Dissipativity of T-periodic Linear Systems

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Abstract

It is proved that the existence of a positive definite storage function is necessary and sufficient for strict dissipativity of linear systems with periodic coefficients. The connection between strict dissipativity of the system and a nonoscillatory property of an associated Hamiltonian system is established.

1 Introduction

The concept of dissipativity for linear and nonlinear systems was introduced by J.C. Willems in 1972 [16]. It is closely related to hyperstability introduced by V.M. Popov [12, 13] and also to absolute stability. An extensive literature is devoted to frequency domain absolute stability criteria. (Let us mention only the survey [17]). In the papers [4, 9], the first results on necessity of frequency domain conditions of absolute stability for systems with several integral constraints were established. Different versions of dissipativity were investigated in detail in [5, 6, 7, 8].

An interest has been renewed recently in the investigation of systems with integral quadratic constraints which have been studied since the 1960's. Various problems of nonconvex optimization [19], analysis [14, 15] and design [10, 2] of nonlinear uncertain systems can be solved by reduction to problems with integral constraints. Recently new generalizations of dissipativity have appeared, e.g. structured dissipativity [15], quasidissipativity [11].

The concept of dissipativity is intimately connected to that of a storage function. These functions provide convenient Lyapunov functions in stability analysis of the system. Moreover it has been shown for linear time-invariant systems [16] that dissipativity is equivalent to existence of a storage function and satisfaction of a frequency domain condition. This result can be derived from the so called frequency theorem (otherwise called the Kalman-Yakubovich-Popov (KYP) lemma) giving necessary and sufficient conditions for solvability of some matrix inequalities. Conditions for existence of a storage function were extended to the time-varying and nonlinear systems [5, 1]. However, some gap between the necessary and sufficient conditions for the time-varying case still remains. Therefore it is of interest to examine for which classes of systems it is possible to remove this gap. It has turned out that the gap between the necessary and sufficient conditions of dissipativity can be completely cancelled for T-periodic linear systems by using an appropriate extension of the KYP lemma [23, 24]. This idea will be pursued in the present paper.

The main result of the paper gives necessary and sufficient conditions for strict dissipativity of T-periodic linear systems. The convenient way of formalizing these conditions is based on a nonoscillatory property of some associated Hamiltonian system. (The concept of nonoscillatority for Hamiltonian systems was introduced in [18], while the connection between nonoscillatority and absolute stability for T-periodic systems was established in [21, 22]). It is shown that, as for the time-invariant case, strict dissipativity is equivalent to the existence of a strong storage function. Also it is shown that strict dissipativity is equivalent to the solvability of the corresponding Riccati equation or to the strong nonoscillatory property of the associated Hamiltonian system.

It worth noting that according to the recent tradition in control theory, we use the term “dissipativity” in sense introduced by Willems [16]. However this term is still being used in the theory of differential equations with a different meaning as all the trajectories falling into some bounded set after some time. This second meaning could be referred to as “dissipativity in sense of N. Levinson” or “ultimate boundedness”, see e.g. [3, 26]).

The structure of the paper is as follows. In Section 2, the main definitions and the problem formulation are given. In Section 3, the associated Hamiltonian
system is introduced and studied. The main results are formulated and the idea of the proof is discussed in Section 4.

2 Problem Statement

Consider a linear periodic controlled system

\[
\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) \tag{2.1}
\]

with the associated quadratic form

\[
F(t, x, u) = \left[ \begin{array}{c} x \\ u \end{array} \right]^T \left[ \begin{array}{cc} G(t) & g(t) \\ g(t)^* & \Gamma(t) \end{array} \right] \left[ \begin{array}{c} x \\ u \end{array} \right] \tag{2.2}
\]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( u(\cdot) \) is locally integrable, \( A, B, G, g, \Gamma \) are bounded (\( \in L_\infty \)), real \( T \)-periodic matrix functions defined on \( \mathbb{R} \), \( G(t) = G(t)^* \), \( \Gamma(t) = \Gamma(t)^* \) and

\[
\Gamma(t) \geq \gamma_0 I > 0. \tag{2.3}
\]

The pair \((A, B)\) is assumed to be controllable on \([0, T]\). (This means that for any \( t_1, t_2, a_1 \in \mathbb{R}^n, a_2 \in \mathbb{R}^n \), such that \( 0 \leq t_1 < t_2 \leq T \) there exists a function \( u(\cdot) \) such that the solution of (2.1) with \( x(t_1) = a_1 \) satisfies \( x(t_2) = a_2 \).)

**Definition 1** System (2.1) is called dissipative with the supply rate (2.2) if

\[
\int_{t_0}^{t_1} F(t, x(t), u(t)) \, dt \geq 0 \tag{2.4}
\]

to any \( t_1 \geq t_0 \) and for any \( x(t), u(t) \) satisfying (2.1) and \( x(t_0) = 0 \).

**Definition 2** System (2.1) is called strictly dissipative if for some \( \delta > 0 \) it is dissipative with the supply rate \( F_\delta = F - \delta(x^2 + u^2) \).

**Definition 3** System (2.1) is called \((t_0, T)\)-dissipative with the supply rate \( F_\delta \), if (2.4) is valid for any \( t_1 = t_0 + kT, k = 0, 1, \ldots \)

In [16], the supply rate is defined as a function of input \( u \) and an output \( y \). This reduces to the current formulation after substituting for \( y \) as a function of \( x, u \).

**Definition 4** System (2.1) is called strictly \((t_0, T)\)-dissipative with the supply rate \( F_\delta \) if it is \((t_0, T)\)-dissipative with the supply rate \( F_\delta = F - \delta(x^2 + u^2) \) for some \( \delta > 0 \).

The problem is to find conditions (necessary and sufficient) for dissipativity, strict dissipativity of system (2.1).

Below only the cases of strict dissipativity and strict \((t_0, T)\)-dissipativity will be studied.

**Definition 5** Function \( V(t, x) \) is called a storage function for system (2.1) with supply rate \( F(t, x, u) \) if \( V(t, x) \geq 0 \) and

\[
\int_{t_0}^{t_1} F(s, x(s), u(s)) \, ds \geq V(t_1, x(t_1)) - V(t_0, x(t_0)) \tag{2.5}
\]

to all \( t_1 \leq t_2 \) \( x(t) \), satisfying (2.1). \( V(t, x) \) is called a strong storage function if, additionally, \( V(t, x) > 0 \) for \( x \neq 0 \).

**Definition 6** Function \( V(t, x) \) is called \((t_0, T)\)-storage function for system (2.1) with supply rate \( F(t, x, u) \), if \( V(t_0 + kT, x) \geq 0 \) for \( k = 0, 1, 2, \ldots \) and (2.5) holds.

Similarly, \( V(t, x) \) is called a strong \((t_0, T)\)-storage function, if \( V(t_0 + kT, x) > 0 \) for \( x \neq 0, k = 0, 1, 2, \ldots \) and (2.5) holds.

Obviously, the existence of the storage function \((t_0, T)\)-storage function implies the dissipativity \((t_0, T)\)-dissipativity of the system (2.1).

The opposite is not, in general, true. However, for a \( T \)-periodic linear system the existence of a storage function with supply rate \( F_\delta = F - \delta(x^2 + u^2) \) is necessary and sufficient for strict dissipativity, as follows from the results of this paper.

3 Associated Hamiltonian system and its properties

We introduce the function

\[
\mathcal{H}(t, x, u, \psi) = \psi^*(Ax + Bu) - F(t, x, u), \tag{3.1}
\]

where \( \psi \in \mathbb{R}^n \), and consider the system

\[
\frac{dx}{dt} = \left( \frac{\partial \mathcal{H}}{\partial \psi} \right)^*, \quad \frac{d\psi}{dt} = -\left( \frac{\partial \mathcal{H}}{\partial x} \right)^*, \quad \frac{\partial \mathcal{H}}{\partial u} = 0. \tag{3.2}
\]

Since \( \det \Gamma \neq 0 \), the last equation (3.2) can be solved for \( u \):

\[
u = \Gamma^{-1} \left( \frac{1}{2} B^* \psi - g^* x \right) =: \mathcal{U}_*(x, \psi). \tag{3.3}
\]

Denote

\[
\mathcal{H} \big|_{u=\mathcal{U}_*(x, \psi)} = \mathcal{H}_0(x, \psi). \tag{3.4}
\]

Then system (3.2) can be rewritten in the form

\[
\frac{dx}{dt} = \left( \frac{\partial \mathcal{H}_0}{\partial \psi} \right)^*, \quad \frac{d\psi}{dt} = -\left( \frac{\partial \mathcal{H}_0}{\partial x} \right)^*. \tag{3.5}
\]

The system (3.5) is called an associated Hamiltonian system for the above problems.
Let
\[ z = \begin{bmatrix} x \\ \psi \end{bmatrix} \in \mathbb{R}^{2n}, \quad J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \]

\[ H(t) = \begin{bmatrix} g \Gamma^{-1}x - G & (A^* - g \Gamma^{-1}B^*) \\ (A^* - g \Gamma^{-1}B^*)^* & BT^{-1}B^* \end{bmatrix}. \] (3.6)

Then the system (3.5) takes the form
\[ J \frac{dz}{dt} = H(t)z. \] (3.8)

Let \( Z(t) \) be the evolution matrix of (3.8);
\[ J = H(t)Z, \quad Z(0) = I_{2n}. \] (3.9)

Introduce the “frequency domain” condition
\[ \det[Z(T) - e^{i\omega}I_{2n}] \neq 0 \quad \forall \omega \in [0, 2\pi]. \] (3.10)

This means that system (3.8) has no multipliers on the unit circle. Due to symmetry of spectrum \( Z(T) \) with respect to the unit circle, condition (3.10) implies that the Hamiltonian equation (3.8) has \( 2n \) real linearly independent solutions \( z_j^+, z_j^-; j = 1, \ldots, n; \) such that
\[ z_j^+(t) = \begin{bmatrix} z_j^+(t) \\ \psi_j^+(t) \end{bmatrix} \rightarrow 0 \quad \text{if} \quad t \rightarrow +\infty, \] (3.11)
\[ z_j^-(t) = \begin{bmatrix} z_j^+(t) \\ \psi_j^-(t) \end{bmatrix} \rightarrow 0 \quad \text{if} \quad t \rightarrow -\infty. \] (3.12)

Introduce \( n \times n \) matrix functions
\[ X(t) = [x_1^+, \ldots, x_n^+], \quad \Psi(t) = [\psi_1^+, \ldots, \psi_n^+]. \] (3.13)

**Definition 7** The Hamiltonian system (3.8) is called **strongly nonoscillatory** if the frequency domain condition (3.10) holds and
\[ \det X(t) \neq 0 \quad \forall 0 \leq t \leq T. \] (3.14)

Note that (3.14) implies that \( \det X(t) \neq 0 \) for all \( t \in \mathbb{R} \).

Suppose the frequency domain condition (3.10) holds. Then it can be easily shown that
\[ X(t) = P(t)e^{Kt}, \quad \Psi(t) = Q(t)e^{Kt}, \] (3.15)
where \( P(t), Q(t) \) are \( T \)-periodic functions, \( K \) is Hurwitz matrix \([P, Q, K]\) may be complex valued, and in that case also the real matrix
\[ R(t) = -\Psi(t)X(t)^{-1} \] (3.16)
is defined for \( t \) such that \( \det X(t) \neq 0 \), is symmetric and satisfies the Riccati equation
\[ \frac{dR}{dt} + RA + A^*R + G = (RB + g)\Gamma^{-1}(RB + g)^*. \] (3.17)

It follows from (3.15) that \( R(t + T) = R(t) \) for mentioned \( t \).

### 4 Main Results

**Theorem 1** System (2.1) is strictly dissipative if and only if the associated Hamiltonian system (3.8) is strongly nonoscillatory and the matrix \( R(t) \) defined by (3.16) is positive definite for all \( t \in [0, T] \).

**Theorem 2** System (2.1) is strictly \((t_0, T)\)-dissipative if and only if the associated Hamiltonian system (3.8) is strongly nonoscillatory and \( R(t_0) > 0 \), where \( R(t) \) is defined by (3.16).

**Theorem 3** Let the Hamiltonian system (3.8) be strongly nonoscillatory (and, consequently, matrix \( R(t) \) in (3.16) is well defined and absolutely continuous). Define function \( W(s, a) \) as
\[ W(s, a) = \inf_{u(.)} \int_{t_1}^{t_2} F(t, z(t), u(t)) dt \]
where \( \inf \) is taken over all \( t > 0 \) and \( u(.) \) such that
\[ \dot{x} = A(t)x + B(t)u, \quad x(0) = 0, \quad x(s + \tau) = a. \] (4.2)

Then
\[ W(s, a) = a^*R(s)a \] (4.3)
where \( R(s) \) is absolutely a continuous function and defined by (3.16). Moreover,
\[ \int_{t_1}^{t_2} F dt \geq W(t_2, x(t_2)) - W(t_1, x(t_1)) \] (4.4)
for all \( t_1 \leq t_2 \) and \( x(t) \) satisfying (2.1).

Function \( W(s, a) \) was called the required supply by Willems [16].

**Corollary 1** Let the system (2.1) be strictly dissipative. Then \( W(s, a) \) defined in (4.1) is a strong storage function, i.e., \( W(s, a) > 0 \) for all \( a \neq 0 \) and (4.4) holds.

**Corollary 2** Let the system (2.1) be strictly \((t_0, T)\)-dissipative. Then \( W(s, a) \), defined in (4.1), is a strong \((t_0, T)\)-storage function, i.e., \( W(t_0, a) > 0 \) for \( a \neq 0 \) and (4.4) holds.

**Theorem 4** Let the Riccati equation (3.17) have for all \( t \) the absolutely continuous solution \( R(t) \), satisfying
\[ R(t + T) = R(t) = R(t)^* > 0 \quad \forall t. \] (4.5)

Then the system (2.1) is strictly dissipative. Conversely, if the system (2.1) is strictly dissipative, then the Riccati equation (3.17) has absolutely continuous solution \( R(t) \), defined for all \( t \), satisfying (4.5) and such that equation \( \dot{x} = A_0(t)x \) is asymptotically stable, where
\[ A_0(t) = A(t) + B(t)r(t)^*, \quad r(t) = -[R(t)B(t) + g(t)]\Gamma(t)^{-1}. \] (4.6)
Theorem 5  Let the Riccati equation (3.17) have the absolutely continuous solution $R(t)$, such that

$$R(t + T) = R(t) = R(t)^*, \quad R(t_0) > 0.$$  \hspace{.5cm} (4.7)

Then the system (2.1) is strictly $(t_0, T)$-dissipative.

Conversely, if the system (2.1) is strictly $(t_0, T)$-dissipative, then the Riccati equation (3.17) has absolutely continuous solution $R(t)$, satisfying (4.7) and such that the equation $\dot{x} = A_0(t)x$ is asymptotically stable where $A_0(t)$ is defined in (4.6).

The proofs of theorems 1–5 rely heavily on the results of papers [23, 24].

5 Conclusions

This paper has presented results on criteria for dissipativeness of $T$-periodic linear systems. A concept of $(t_0, T)$-dissipativeness is introduced as dissipativeness over intervals $kT$, $k = 1, 2, \ldots$ Dissipativeness and $(t_0, T)$-dissipativeness are connected to nonoscillatory Hamiltonian systems and existence of storage functions with particular positivity properties.

References


