On sequences resulting from iteration of modified quadratic and palindromic mappings

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Abstract

Blanchard and Fabre (1994) have considered description of symbolic systems by sequences obtained by iteration of two mappings $g_j$ and $d_j$, defined by $g_j(u) = u[u_j]$ and $d_j = u[u_j]^R$, where the word $[u]_j$ is obtained from a word $u$ by removing the last $j$ symbols. They stated a conjecture that the iteration of $d_j$ results in an automatic sequence. We prove here that $d_j$ is, while $g_j$ is not, a 2-automatic sequence. Our result is obtained by investigation of two wider classes of sequences.

1. Introduction

The area of symbolic dynamics remains for a long time one of the important sources of challenging problems stimulating investigation of infinite sequences of symbols (see e.g. [9]). One of the essential questions consists in finding a simple, but powerful enough, finite tool for description of single infinite sequences, and/or of sets (languages) consisting of infinite sequences. Finite state automata, because of their simple and well-known properties from the finitary case, have been applied in various ways in description and recognition of infinite sequences. One-way infinite automatic sequences [4], originally introduced in [5] as uniform tag sequences, are a particularly important class described by finite-state automata, providing a quasi random, but close to periodic (or periodic), order of symbols in the sequence (cf. [2]). They have been subject of several generalizations [1], and, among lots of other applications, proved to be one of the principal tools for description of symbolic systems, being shift-invariant sets of double-infinite sequences.

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In [3] the authors considered description of symbolic systems based on sequences resulting from iteration of mappings \( g_j \) and \( d_j \), defined by \( g_j(u) = u[1:j] \) and \( d_j = u[0:j] \) where \( [1:j] \) is obtained from a word \( u \) by removing the last \( j \) symbols. They have shown that, for \( j = 1 \), these sequences define symbolic systems. They stated a conjecture that for \( j > 1 \) the iteration of \( d_j \) results in an automatic sequence, and hence the result can be extended. We will prove the conjecture here, by showing that the limit of iteration of \( d_j \) is always a 2-uniform tag sequence in the sense of [5].

A tag sequence is obtained by applying a length-preserving morphism to a substitutional sequence, being a result of iteration of a morphism. Such a sequence can be equivalently described by a tag machine [5]. In Section 3, we will describe a modification of the tag machine model, suitable to generate iterations of \( g_j \) and \( d_j \), which will bring us to consideration of two more general classes of mappings, one including \( g_j \), the other including \( d_j \). We will show that the iteration of the mappings of the latter class leads to 2-uniform tag sequences, while the same is true just for a limited subclass of the former class, not including the mapping \( g_j \).

2. Preliminaries

We denote \( \mathbb{N} = \{0, 1, 2, \ldots \} \) the set of all natural numbers. By \( I \) we denote the identity mapping (on any set).

An infinite sequence \( s = (s_i)_{i \in \mathbb{N}} \) of elements is periodic if, for some \( p \geq 0, q > 0 \), and for all \( i \in \mathbb{N} \), \( s_{p+i} = s_{q+i} \). In such case \( p \) is the length of pre-period and \( q \) the length of period of \( s \). The values of \( p \) and \( q \) are not unique, \( p \) can be replaced by any greater number, and \( q \) by any its multiple.

Let \( \Sigma \) be an alphabet. We denote by \( \Sigma^* \) the set of all (finite) words over \( \Sigma \), by \( |x| \) the length and by \( x^\lambda \) the reverse (mirror image) of a word \( x \), by \( \lambda \) the empty word. Further, for \( p \in \mathbb{N} \), we denote \( \Sigma^p = \{ x \in \Sigma^*; |x| = p \} \) and \( \Sigma^{p^*} = \{ x \in \Sigma^*; |x| \text{ is a multiple of } p \} \). A (one-way) infinite word or a sequence over \( \Sigma \) is an infinite sequence of symbols from \( \Sigma \), i.e. a mapping \( \mathbb{N} \to \Sigma \). The concatenation of two words \( x \) (finite) and \( y \) (possibly infinite) is denoted as \( x.y \). Then \( x \) is a [proper if \( y \neq \lambda \)] prefix and \( y \) is a [proper if \( x \neq \lambda \)] suffix of \( x.y \). We will omit the operational symbol "." and use it just to improve readability. For \( k, m \in \mathbb{N} \) and a word \( x \), \( |x| \geq k \), we denote by \( \text{pref}_k(x) \) the prefix and, for finite \( x \), by \( \text{suffix}_m(x) \) the suffix of \( x \) of length \( k \). If \( |x| \geq k + m \), we denote by \( k[x]_m \) the word \( x \) without its prefix of length \( k \) and suffix of length \( m \); hence \( x = \text{pref}_k(x)[x]_m \text{ suffix}_m(x) \). We omit subscripts equal to 0, and use the notation \( k[x] \) for infinite \( x \) as well.

Let \( \Sigma, \Gamma \) be alphabets, \( S \) a set.

For a mapping \( f: S \to \Gamma^* \), we denote \( f^R: S \to \Gamma^* \) the mapping defined by \( f^R(x) = (f(x))^R \). \( f \) is \([p]-\text{uniform} \), if for some \( p \in \mathbb{N} \) and for all \( x \in S \), \( |f(x)| = p \), we denote \( |f| = p \) in that case.

We call a mapping \( \alpha: \Gamma^p \to \Gamma^p \), for some \( p \in \mathbb{N} \), to be a fixed-length mapping (or \( k \)-mapping) on \( \Gamma \). Then \( \alpha \) is \( p \)-uniform, \( |\alpha| = p \). The concatenation of two \( k \)-mappings
\( \alpha, \beta \) on \( \Gamma \) is the \( \alpha \)-mapping \( \beta : \Gamma^{[x]+|\beta|} \rightarrow \Gamma^{[x]+|\beta|} \) defined by \( \beta(x) = \alpha(\text{pref}_{|\beta|}(x)).\beta \) \((\text{suffix}_{|\beta|}(x))\). For \( n \in \mathbb{N} \), \( \alpha^{(n)} : \Gamma^{[x]} \rightarrow \Gamma^{[x]} \) is the \( n \)-th power of \( \alpha \), defined inductively: \( \alpha^{(0)} = I \) on \( \Gamma^{0} = \{ \lambda \} \), and \( \alpha^{(n+1)}(x) = \alpha^{(n)}(\text{pref}_{|x|}.\alpha(\text{suffix}_{|x|}(x))) \). The closure of \( \alpha \) is the mapping (but not an \( \alpha \)-mapping) \( \alpha^{*} : \Gamma^{[x]} \rightarrow \Gamma^{[x]} \) defined by \( \alpha^{*}(x) = \alpha^{(|x|)}(x) \). We will perform the mapping-concatenation of a closure of an \( \alpha \)-mapping with \( \beta \)-mappings as well.

A morphism \( h : \Sigma^{*} \rightarrow \Gamma^{*} \) is an extension of a mapping \( h : \Sigma \rightarrow \Gamma^{*} \), defined for \( x, y \in \Sigma^{*} \) by \( h(xy) = h(x)h(y) \); it can be further extended to map \( \Sigma^{+} \) to \( \Gamma^{+} \) as well. \( h \) is \([p]-\)uniform, if it is \([p]-\)uniform on \( \Sigma \).

A mapping \( m : \Sigma^{*} \rightarrow \Sigma^{*} \) is extendible in \( w \in \Sigma^{*} - \{ \lambda \} \), if for each \( n \geq 0 \), \( m^{n}(w) \) is a proper prefix of \( m^{n+1}(w) \). In such case the limit of the iteration of \( m \) on \( w \) is an infinite word denoted by \( m^{*}(w) \). Hence, a non-erasing morphism \( h \) is extendible in a one-symbol word \( a \), iff \( \text{pref}_{1}(h(a)) = a \). Following [5], an infinite word is a \([p]-\)uniform tag sequence \((p \geq 2)\), if it can be expressed as \( h_{2}(h_{1}^{p}(a)) \) for some symbol \( a \), \([p]-\)uniform morphism \( h_{1} \) extendible in \( a \), and a \( 1 \)-uniform morphism \( h_{2} \). The \( i \)-th symbol of a \( p \)-uniform tag sequence can be generated by a finite-state automaton by processing the notation of \( i \) in base \( p \). We will use the following properties of uniform tag sequences.

**Proposition 1.** For all \( p \geq 2 \) the class of all \( p \)-uniform tag sequences is closed under

(i) mapping by a uniform morphism,

(ii) mapping by an inverse of a one-to-one uniform morphism,

(iii) replacing a prefix of a sequence by another word (not necessarily being of the same length)

Throughout the rest of the paper, let \( \Gamma \) denote a fixed (but arbitrary) alphabet, and \( \Sigma_{a,b} \) the alphabet defined for given numbers \( p_{a,b}, q_{a,b}, p_{b}, q_{b} \in \mathbb{N}, q_{a,b}, q_{b} > 0 \) as follows. Denote

\[
\Sigma_{a} = \{ a_{i} ; i = 0, \ldots, p_{a,b} + q_{a,b} - 1 \},
\]

\[
\Sigma^{(k)}_{b} = \{ b_{i}^{k} ; i = 0, \ldots, p_{a,b} + q_{a,b} - 1 \}, \quad k = 0, \ldots, p_{b} + q_{b} - 1
\]

alphabets consisting of pairwise distinct symbols. Further denote

\[
\Sigma_{b} = \bigcup_{k=0}^{p_{b}+q_{b}-1} \Sigma^{(k)}_{b}
\]

\[
= \{ b_{i}^{k} ; i = 0, \ldots, p_{a,b} + q_{a,b} - 1, \quad k = 0, \ldots, p_{b} + q_{b} - 1 \},
\]

\[
\Sigma_{a,b} = \Sigma_{a} \cup \Sigma_{b}.
\]

For each of the alphabets \( \Sigma_{a} \), \( \Sigma^{(k)}_{b} \), consider the finite sequence of its symbols written in the increasing order of indices. We periodically extend these finite sequences by introducing equivalent notation for the alphabet symbols defined as \( a_{i+q_{a,b}} = a_{i} \), \( b_{i+q_{a,b}}^{k} = b_{i}^{k} \) for \( i \geq p_{a,b}, k \geq 0 \), and \( b_{i}^{k+q_{b}} = b_{i}^{k} \) for \( k \geq p_{b}, i \geq 0 \). In this way we obtain
the periodic sequences of symbols $\mathbf{a} = (a_i)_{i=0}^{\infty}$, $\mathbf{b}^k = (b^k_i)_{i=0}^{\infty}$, $0 \leq k < p_b + q_b$, each of them having the pre-period length $p_{a,b}$ and the period length $q_{a,b}$. The sequence of sequences $\mathbf{b} = (\mathbf{b}^k)_{k=0}^{\infty}$ is then periodic as well, with the pre-period length $p_b$ and the period length $q_b$.

Finally, let the morphism $\psi : \Sigma_{a,b}^* \to \Sigma_{a,b}^*$ be defined for $i, k \geq 0$ as

$$\psi : a_i \mapsto a_{i+1}, \quad b^k_i \mapsto b_{i+1}^k,$$

and let $\psi^{-1}, \psi^{-R} : \Sigma_{a,b}^* \to \Sigma_{a,b}^*$ be (possibly partial) mappings, $\psi^{-1}$ being a right inverse of $\psi$, and $\psi^{-R} = (\psi^{-1})^R$. If $p_{a,b} \neq 0$, then there are two ways to choose $\psi^{-1}$ (and $\psi^{-R}$), in this case $\psi^{-1}$ is not defined for symbols with the lower index equal to 0. On its domain, $\psi^{-1}$ is a morphism. The mappings $\psi^R, \psi^{-R}$ have the following properties.

**Proposition 2.** For all $x, x' \in \Sigma_{a,b}^*$, $i, k \in \mathbb{N}$,

(i) $\psi^R(xx') = \psi^R(x')\psi^R(x)$, \hspace{1cm} $\psi^{-R}(xx') = \psi^{-R}(x')\psi^{-R}(x)$,

(ii) $\psi^R(\{x\})_k = \psi^R(\{x\})_k$, \hspace{1cm} $\psi^{-R}(\{x\})_k = \psi^{-R}(\{x\})_k$.

3. Modified quadratic and palindromic mappings

A substitutional sequence obtained by iteration of a morphism $h$ starting from some initial letter $a$, being a prefix of $h(a)$, can be described by a *tag machine* [5], consisting of a one-way infinite tape, a reading head and a writing head. Initially, both heads are positioned at the first tape field containing the symbol $a$, the other tape fields are empty. In one step the reading head reads the contents $b$ of the scanned field and moves to the next field. The writing head moves to the right while writing the word $h(b)$. Each step results in a longer initial portion of the substitutional sequence.

The tag machine may be modified to compute iterations of the mappings $g_j$ or $d_j$. Assume the tape of the machine contains a word $x$. In one step, simulating one application of $g_j$ or $d_j$ to $x$, the reading head makes a move along the whole word $x$ and backwards. The writing head moves during the backward move of the reading head only, copying the scanned symbols. When simulating $g_j$, the reading head starts at the right end of $x$, moving to the beginning of the tape and back. The writing head is writing with the delay of $j$ symbols, to avoid copying the suffix of length $j$. When simulating $d_j$, the head starts at the left end of $x$ moving into the right end and back. During the backward move the writing head copies the scanned symbols, skipping the first $j$ of them. At the beginning of each step, the writing head sets a marker on the current field. This marker limits the movement of the reading head and is erased by it, when found.

To delay copying, the tag machine is equipped with a finite buffer. The existence of such a buffer enables the machine to perform a more complex transformation of the input word. Instead of simple copying the symbols, we will consider here the machine to map fixed-sized portions of the input. We will consider just length-preserving (i.e. fl-)mappings, thus the next iteration will be easily applicable. Using the buffer during
the first pass of the reading head through the input word, the machine can store a prefix and a suffix of a fixed length. We will assume, that the machine replaces the prefix of length \(i\), and suffix of length \(j\), by a new prefix and suffix (possibly of different, but fixed, length), each depending on both the original prefix and suffix only.

We can describe the action of our modified tag machines more formally by two (partial) mappings \(g_{\lfloor i, j, \pi, \rho, \sigma \rfloor} : \Gamma^* \rightarrow \Gamma^*\) as follows. Let, for some fixed \(i, j \in \mathbb{N}\), \(\pi : \Gamma^i \times \Gamma^j \rightarrow \Gamma^{\lfloor i \rfloor}\), \(\rho : \Gamma^* \rightarrow \Gamma^\lfloor \sigma \rfloor\), \(\sigma : \Gamma^i \times \Gamma^j \rightarrow \Gamma^{\lfloor \sigma \rfloor}\) be mappings (we will specify \(\rho\) more precisely later). Then for \(x \in \Gamma^*\), \(|x| \geq i + j\), we define

1. \(g_{\lfloor i, j, \pi, \rho, \sigma \rfloor}(x) = x.\pi(\text{pref}_i(x), \text{surf}_j(x)).\rho([x]_i).\sigma(\text{pref}_i(x), \text{surf}_j(x)).\)
2. \(d_{\lfloor i, j, \pi, \rho, \sigma \rfloor}(x) = x.\pi(\text{pref}_i(x), \text{surf}_j(x)).\rho^R([x]_i).\sigma(\text{pref}_i(x), \text{surf}_j(x)).\)

When the mappings from (1) are repeatedly applied, starting from some initial word \(u \in \Gamma^*\), \(|u| \geq i + j\), the first argument of \(\pi\) and \(\sigma\) is always equal to \(\text{pref}_i(u)\). We can therefore make the definition of \(\pi\) and \(\sigma\) independent on \(i\) by fixing the first argument, then start the iteration of \(g\) or \(d\) from \([u]\) instead of \(u\), and apply (1) with \(i = 0\). The resulting new sequence will differ from that obtained starting from \(u\) by the missing prefix \(\text{pref}_i(u)\) only. Following (iii) of Proposition 1.1, the new sequence is a uniform tag sequence iff the original one is. Hence, it is sufficient to adopt the following simplified definition of our mappings.

Let for a fixed number \(j \in \mathbb{N}\), \(\pi : \Gamma^j \rightarrow \Gamma^{\lfloor \sigma \rfloor}\), \(\rho : \Gamma^* \rightarrow \Gamma^{\lfloor \sigma \rfloor}\), \(\sigma : \Gamma^j \rightarrow \Gamma^{\lfloor \sigma \rfloor}\) be mappings. Let the mappings \(g_{\lfloor j, \pi, \rho, \sigma \rfloor}, d_{\lfloor j, \pi, \rho, \sigma \rfloor} : \Gamma^* \rightarrow \Gamma^*\) be defined for \(w \in \Gamma^*, \ |w| > j\) as

1. \(g_{\lfloor j, \pi, \rho, \sigma \rfloor}(w) = w.\pi(\text{surf}_j(w)).\rho([w]_j).\sigma(\text{surf}_j(w)).\)
2. \(d_{\lfloor j, \pi, \rho, \sigma \rfloor}(w) = w.\pi(\text{surf}_j(w)).\rho^R([w]_j).\sigma(\text{surf}_j(w)).\)

We will omit the subscript \(\lfloor j, \pi, \rho, \sigma \rfloor\) whenever there is no danger of confusion.

Let us now have a closer look at the mapping \(\rho\). If \(j = \lfloor \sigma \rfloor\) and if \(\rho = \pi\), then both \(g^w(u)\) and \(d^w(u)\) have the shape \([u]_j^\sigma.s.p.[u]_j^\sigma.s.p.\ldots.[u]_j^\sigma.s.p.[u]_j^\sigma.s\) where the parts denoted as \(s.p\) (not necessarily all the same) are formed by the newly added suffixes and prefixes, each being of length \(|\sigma| + |\pi|\), and \([u]_j^\sigma\) denotes \( [u]_j \) for \(g\), and \([u]_j \) or \(([u]_j)^R\) for \(d\). We will assume \(\rho\) to be a composition of local mappings, preserving the above internal structure of the word \(g^w(u)\) and \(d^w(u)\), i.e. mapping independently the parts \(s.p\) and the parts \([u]_j^\sigma\). Then \(\rho\) has to be composed as a length-preserving mapping

\[\rho = \pi.(\beta, x)^\sigma,\]

where \(\pi, \beta\) are two fl-mappings on \(\Gamma\) satisfying \(|\beta| = |\sigma| + |\pi|\). The mappings \(g\) and \(d\) are then applicable to \(u\) iff \(|u| = (j + |x|)\) is a multiple of \(|x| + |\beta|\). In that case it is easy to verify, that if the mapping \(g\), \([d]\) is defined by (2) using (3), then whenever \(g\) and \(d\) are applicable to a word \(u\), i.e. whenever \(|u| = (j + |x|)\) is a multiple of \(|x| + |\beta|\), then all powers of \(g\) and of \(d\) are applicable to \(u\) as well. This remains true, even if we allow a finer decomposition of \(\rho\) of the form (3) satisfying the condition

\[|\sigma| + |\pi| - |\beta|\] is a multiple of \(|x| + |\beta|\).
Then the \([u]^j\) parts are mapped by \(z.(\beta,x)^*\), the \(s,p\) parts by \(\beta.(x,\beta)^*\). Such form of \(\rho\) may be applicable even in the case \(j \neq |\sigma|\). We will discuss this possibility in Section 4.2.

For the rest of the paper, we will assume that the mappings \(g, d\) are defined by (2), where \(\rho\) satisfies (3) and (4). We will call mappings of this form to be **modified quadratic mapping**, and **modified palindromic mapping**, respectively, since for \(j = 0\), \(g_j(u)\) is the square of \(u\), and for \(j = 0\) or \(j = 1\), \(d_j(u)\) is a palindrome. For \(|x| - 1, x = 1, \beta = |\pi| = |\sigma| = 0\), we have the original mappings \(g_j\) and \(d_j\).

### 4. The sequences \(g^\sigma(u)\) and \(d^\sigma(u)\)

Based on analogical shape of the mappings \(g\) and \(d\), we will deal with them in parallel, proving several assertions being common or similar for both of the mappings. We therefore introduce a common notation, having however, slightly different meaning for the two mappings. We will distinguish two cases. First, in the case \(|\sigma| = \pm j\), we describe two 2-uniform tag sequences over \(\Sigma_{a,b}\), denoted as \(g\) and \(d\), closely matching the shape of \(g^\sigma(u)\) and \(d^\sigma(u)\), respectively. Later we consider the case \(|\sigma| \neq \pm j\).

#### 4.1. The case \(|\sigma| = j\)

For \(g\) or \(d\) to be applicable, the initial word \(u\) has to be decomposable as \(u = u_1 u_2\) where \(|u_1| - |x|\) is a multiple of \((|x| + |\beta|), |u_2| = j\).

We denote for both mappings \(g\) and \(d\), for \(i, k \geq 0\),

\[
\begin{align*}
s_0 &= u_2, & s_{i+1} &= \sigma(s_i), & p_i &= \pi(s_i)
\end{align*}
\]

and separately

for the mapping \(g\):

\[
\begin{align*}
a_0 &= u_1, & a_{i+1} &= z.(\beta,x)^*(a_i), & a_0 &= u_1, & a_{i+1} &= z.R.(\beta.R.x.R)^*(a_i),

b^k_0 &= s_k.p_k, & b^k_{i+1} &= \beta.(x,\beta)^*(b^k_i), & b^k_0 &= s_k.p_k, & b^k_{i+1} &= \beta.R.(x.R.\beta.R)^*(b^k_i),
\end{align*}
\]

where the superscripts do not denote powers. Then

for the mapping \(g\):

\[
\begin{align*}
g^0(u) &= a_0 s_0, & d^0(u) &= a_0 s_0,

g^1(u) &= a_0 b^0_0 a_1 s_1, & d^1(u) &= a_0 b^0_0 a_1 s_1,

g^2(u) &= a_0 b^0_0 a_1 b^0_1 a_1 s_2, & d^2(u) &= a_0 b^0_0 a_1 b^0_1 a_2 s_2,
\end{align*}
\]

\[
\cdots
\]

Since \(\sigma, \pi, x, \beta\) are fl-mappings, it is easy to observe that the sequences \(s = (s_0, s_1, \ldots), \)
\(p = (p_0, p_1, \ldots), \)
\(a = (a_0, a_1, \ldots)\), and each \(b^k = (b^k_0, b^k_1, \ldots)\), are periodic. Moreover, the
sequence \((b_0^0, b_0^1, \ldots)\) is periodic, hence the sequence \(\tilde{b} = (\tilde{b}_0^0, \tilde{b}_1^1, \ldots)\) is periodic as well. Let us choose for \(p_b, q_b\) a common length of the pre-period and period, respectively, of \(\tilde{b}\) and \(\tilde{s}\) (and, consequently, of \(\tilde{h}\)), and for \(p_{a,b}, q_{a,b}\) a common length of the pre-period and period, respectively, of \(\tilde{a}\) and \(\tilde{b}\). In the case of the mapping \(d\) we choose \(q_{a,b}\) to be even. Based on these values, we construct the alphabet \(\Sigma_{a,b}\).

We will now describe 2-uniform tag sequences \(\tilde{g}\) and \(\tilde{d}\), and later we will show that they match the structure of \(g^n(u)\) and \(d^n(u)\), respectively.

Let \(\mu: \Sigma_{a,b}^* \rightarrow \Sigma_{a,b}^*\), \(\nu: \Sigma_{a,b}^* \rightarrow \Sigma_{a,b}^*\) be morphisms (if \(p_{a,b} \neq 0\), \(v\) is not defined for symbols with the subscript 0) defined for \(i, k \geq 0\) by

for the mapping \(g\):

\[
\mu: a_i \mapsto a_i b_0^0, \quad b_i^1 \mapsto a_{k+i+1} b_i^k+1, \\
\nu = \psi \circ \mu \circ \psi^{-1},
\]

for the mapping \(d\):

\[
\mu: a_i \mapsto a_{2[i+1]} b_0^0, \quad b_i^k \mapsto a_{2[i+1]} b_i^k, \\
\nu = \psi^R \circ \mu \circ \psi^{-R}.
\]

Actually, in the case of the mapping \(g\), \(\nu\) coincides on its domain with \(\mu\), hence we will use directly \(\mu\) in our further considerations.

Since \(q_{a,b}\) is even, \(2[(p_{a,b} + q_{a,b} + i)/2] = 2[(p_{a,b} + i)/2] + q_{a,b}\), therefore the definition of \(\mu\) for the mapping \(d\) is consistent regarding the periodicity of the sequences of symbols. \(\mu\) is extendible in \(\alpha_0\), so we may define for the mapping \(g\)

\[
\tilde{g} = \mu^n(a_0) = a_0 b_0 a_1 b_0 a_2 b_0 a_3 b_0 a_4 b_0 a_5 b_0 a_6 b_0 a_7 b_0 a_8 b_0 a_9 b_0 a_{10} b_0 \ldots
\]

and for the mapping \(d\)

\[
\tilde{d} = \mu^n(a_0) = a_0 b_0 a_1 b_0 a_2 b_0 a_3 b_0 a_4 b_0 a_5 b_0 a_6 b_0 a_7 b_0 a_8 b_0 a_9 b_0 a_{10} b_0 \ldots
\]

The following lemmas list some basic properties of \(\mu\) and its iterations.

**Lemma 1.** For \(n \in \mathbb{N}\),

for the mapping \(g\): \(\psi \circ \mu^n = \mu^n \circ \psi\), \(\psi^R \circ \mu^n = \nu^n \circ \psi^R\).

**Proof.** The case \(n = 0\) is trivial, the case \(n = 1\) follows from the definition of \(\mu, \nu\), and is used in the inductive proof for \(n \geq 2\). \(\square\)

**Lemma 2.** For \(n, i \geq 0\), for both mappings \(g\) and \(d\),

\[
suf_i(\mu^n(b_i^0)) = b_i^n = suf_i(\mu^{n+1}(a_i)).
\]

**Proof.** By induction on \(n\). \(\square\)

In the case of the mapping \(d\) we will need two additional properties.
Lemma 3. For the mapping $d$, for $n, k \geq 0$, $c \in \Sigma_{a,b}$,

(i) $\text{suf}_i(\mu^n(c)) = \text{pref}_i(v^n(c))$,

(ii) $[\mu^n(a_{2k+1})]_1 = [\mu^n(a_{2k+2})]_1$, $1[\mu^n(a_{2k+1})] = 1[\mu^n(a_{2k})]$.

Proof. By induction on $n$. □

Lemma 4. For $n, i, k \geq 0$,

for the mapping $g$:

\[ [\mu^n(b^k_i)]_1 = [\mu^n(a_{k+i+1})]_1, \quad [\mu^n(b^k_{n-i})]_1 = 1[v^n(a_{2[(n+i)/2]+1})], \]

or the mapping $d$:

\[ 1[v^n(b^k_{n+i})] = [\mu^n(a_{2[(n+i+1)/2]+1})]. \]

Proof. By induction on $n$. The assertion is trivial for $n = 0$. Let it be valid for some $n \geq 0$. Then in the case of the mapping $g$ we obtain

\[ \mu^{n+1}(b^k_i) = [\mu^n(a_{k+i+1})b^k_{i+1})]_1 \]

\[ = [\mu^n(a_{k+i+1})]\cdot [\mu^n(b^k_{i+1})]_1 \]

\[ = \mu^n(a_{k+i+1})[\mu^n(b^k_{i+1})]_1 \]

\[ = \mu^n(a_{k+i+1})[\mu^n(a_{k+i+2})]_1 \text{ ind. hyp.} \]

\[ = \mu^n(a_{k+i+1})[\mu^n(b^0_{k+i+1})]_1 \text{ ind. hyp.} \]

\[ = [\mu^n(a_{k+i+1})b^0_{k+i+1})]_1 \]

In the case of the mapping $d$, denote $c_k = \text{suf}_i(\mu^n(a_k))$ [$= \text{pref}_i(v^n(a_k))$ by Lemma 3].

Then

(i) $[\mu^{n+1}(b^k_{n+i+1})]_1$

\[ = [\mu^n(a_{2[(n+i+1)/2]+1})b^k_{n+i+1})]_1 \]

\[ = [\mu^n(a_{2[(n+i+1)/2]+1})]\cdot [\mu^n(b^k_{n+i+1})]_1 \]

\[ = [\mu^n(a_{2[(n+i+1)/2]+1})]_1 \cdot [\mu^n(b^k_{n+i+1})]_1 \]

\[ = 1[v^n(b^k_{n+i+1})] \cdot [\mu^n(a_{2[(n+i+1)/2]+1})]_1 \text{ ind. hyp.} \]

\[ = 1[v^{n+1}(a_{2[(n+i+1)/2]+1})] \]

(ii) $1[v^{n+1}(b^k_{n+i+1})]$

\[ = 1[v^n(b^k_{n+i+1})] \cdot v^n(a_{2[(n+i)/2]+2}) \]

\[ = 1[v^n(b^k_{n+i+1})] \cdot [\mu^n(a_{2[(n+i)/2]+2})]_1 \]

\[ = [\mu^n(a_{2[(n+i+1)/2]+2})]_1 \cdot [\mu^n(b^k_{n+i+1})]_1 \text{ ind. hyp.} \]
We will now show that there is a correspondence between the sequences \( \bar{g}, \bar{d} \) and \( g^R(u), d^R(u) \), respectively, described by the morphism \( \varphi : \Sigma^*_a \rightarrow \Gamma^* \) defined (for both \( g \) and \( d \)) as

\[
\varphi : a_i \mapsto a_i, b^\ell_i \mapsto b^\ell_i.
\]

This morphism need not be uniform, since \( a_i \) and \( b^\ell_i \) may be of different length. We will use the following two important properties of \( \varphi \) (\( \rho \) is defined by (3) and (4)).

**Lemma 5.**

- For the mapping \( g \): for the mapping \( d \):
  - (i) \( \rho \circ \varphi = \varphi \circ \psi \) on \( \Sigma_a.(\Sigma_b.\Sigma_a)^* \).
  - (ii) \( \varphi \circ \mu \) is a uniform morphism, \( \varphi \circ \mu \) is a uniform morphism.

**Proof.** (i) Induction on the length of the mapped word. (ii) follows from the definition of \( \mu \) and \( \varphi \). \( \square \)

To prove our main result, being actually a corollary of Lemma 7, we need one more supporting lemma which can be easily proved by induction.

**Lemma 6.** For \( n \geq 0 \)

- For the mapping \( g \): for the mapping \( d \):
  - (i) \( \text{suf}_j(g^n(u)) = s_n \).
  - (ii) \( g^{n+1}(u) = g^n(u).p_n.\rho([g^n(u)]_1).s_{n+1} \).
  - \( d^n(u) = d^n(u).p_n.\rho^R([d^n(u)]_1).s_{n+1} \).

**Lemma 7.** For \( n \geq 0 \)

- For the mapping \( g \): for the mapping \( d \):
  - \( g^n(u).p_n = \varphi(\mu^{n-1}(a_0)) \)
  - \( d^n(u).p_n = \varphi(\mu^{n-1}(a_0)) \)

**Proof.** Induction on \( n \). For \( n = 0 \) we have

- For the mapping \( g \): for the mapping \( d \):
  - \( g^0(u).p_0 = a_0.s_0.p_0 \)
  - \( d^0(u).p_0 = a_0.s_0.p_0 \)
  - \( = \varphi(a_0).\varphi(h^0_0) \)
  - \( = \varphi(a_0).\varphi(h^0_0) \)
  - \( = \varphi(\mu^1(a_0)) \)
  - \( = \varphi(\mu^1(a_0)) \).
Let now the assertion be true for some \( n \). Denote \( p = |\pi| \). Then

**for the mapping \( g \):**

\[
\rho([g^p(u)]_j) = \rho([g^p(u) \cdot p_n]_{j+p}) = \rho([\phi(\mu^{n+1}(a_0))]_{j+p}) = \rho([\phi([\mu^{n+1}(a_0)] \cdot b_0^0]_{j+p})
\]

**for the mapping \( d \):**

\[
\rho^R([d^n(u)]_j) = \rho^R([d^n(u) \cdot p_n]_{j+p}) = \rho^R([\phi(\mu^{n+1}(a_0))]_{j+p}) \text{ ind. hyp.}
\]

**Lemma 2**

\[
\rho([\phi([\mu^{n+1}(a_0)] \cdot b_0^0]_{j+p})
\]

**Lemma 5(i)**

\[
\rho([\phi([\mu^{n+1}(a_0)] \cdot b_0^0]_{j+p})
\]

**Lemma 6(ii)**

\[
\rho([\phi([\mu^{n+1}(a_0)] \cdot b_0^0]_{j+p})
\]

and, consequently

**for the mapping \( g \):**

\[
g^{n+1}(u) \cdot p_{n+1} = g^n(u) \cdot p_n \cdot \rho([g^n(u)]_j) \cdot s_{n+1} \cdot p_{n+1}
\]

**for the mapping \( d \):**

\[
d^{n+1}(u) \cdot p_{n+1} = d^n(u) \cdot p_n \cdot \rho^R([d^n(u)]_j) \cdot s_{n+1} \cdot p_{n+1}
\]

**Corollary 1.**

for the mapping \( g \):

\[
g^{n}_o(u) = \phi(\mu^n(a_0)),
\]

for the mapping \( d \):

\[
d^{n}_o(u) = \phi(\mu^n(a_0)).
\]
4.2. The case $|\sigma| \neq j$

Denote $s = |\sigma|$.

Let us consider the mapping $y$ first. We will just provide two examples, one for $j < s$, one for $j > s$, when $g^a(u)$, for a particular word $u$, is not a 2-uniform tag sequence.

Let us define on the alphabet $\Sigma = \{a, b\}$ the mappings $g_1 : x \mapsto [x]_1$, $g_{-1} : x_1 x_2 b$, i.e., for $g_1$ $j = 1$, $s = 0$, and for $g_{-1}$ $j = 0$, $s = 1$. Let us consider the sequences $g_1^a(ab)$ and $g_{-1}^a(a)$. We will prove that neither of them is a 2-uniform tag sequence by using the following lemma.

Lemma 8. Let for alphabets $\Delta, \Sigma$, $h_1 : \Delta^* \rightarrow \Sigma^*$ be a 2-uniform morphism extendible in $a_0 \in \Delta$, $h_2 : \Delta^* \rightarrow \Sigma^*$ be a 1-uniform morphism. Let $s = h_2(h_1^a(a_0))$. Then for $r \geq 0$ the sequence $(\tilde{s}(2^n + r))_{n=0}^\infty$, and for $r \geq 1$ the sequence $(\tilde{s}(2^n + r))_{n=\lceil \log r \rceil}^\infty$, are periodic with the sum of the lengths of the pre-period and of the period not exceeding $|\Delta|$.

Proof. Let us denote $\tilde{c} = h_1^a(a_0)$, and further, for $r \geq 0$, $k \geq 0$, $c_{r,k} = \tilde{c}(2^k + r)$ and, for $r \geq 1$, $c_{r,k} = \tilde{c}(2^k + \lceil \log r \rceil - r)$ (the positions in a sequence are numbered starting from 0). Then for $r \geq 0$, $n \geq 0$, $s(2^n + r) = h_2(c_{r,n})$, and for $r \geq 1$, $n \geq \lceil \log r \rceil$, $\tilde{s}(2^n - r) = h_2(c_{r,n-\lceil \log r \rceil})$. The symbols $c_{r,k}$ can be computed by the recurrent relations $c_{0,0} = \text{suf}_1(h_1(a_0))$, $c_{0,k+1} = \text{pref}_1(h_1(c_{k,0}))$, and $c_{1,0} = a_0$. $c_{r,k+1} = \text{suf}_1(h_1(c_{r,k}))$. The sequences $(c_{0,k})_{k=0}^\infty$, $(c_{1,k})_{k=0}^\infty$ are therefore periodic, in their pre-period and period no symbol of $\Delta$ is repeated. Since $c_{r,k} = \text{suf}_1(\text{pref}_{r+1}(h_1(\lceil \log r + 1 \rceil)(c_{k,0})))$, $c_{r,k} = \text{pref}_1(\text{suf}_1(h_1^a(c_{r,k}))$, the sequences $(\tilde{s}(2^n + r))_{n=0}^\infty$, and $(\tilde{s}(2^n - r))_{n=\lceil \log r \rceil}^\infty$ are periodic with the sum of the lengths of the pre-period and of the period not exceeding $|\Delta|$. □

In the following lemmas we state some properties of $g_1$ and $g_{-1}$. The first two of them are trivial observations.

Lemma 9. For $n \geq 0$

(i) $|g_1^a(ab)| = 2^n + 1$, $|g_{-1}^a(a)| = 2^n - 1$,

(ii) $\text{pref}_{2}(g_1^a(ab)) = ab$, $\text{suf}_{n+1}(g_{-1}^a(a)) = ab^n$.

Lemma 10. For any word $w$, and any $1 \leq i \leq |w|$, $\text{suf}_{2}[w]_{i-1}(g_1^a(w)) = w[w]_i$. In particular, $\text{suf}_{|w|}(g_1^a(w)) = w$.

Lemma 11. For any $k \in \mathbb{N}$ there is some $m \in \mathbb{N}$ and a word $x$ such that $g_1^a(ab) = x \cdot a^k b$.

Proof. Induction on $k$. For $k = 0$ the assertion is trivial. Let for $k \geq 1$, $g_1^a(ab) = x \cdot a^k b$. Then, following Lemma 9(ii), $g_1^{n+2}(ab) = z \cdot a^k ab \cdot t$ for some word $z$, and following Lemma 10, $g_1^{n+2+|t|}(ab) = x' \cdot a^{k+1} b$ for some word $x'$.
Lemma 12. Neither $g_1^o(ab)$ nor $g_{-1}^o(a)$ is a 2-uniform tag sequence.

Proof. To prove the assertion for $g_1^o(ab)$, we will apply Lemma 8 after proving that the sequence $(g_1^o(ab)(2^n))_{n=0}^{\infty}$ is not periodic. Assume it is, the lengths of its preperiod and period being $p, q$, respectively. According to Lemma 9(i), $g_1^o(ab)(2^n) = \text{suf}_1(g_1^o(ab))$, hence, for $k \geq p$, $\text{suf}_1(g_1^{k+q}(ab)) = \text{suf}_1(g_1^q(ab))$. Let us apply Lemma 11 and choose $m > p$ such that $g_1^m(ab) = xa^yb$ for some word $x$. Then $b - \text{suf}_1(g_1^m(ab)) = \text{suf}_1(g_1^{m+q}(ab)) = a$ -- a contradiction.

To prove the assertion for $g_{-1}^o(a)$, we will again apply Lemma 8. Using the notation from there, assume $g_{-1}^o(a) = h_2(h_1^o(a_0))$. Let us choose $n_0 \in N$ big enough to satisfy $2^{n_0} > 2|d|n_0 + 3$. Denote $r = n_0 + 3$. Let us assume that the sequence $(g_{-1}^o(a)(2^n - r))_{n=\lceil \log r \rceil}^{\infty}$ is periodic with the lengths of its period and preperiod being $p, q$, respectively, $p + q < l_1, q > 1$ . We have $n_0 \geq \lceil \log(n_0 + 3) \rceil + |d| \geq \lceil \log r \rceil + p$. Then, applying Lemma 9(ii) we obtain $a = \text{pref}_1(\text{suf}_{n_0+1}(g_{-1}^o(a))) = g_{-1}^o(a)(2^{n_0} - r) = g_{-1}^o(a)(2^{n_0+q} - r) = \text{pref}_1(\text{suf}_{n_0+1}(g_{-1}^o(a))) = b$ -- a contradiction. \(\square\)

Using the property of Proposition 1(ii), one can easily construct examples of the mapping $g$ for any values $j, s$, $s \neq j$ such that the result of infinite iteration of $g$ on some input word is not a 2-uniform tag sequence.

Let us now consider the mapping $d$. This mapping defined by (2), (3), (4) can be applied iteratively to some initial word $u$ (of a suitable length) just in the case when $|j - s|$ is a multiple of $|\alpha| + |\beta|$. Let it be the case. We distinguish two cases

I. $s \geq j$: Let $\pi' : \Gamma^s \to \Gamma^{s-j+p}$, $\sigma' : \Gamma^s \to \Gamma^s$ be defined for $x \in \Gamma^s$ by

$$\pi'(x) = \pi(\text{suf}_1(x)), ((x, \beta)^s(\text{pref}_{s-j}(x)))^R,$$

$$\sigma'(x) = \sigma(\text{suf}_1(x)).$$

Then for any word $w, |w| > s$

$$d_{[j, \pi, \sigma]}(w)$$

$$= w \cdot \pi(\text{suf}_1(w)), \rho^R(\lceil [w]_j \rceil), \sigma(\text{suf}_1(w))$$

$$= w \cdot \pi(\text{suf}_1(\text{suf}_1(w))), (\rho(\lceil [w]_s \rceil)(\alpha, \beta)^s(\text{suf}_{s-j}(\lceil [w]_j \rceil)))^R, \sigma(\text{suf}_1(\text{suf}_1(w)))$$

$$= w \cdot \pi(\text{suf}_1(\text{suf}_1(w))), ((\alpha, \beta)^s(\text{pref}_{s-j}(\text{suf}_1(w))))^R, \rho^R(\lceil [w]_s \rceil), \sigma'(\text{suf}_1(w))$$

$$= w \cdot \pi(\text{suf}_1(w)), \rho^R(\lceil [w]_s \rceil), \sigma'(\text{suf}_1(w))$$

$$= d_{[s, \pi', \sigma']}(w),$$

where $|\sigma'| = s$; therefore Corollary 1 applies to $d_{[j, \pi, \sigma]}$ as well.

II. $s < j$: For a sufficiently long word $w$

$$\text{pref}_{j-s}(\text{suf}_1(\text{pref}_{j-s}(d_{[j, \pi, \sigma]}(w))))$$

$$= \text{pref}_{j-s}(\text{suf}_1(\pi(\text{suf}_1(w))), \rho^R(\lceil [w]_j \rceil), \sigma(\text{suf}_1(w)))$$

$$= \text{pref}_{j-s}(\text{suf}_{j-s}(\rho^R(\lceil [w]_j \rceil), \sigma(\text{suf}_1(w))))$$
\[\begin{align*}
\text{Let } & \sigma' : \Gamma^j \to \Gamma^j \text{ be defined for } x \in \Gamma^j \text{ as } \sigma'(x) = ((\alpha.\beta)\sigma)^{(\sigma_j(x))}.p(\sigma_j(x)).s(\sigma_j(x)).
\text{Consequently } d'_{\rho_j,\sigma_j}(u) = \text{pref}_j-\beta\sigma_j(u) = \text{pref}_j-\beta\sigma_j(u).d'[j,\eta,\sigma_j][j-\beta].
\end{align*}\]

The following theorem, where, generally, the assertion (i) cannot be extended to the case \(|\sigma| \neq j\), summarises our main result.

**Theorem 1.** Let \(g,d\) be defined by (2), (3) and (4), and let \(u\) be a word such that \(|u| \geq (j + |x|)\) and \(|u| - (j + |x|)\) is a multiple of \(|x| + |\beta|\). Then

1. If \(|\sigma| = j\) then \(d'(u)\) is a 2-uniform tag sequence.
2. If \(|\sigma| = j\) is a multiple of \((|x| + |\beta|)\) then \(d^*(u)\) is a 2-uniform tag sequence.

4.3. The mapping \(d_j\)

The mappings \(g_j, d_j\) from [3] can be characterized by (2)–(4) where we chose \(x = \eta, |x| = 1, |\beta| = |\eta| = |\sigma| = 0\). In the construction of the sequence \(\bar{d}\) corresponding to the mapping \(d_j\), the only symbols belonging to \(\Sigma_{a,b}\) will be \(a_0, a_1, b_0, b_1\). For our convenience, we will rename them and use the alphabet \(\Sigma_{a,b} = \{a,\bar{a},\bar{b}\}\). The morphisms \(\psi, \mu : \Sigma^*_{a,b} \to \Sigma^*_{a,b}\) from Sections 3 and 4 are defined as

\[\psi : a \mapsto \bar{a}, \quad \bar{a} \mapsto a, \quad b \mapsto \bar{b}, \quad \bar{b} \mapsto b,\]
\[\mu : a \mapsto ab, \quad \bar{a} \mapsto \bar{a}b, \quad b \mapsto \bar{b} \mapsto \bar{a}b.\]
We denote
\[ d_{a,b} = \mu^o(a) = ab\bar{a}bab\bar{a}bab\bar{a}bab\bar{a}bab\bar{a}bab\bar{a}bab\ldots \]

Given an arbitrary initial word \( u \) of a suitable length, we may, if necessary, start the iterative applications of \( d_j \) from \( d_j(u) \) instead from \( u \). We may therefore assume, that \( u \) can be decomposed as \( u = xyx^R \), where \( |x| = j \). Then we have \( d_j^o(u) = x \cdot \phi(d_{a,b}) \), where the morphism \( \phi : \Sigma^* \rightarrow \Sigma^* \) is defined as
\[ \phi : a \mapsto y \quad \bar{a} \mapsto y^R \quad b \mapsto x^R \quad \bar{b} \mapsto x. \]

The sequence \( d_{a,b} \) is interesting for its internal structure. We may define the shuffle operation on two sequences \( \hat{x} = x_0x_1x_2\ldots \), \( \hat{y} = y_0y_1y_2\ldots \) of symbols as \( \hat{x} \downarrow \hat{y} = x_0y_0x_1y_1x_2y_2\ldots \). (The operation \( \downarrow \) is neither commutative nor associative.) For example, for the well-known sequence of Thue \([7,8]\), (for some properties see \([6]\)) \( \hat{t} \) and its dual sequence \( \hat{t}' \) where \( \hat{t} = 011010011011011011 \), \( \hat{t}' = 1001101011010011 \ldots \) we have \( \hat{t} \downarrow \hat{t}' = \hat{t} \), and \( \hat{t}' \downarrow \hat{t} = \hat{t}' \). If we denote by \( \phi^1 \hat{x} \) the limit sequence (fixed point) of iterated left shuffling of the sequence \( \hat{x} \), i.e. the sequence satisfying \( \hat{x} \downarrow \phi^1 \hat{x} = \hat{x} \), then \( \hat{t} = 0.(\phi^1 \hat{t}') \), \( \hat{t}' = 1.(\phi^1 \hat{t}) \).

In our case \( d_{a,b} \) can be decomposed as \( d_{a,b} = (a\bar{a})^o \downarrow o^1((b\bar{b})^o) \).

References