hypermaps will be considered in the following section.

4. **Constructing $\frac{1}{2}$-arc-transitive graphs from chiral hypermaps with large planar width**

Results proved in the previous sections suggest that chiral hypermaps, that is orientably regular hypermaps without mirror symmetries, can be used to construct $\frac{1}{2}$-arc-transitive graphs with cyclic stabilisers of vertices. To verify that the medial graph of such a hypermap $\mathcal{H}$ is $\frac{1}{2}$-arc-transitive we have to prove that all automorphisms of the graph are hypermap automorphisms.

By Theorem 3.1 we also have to take control of possible self-dualities of $\mathcal{H}$. Before we proceed further let us inspect two infinite toroidal families of hypermaps demonstrating in a condensed form the principal ideas that we are going to employ.

**Example 1** (See [22]). Let us consider the infinite family of regular maps of type (3, 6) on a torus classified by Coxeter and Moser in [8]. It is known that these maps are determined by two non-negative integer parameters $a$, $b$ determining the fundamental region in the tessellation of the plane of type (3, 6) giving rise to a map $\mathcal{M} = M_{a,b}$. It is known also that $M_{a,b}$ is chiral if and only if $ab(a - b) \neq 0$; see [8]. Let us consider the medial map $\text{Med}(M_{a,b})$. It has faces of two lengths 3 and 6: the first family corresponds to the faces of the original map while the second corresponds to the vertex set of $M_{a,b}$. It follows from the construction of $M_{a,b}$ that as long as $a + b$ is large enough, there are no other 6-cycles and 3-cycles in $\text{Med}(M_{a,b})$ except those bounding the faces. Obviously any graph automorphism maps a $k$-cycle onto a $k$-cycle and, in particular, it permutes the families of 3-cycles and 6-cycles in $\text{Med}(M_{a,b})$ except those bounding the faces. Since every edge is incident with one face of length 3 and one face of length 6 and there are no other cycles of length at most 6, every graph automorphism
extends to a self-homeomorphism of the surface. Finally, since the face-length differs from the valency of $M_{a,b}$, the map does not admit a self-duality.

Summing up, by Theorem 3.1 we get that with a few small exceptions all the maps $M_{a,b}$, where $ab(a - b) \neq 0$, give rise to $4$-valent $\frac{1}{2}$-arc-transitive medial graphs with stabiliser $Z_2$ (see [22]).

**Example 2.** There is a family of (clean) hypermaps $\mathcal{H} = \mathcal{H}_{a,b}$ of type $(3, 3, 3)$ on a torus arising by colouring faces of some regular maps of type $(6, 3)$ with three colours; see [12]. We can control the chirality of $\mathcal{H}$ by using a similar numerical condition in terms of $a$, $b$ as above. Its medial map has all the faces of length 6. As above, if $\mathcal{H}$ is sufficiently large there are no other 6-cycles in the medial graph except those bounding the faces. However, the graph is not $\frac{1}{2}$-arc-transitive since the original hypermap admits a positive self-duality rotating a fixed hexagon. This mapping induces an automorphism of the medial map cyclically permuting the 6 arcs based at a fixed vertex. By Theorem 3.1 the medial graph is one-regular with cyclic stabiliser $Z_b$. Let us note that by Corn and Singerman [12] no other clean regular hypermaps on the torus exist.

In what follows we shall make the ideas exposed above more precise. To do this we shall introduce one new topological invariant for hypermaps together with some basic observations related to it. Finally, combining some known results we shall construct chiral hypermaps satisfying certain constraints and giving rise to 1-arc-transitive medial graphs.

Given a topological map $M$ the planar width $w(M)$ of $M$ is the size of the minimum subset of faces of $M$ whose closure contains a non-contractible closed cycle (for a survey on planar width see [7]). Given a hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ of genus at least one we shall define its planar width $w(\mathcal{H})$ to be the planar width of the associated 3-valent 3-coloured topological map representing $\mathcal{H}$.

Given a map on a surface $S$, for each cycle $C$ on $S$, denote by $|C|$ the cardinality of the set $C \cap G$, where $G$ is the embedded graph. Alternatively, a planar width of a map is the minimum number of $|C \cap G|$ taken over all non-contractible cycles on the surface. It is known that the minimum is always achieved by simple cycles intersecting the map only at vertices and meeting each face at most once. There is a one–one correspondence between such sets of cycles in $\mathcal{H}$ and in Med($\mathcal{H}$); a cycle $C$ in $\mathcal{H}$ gives rise to a cycle $C'$ in Med($\mathcal{H}$) by contracting each segment of $C$ passing through a face coloured by 1 to a point. Vice versa, each such a cycle in Med($\mathcal{H}$) arises by expanding a certain finite set of points, being a subset of the set of vertices met by $C'$, into line segments. Applying this operation to $C'$ to get $C$, the number of vertices met by $C$ cannot exceed $2|C'|$. Hence $2|C'| \geq |C|$ and the result follows by taking $C'$ with $w(\text{Med}(\mathcal{H})) = |C'|$. Thus we have the following lemma.

**Lemma 4.1.** Let $\mathcal{H}$ be a hypermap. Then $w(\text{Med}(\mathcal{H})) \geq w(\mathcal{H})/2$.

Clearly, if the planar width of $\mathcal{H}$ is greater than two times the maximum of the face-lengths of Med($\mathcal{H}$), then the medial graph Med($M$) admits only automorphisms which extend to self-homeomorphisms of the supporting surface.

It is known that for any hyperbolic type $(m, n)$ there exist regular maps of arbitrarily large planar width $r$ (see [25, 26]). Moreover, Širáň in [26] constructs regular maps of type $(m, n)$ such that the size of the automorphism group is bounded by $C'$, where $C$ is of the form $2^p(m,n)$ for some polynomial in variables $m$, $n$ and it does not depend on $r$. It is also
noted there that, using the same method, one can generalise the statement to hypermaps. Hence we have the following result.

**Theorem 4.2** (Širáň [27]). For every \( r \) and for every type \((k, m, n)\) there are orientably regular hypermaps of type \((k, m, n)\), with planar width at least \( r \) and with the number of darts bounded by \( C^r \) where \( C \) does not depend on \( r \) and is of the form \( C = 2^{P(k,m,n)} \) for some polynomial in \( k, m \) and \( n \).

Generally, controlling both the planar width and chirality of a hypermap is difficult. In fact, we do not know whether the above result holds true if we restrict ourselves to chiral hypermaps. However, as we can see, for some types we can construct chiral hypermaps of large planar width. Let \( \mathcal{H} \) and \( \mathcal{K} \) be two regular hypermaps. Let us denote by \( \mathcal{H} \lor \mathcal{K} \) the least orientable regular hypermap covering both \( \mathcal{H} \) and \( \mathcal{K} \). It is known that if an orientable regular hypermap \( \mathcal{N} \) covers both \( \mathcal{H} \) and \( \mathcal{K} \), then it covers \( \mathcal{H} \lor \mathcal{K} \), as well. Moreover, it is possible to bound the size of the join \( \mathcal{H} \lor \mathcal{K} \) in terms of its factors. The following proposition summarises some important properties of the joins of hypermaps. We denote by \(|\mathcal{H}| = |F^+|\) the number of darts of an orientable hypermap \( \mathcal{H} \). We shall say that \(|\mathcal{H}| \) is the size of \( \mathcal{H} \).

**Proposition 4.3** ([4]). Let \( \mathcal{H} \) and \( \mathcal{K} \) be arbitrary orientably regular hypermaps. Let the type of \( \mathcal{H} \) and \( \mathcal{K} \) be \((h_0, h_1, h_2)\) and \((k_0, k_1, k_2)\), respectively. Then the join \( \mathcal{H} \lor \mathcal{K} \) satisfies the following properties:

- Any orientably regular hypermap covering both \( \mathcal{H} \) and \( \mathcal{K} \) covers \( \mathcal{H} \lor \mathcal{K} \) as well.
- \(|\mathcal{H} \lor \mathcal{K}| \leq |\mathcal{H}| \cdot |\mathcal{K}|\).
- The type of the join \( \mathcal{H} \lor \mathcal{K} \) is \((n_0, n_1, n_2)\), where \( n_i = \text{lcm}(h_i, k_i) \) for \( i = 0, 1, 2 \).

A particular instance of the join arises if the two hypermaps are mirror images of each other. Given hypermap \( \mathcal{H} = (F; r_0, r_1, r_2) \), denote by \( \mathcal{H}' = (F; r_2r_0r_2, r_2r_1r_2, r_2) \) the mirror image of \( \mathcal{H} \). Let us call an orientably regular hypermap \( \mathcal{H} \) totally chiral if the join \( \mathcal{H} \lor \mathcal{H}' \) has the maximum size, that is, if \(|\mathcal{H} \lor \mathcal{H}'| = |\mathcal{H}|^2\). In [3] several methods of construction for totally chiral hypermaps are developed. It is proved that a chiral hypermap with a non-Abelian simple automorphism group is necessarily totally chiral.

We are now ready to prove the following result.

**Proposition 4.4.** There exists \( n_0 \) such that for every even \( n \geq n_0 \) there are chiral hypermaps of type \((6, n - 4, n - 1)\) with planar width larger than \( 2n \).

**Proof.** Let \( n \) be an even integer \( n \geq 8 \). Let \( x = (1, 2, 3, \ldots, n - 1) \) and \( y = (0, 1, 2)(3, 4)(5, 6) \) be permutations acting on the set \([0, 1, \ldots, n - 1]\). Since both \( x \) and \( y \) are even permutations, they generate a subgroup \( G = \langle x, y \rangle \) of the alternating group \( A_n \). The action of \( G \) is clearly doubly transitive, and therefore primitive, and since \( y^2 \in G \) is a 3-cycle, by a theorem of Jordan [30, Theorem 13.9] \( G \) is \( A_n \). Since \( xy = (0, 1)(3)(4)(2, 4, 6, 7, \ldots, n - 1) \), its order is \( n - 4 \). Hence \( G \) is a quotient of the triangle group \( T(6, n - 4, n - 1) \) giving rise to a regular hypermap of the type \((6, n - 4, n - 1)\). Since \( n \geq 8 \) any automorphism of \( G \) is induced by a conjugation with elements in \( S_n \). However, the only permutations inverting \( x \) are the reflections of the obvious dihedral group determined by \( x \). But these do not send \( y \) into its inverse.
Hence the assignment \( x \mapsto x^{-1}, y \mapsto y^{-1} \) does not extend to an automorphism of \( G \), and the associated hypermap \( G \) is chiral. As was noted above, \( G \) must be totally chiral since \( A_n \) is simple for \( n \geq 5 \). Now take a Širáň’s hypermap \( \mathcal{H} \) of type \((6, n - 4, n - 1)\) with planar width \( r \) at least \( 2n \). The size of \( \mathcal{H} \) is bounded by a function of the form \( 2^{P(n)} \) where \( P(n) \) is a polynomial in \( n \). Set \( K = G \vee \mathcal{H} \). We claim that \( K \) is chiral. Indeed, by Proposition 4.3 we have \( |K| \leq |G||\mathcal{H}| = (n!/2)2^{P(n)}. \) On the other hand, since \( G \) is totally chiral, the least (reflexible) regular map covering it has size \( |G|^2 = (n!/2)^2 \); this number is for sufficiently large \( n \) clearly larger than the size of \( K \). Since both \( G \) and \( H \) have the same type, the covering \( G \vee \mathcal{H} \to \mathcal{H} \) is smooth. Consequently, the planar width of the join \( w(G \vee \mathcal{H}) \geq w(\mathcal{H}) \geq 2n \). Hence \( K \) has the required properties. \( \square \)

**Corollary 4.5.** For any sufficiently large \( m \), there exists a \( 2m \)-valent \( \frac{1}{2} \)-arc-transitive graph with vertex stabiliser \( Z_m \).

**Proof.** Take the chiral hypermap \( \mathcal{H} \) of type \((6, n - 4, n - 1)\) constructed in the previous proposition. If \( m \) is even, set \( n = m + 4 \) and take the medial graph \( X \) of \( \mathcal{H} \). It has valency \( 2(n - 4) = 2m \) and by Proposition 2.1 it admits a \( \frac{1}{2} \)-arc-transitive action of the automorphism group of \( \mathcal{H} \) with cyclic stabiliser \( Z_{n-4} = Z_m \). Since the planar width of \( \mathcal{H} \) is at least \( 2n \), there no other automorphisms of \( X \).

For \( m \) odd we set \( n = m + 1 \) and we consider the dual of type \((6, n - 1, n - 4)\) of the hypermap \( \mathcal{H} \) constructed in Proposition 4.4. Then the medial graph for this hypermap has the required properties. \( \square \)

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