Computation and application of the paired combinatorial logit stochastic user equilibrium problem

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A B S T R A C T
The paired combinatorial logit (PCL) model is one of the recent extended logit models adapted to resolve the overlapping problem in the route choice problem, while keeping the analytical tractability of the logit choice probability function. However, the development of efficient algorithms for solving the PCL model under congested and realistic networks is quite challenging, since it has large-dimensional solution variables as well as a complex objective function. In this paper, we examine the computation and application of the PCL stochastic user equilibrium (SUE) problem under congested and realistic networks. Specifically, we develop an improved path-based partial linearization algorithm for solving the PCL SUE problem by incorporating recent advances in line search strategies to enhance the computational efficiency required to determine a suitable stepsize that guarantees convergence. A real network in the city of Winnipeg is applied to examine the computational efficiency of the proposed algorithm and the robustness of various line search strategies. In addition, in order to acquire the practical implications of the PCL SUE model, we investigate the effectiveness of how the PCL model handles the effects of congestion, stochasticity, and similarity in comparison with the multinomial logit stochastic traffic equilibrium problem and the deterministic traffic equilibrium problem.

1. Introduction
The stochastic user equilibrium (SUE) principle was suggested by Daganzo and Sheffi [21] more than 30 years ago to relax the perfect knowledge assumption of network travel times of the deterministic user equilibrium (DUE) model. It is defined as follows:

“At SUE, no motorists can improve his or her perceived travel time by unilaterally changing routes” [21].

Specifically, a random error term is incorporated in the route choice decision process to simulate travelers’ imperfect perceptions of network travel times, such that they do not always end up picking the minimum travel time route. The random error term here is interpreted as the perception error of network travel times due to the travelers’ imperfect knowledge of network conditions. In this model, each traveler is assumed to have some perceptions of the mean travel times on each link of the network. Each traveler’s route choice criterion is to minimize the perceived value of the route travel time, which can be obtained by adding up the perceived travel times on all the links belonging to the route.

Route choice models proposed under this approach can have different specifications according to modeling assumptions on the random error term. The two commonly used random error terms are Gumbel [23] and Normal [21] distributions, which result in the logit- and probit-based route choice models, respectively. The logit-based route choice model has a closed-form probability expression and also an equivalent mathematical programming (MP) formulation for the SUE problem under congested networks [25]. The multinomial logit (MNL) SUE MP formulation can be solved using both path enumeration techniques [9,11,46] and column (or path) generation techniques [7,8,12–13,16,18–19,22,27,30,33]. Column generation techniques for the MNL SUE model can be implemented in the link- or path-based domains. The link-based algorithms [7,18,27,33] do not require path storage and often use Dial’s STOCH algorithm [23] or Bell’s alternative [6] as the stochastic loading step, while the path-based algorithms [8,12–13,16,19,22,30,46] require explicit path storage in order to directly compute the logit route choice probabilities. The drawbacks of the logit model are: (1) inability to account for overlapping (or correlation) among routes, and (2) inability to account for perception variance with respect to trips with different lengths. These two drawbacks stem from its underlying assumptions that the random
error terms are independently and identically distributed (IID) with the same and fixed variances [43]. On the other hand, the probit-based route choice model does not have such drawbacks, because it handles the overlapping and identical variance problems between routes by allowing covariance between the random error terms for pairs of routes. However, the probit model does not have a closed-form solution and it is computationally burdensome to solve the multiple integrals when the choice set contains more than a handful of routes. Due to the lack of a closed-form probability expression, solving the probit-based SUE model will require either Monte Carlo averaging (SRA) scheme and quadratic interpolation scheme) and the traditional line search strategies (e.g., method of successive averages (MSA), bisection, and Armijo) are examined in order to efficiently determine a suitable stepsize. A real-size network in the city of Winnipeg is applied to examine the efficiency and robustness of various line search schemes that minimize the computational efforts required to determine a suitable stepsize while guaranteeing convergence. In addition, the practical implications of implementing the PCL SUE model are acquired by examining the effects of congestion, stochasticity, and similarity in comparison with the multinomial logit stochastic traffic equilibrium problem and the deterministic traffic equilibrium problem.

The paper is organized as follows. The PCL route choice model and the equivalent PCL SUE formulation are presented in Section 2. The path-based partial linearization algorithm with various line search schemes is described in Section 3. Computational results and practical implications are then presented in Section 4 and Section 5, respectively. Finally, some conclusions are summarized in Section 6.

2. Paired combinatorial logit stochastic user equilibrium problem

2.1. PCL route choice model

The paired combinatorial logit (PCL) model was originally proposed by Chu [20], further developed by Koppleman and Wen [29] to examine the structure, properties, and estimation, and adapted to model route choice decisions by Bekhor and Prashker [2], Gliebe et al. [26], and Pravinvongvuth and Chen [40]. In contrast to the simple structure of the logit model, the PCL model has a hierarchical structure that decomposes the choice probability into two levels represented by the marginal and conditional probabilities (see Fig. 1). Thus, the PCL choice probability can be expressed as

\[ P(k) = \sum_{j \neq k} P(k|kj) \cdot P(k|kj), \]

where

\[ P(k|kj) = \frac{e^{\sigma_{kj}(V_{kj}/(1-\sigma_{kj}) + V_{kj}/(1-\sigma_{kj}))}}{\sum_{m=1}^{n} \sum_{l=1}^{m} (1-\sigma_{lm})(V_{lm}/(1-\sigma_{lm}) + V_{lm}/(1-\sigma_{lm}))^{-\sigma_{mj}}}, \]

\[ P(k|kj) = \frac{(1-\sigma_{kj})(V_{kj}/(1-\sigma_{kj}) + V_{kj}/(1-\sigma_{kj}))^{-\sigma_{kj}}}{\sum_{m=1}^{n} \sum_{l=1}^{m} (1-\sigma_{lm})(V_{lm}/(1-\sigma_{lm}) + V_{lm}/(1-\sigma_{lm}))^{-\sigma_{mj}}}, \]

where \( V_{kj} \) is the observable component of the utility for alternative \( k \) (i.e., \( U_k = V_k + \epsilon_k, \forall k \)); \( \sigma_{kj} \) is a similarity index between alternatives \( k \) and \( j \); and \( n \) is the number of alternatives. \( P(k|kj) \) is the conditional probability of choosing alternative \( k \) given that the alternative pair \( kj \) has been chosen, and \( P(k) \) is the marginal (unobserved) probability objective function (to be shown later in Section 3). For the line search step, recent advances in line search strategies (e.g., self-regulated averaging (SRA) scheme and quadratic interpolation scheme) and the traditional line search methods (e.g., method of successive averages (MSA), bisection, and Armijo) are examined in order to efficiently determine a suitable stepsize. A real-size network in the city of Winnipeg is applied to examine the efficiency and robustness of various line search schemes that minimize the computational efforts required to determine a suitable stepsize while guaranteeing convergence. In addition, the practical implications of implementing the PCL SUE model are acquired by examining the effects of congestion, stochasticity, and similarity in comparison with the multinomial logit stochastic traffic equilibrium problem and the deterministic traffic equilibrium problem.

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for the alternative pair kj. For a choice set of n alternatives, there are a total of n(n−1)/2 pairs of alternatives as shown in the double summation of the marginal probability in Eq. (3).

Fig. 1 provides an illustration of the hierarchical tree structure of the PCL model for a simple network with three alternative routes connecting node X and node Y. If θj is equal to zero for all kj pairs, the PCL model reduces to the multinomial logit (MNL) model [29] as follows:

\[
P(k) = \frac{e^{\theta_k}}{\sum_{j=1}^n e^{\theta_j}}
\]

(4)

When the similarity index is zero for all route pairs, this implies that the routes in the choice set are uniquely distinct.

2.2. Equivalent mathematical programming formulation

The PCL equivalent mathematical programming (MP) formulation presented here is due to Bekhor and Prasheker [2]. They showed that the marginal and conditional probabilities of the PCL model could be represented by two entropy terms in the objective function as follows:

\[
\min Z = Z_1 + Z_2 + Z_3,
\]

(5)

where

\[
Z_1 = \sum_{k} \frac{f_{kij}}{n} t_{ij}(\theta) d\theta,
\]

(6)

\[
Z_2 = \frac{1}{\theta} \sum_{rs} \sum_{k} \sum_{j=k} \theta^2 f_{rs}^r k_{kij} \ln \left( \frac{f_{rs}^r k_{kij}}{\beta_{rs}^k} \right),
\]

(7)

\[
Z_3 = \frac{1}{\theta} \sum_{rs} \sum_{k} \sum_{j=k} (1-\theta^2) f_{rs}^r k_{kij} + f_{rs}^r k_{kij} \ln \left( \frac{f_{rs}^r k_{kij}}{\beta_{rs}^k} \right),
\]

(8)

subject to

\[
\sum_{k} f_{rs}^r k_{kij} = q^r, \quad \forall \; r, s,
\]

(9)

\[
f_{rs}^r k_{kij} \geq 0, \quad \forall \; k, (kj), r, s,
\]

(10)

where \( t_{ij}(\cdot) \) is the travel time on link \( a, x_i \) is the flow on link \( a, \theta \) is the dispersion coefficient, \( \beta^k \) is a measure of dissimilarity index, defined as \( \beta^k = 1-\sigma^2 f_{rs}^r k_{kij} \), \( f_{rs}^r k_{kij} \) is the flow on route \( k \) (of route pair \( kj \)) between origin \( r \) and destination \( s \); \( |K|_1 \) is the number of routes between O and D pair rs; \( q^r \) is the travel demand between O and D pair rs.

The objective function in Eq. (5) consists of three terms: \( Z_1 \) given in Eq. (6) is the well-known UE term reflecting the congestion effect; \( Z_2 \) given in Eq. (7) is a modification of Fisk's entropy term reflecting the stochasticity effect; and \( Z_3 \) given in Eq. (8) is another entropy term reflecting the similarity effect. These two entropy terms represent the two-level tree representation of the PCL model shown in Fig. 1. \( Z_2 \) represents the conditional probability of the lower level and \( Z_3 \) represents the marginal probability of the upper level. Note that the expression \( \beta^k = 1-\sigma^2 f_{rs}^r k_{kij} \), \( \ln(f_{rs}^r k_{kij} / \beta^k) \) is defined as zero if either \( f_{rs}^r k_{kij} = 0 \) or \( \beta^k = 0 \) in Eq. (7). Eq. (9) ensures that the route flows add up to the travel demand between each O-D pair. To ensure meaningful solutions, non-negativity constraints are imposed in Eq. (10). For the equivalency and uniqueness of the above model, interested readers may refer to Bekhor and Prasheker [2].

The uniqueness means that the above MP formulation is a strictly convex program. This property is beneficial for both algorithmic development and route flow analysis. We can adopt existing efficient algorithms to solve the MP formulation. In addition, the uniqueness of route flows is helpful in several important applications (e.g., optimal routing in the route guidance system, environmental impact analysis and fuel consumption estimation based on route profiles of travel speeds, route-based congestion pricing, select link analysis, sensitivity analysis, and O-D trip table estimation from traffic counts).

The PCL model accounts for route overlapping problem by incorporating a similarity index into the objective function (i.e., \( Z_2 \) and \( Z_1 \)). This enables the PCL model to treat the similarity effect separately from the congestion effect (\( Z_1 \)) and the stochasticity effect (\( Z_2 \) in Fisk's formulation). Hence, the PCL SUE formulation is theoretically capable of accounting for congestion, stochasticity, and similarity effects simultaneously.

3. Path-based partial linearization algorithm

3.1. Algorithmic description

In this section, we provide a path-based partial linearization method for solving the PCL SUE formulation. The method is a descent algorithm for continuous optimization problems [37]. Conceptually, this method consists of two main steps: search direction and line search. A search direction is obtained from the solution of a convex auxiliary problem, defined by an approximation of the original objective function through a first-order approximation of an additive part of the PCL SUE objective function. A line search is made in the search direction obtained with respect to the original objective function, and then the resulting stepsize defines a new solution with a reduced objective value.

Suppose that a feasible solution \( f_{kij}^r(n) \) is given in iteration \( n \), and its corresponding route travel time is \( t_{ij}^r(n) \). The partial linearized subproblem is

\[
\min Z_{1,sub} = Z_{1,sub} + Z_2 + Z_3,
\]

(11)

where

\[
Z_{1,sub} = \sum_k c_{ij}^r(n) \cdot h_{ij}^r(n),
\]

(12)

\[
Z_2 = \frac{1}{\theta} \sum_{rs} \sum_{k} \sum_{j=k} \beta^k h_{ij}^r k_{kij} \ln \left( \frac{h_{ij}^r k_{kij}}{\beta^k} \right),
\]

(13)

\[
Z_3 = \frac{1}{\theta} \sum_{rs} \sum_{k} \sum_{j=k} (1-\theta^2) h_{ij}^r k_{kij} + h_{ij}^r k_{kij} \ln \left( \frac{h_{ij}^r k_{kij} + h_{ij}^r k_{kij}}{\beta^k} \right),
\]

(14)

subject to

\[
\sum_{k} h_{ij}^r k_{kij} = q^r, \quad \forall \; r, s,
\]

(15)

\[
h_{ij}^r k_{kij} \geq 0, \quad \forall \; k, (kj), r, s.
\]

(16)

The search direction is obtained by solving a first-order approximation of an additive part of the original objective function (i.e., \( Z_1 \) of the PCL SUE formulation). This can be done by updating the route travel times, calculating the PCL route choice probabilities, and assigning the auxiliary route flows (and then auxiliary link flows) according to the obtained PCL probabilities. The solution to the above subproblem is

\[
P_{ij}^r(n) = \frac{\beta^k \exp \left( -\frac{\alpha_{ij}^{\text{wfs}}(n)}{\beta_{ij}} \right) + \exp \left( -\frac{\alpha_{ij}^{\text{wes}}(n)}{\beta_{ij}} \right)^\beta_{ij}}{\sum_m=1^{\alpha_{ij}^{\text{efs}}(n)} \sum_l=1^{\alpha_{ij}^{\text{wes}}(n)} \left( \exp \left( -\frac{\alpha_{ij}^{\text{wes}}(n)}{\beta_{ij}} \right) + \exp \left( -\frac{\alpha_{ij}^{\text{wes}}(n)}{\beta_{ij}} \right)^\beta_{ij} \right)},
\]

(17)

\[
P_{ij}^r(kj)(n) = \frac{\exp \left( -\frac{\alpha_{ij}^{\text{wfs}}(n)}{\beta_{ij}} \right) + \exp \left( -\frac{\alpha_{ij}^{\text{wes}}(n)}{\beta_{ij}} \right)^\beta_{ij}}{\sum_j=1^{\alpha_{ij}^{\text{wfs}}(n)} \sum_l=1^{\alpha_{ij}^{\text{wes}}(n)} \left( \exp \left( -\frac{\alpha_{ij}^{\text{wes}}(n)}{\beta_{ij}} \right) + \exp \left( -\frac{\alpha_{ij}^{\text{wes}}(n)}{\beta_{ij}} \right)^\beta_{ij} \right)},
\]

(18)

\[
P_{ij}^r(n) = \sum_{j=1}^{\alpha_{ij}^{\text{wes}}(n)} P_{ij}^r(n) \cdot P_{ij}^r(kj)(n), \quad \forall \; k, r, s.
\]

(19)
If the vector \( \mathbf{h}(n) - f(n) \) is a non-zero, it defines a descent direction for the original objective function. A line search in this descent direction is then conducted to find a suitable stepsize
\[
\alpha(n) = \arg \min_{0 \leq \alpha \leq 1} Z[\mathbf{h}(k|k) + \alpha \mathbf{h}(k|k)(n) - f(k|k)(n)].
\]
We will elaborate the line search schemes in Section 3.2.

The new iterative solution can then be obtained by
\[
f_{k|k}(n+1) = f_{k|k}(n) + \alpha(n) \cdot \mathbf{h}_{k|k}(n) - f_{k|k}(n), \quad \forall \; k, (k), r, s.
\]

The partial linearization method iterates between the search direction and line search steps until some convergence criterion is satisfied. For completeness, the solution procedure is provided in Table 1.

### 3.2. Line search schemes

Before presenting the line search schemes, we analyze the difficulty of stepsize determination when solving the above MP formulation.

- The solution variable is the route pair flows \( f_{k|k}^{rs} \), which have a large dimension for large-scale realistic networks.
- The objective function has one integral term (i.e., summation of integral of link cost functions) and two entropy terms (i.e., conditional probability representing the stochasticity effect in the lower-level tree structure and marginal probability representing the similarity effect in the upper-level tree structure).
- Thus, it is quite computationally expensive to evaluate the objective function and its derivatives.

In the literature, there are three main types of line search schemes: predetermined, exact, and inexact. The method of successive averages (MSA) is the most widely used method in the predetermined line search scheme. It predetermines a diminishing stepsize sequence that satisfies \( \alpha(n) \rightarrow 0 \) and \( \sum_{n=0}^{\infty} \alpha(n) = \infty \), such as \( \{1, 1/2, 1/3, ..., 1/n\} \). This scheme is easy to implement since it does not need to evaluate the complex objective function and/or its derivative. However, it suffers from a sublinear convergence rate [36]. To overcome the sublinear convergence rate issues of the MSA scheme, McCormick [35] suggested determining the stepsize by exactly solving a one-dimensional minimization problem. Golden section and bisection are two widely used exact line search schemes (see [43]). They are simple and robust. However, both are computationally expensive because they need to evaluate the objective function or its derivatives several times in order to solve the one-dimensional minimization problem exactly. In practice, inexact line search schemes are usually more practical and efficient. Among others, the well-known Armijo-type methods are perhaps the most popular inexact line search scheme [1]. Note that the Armijo scheme was originally proposed for the steepest descent approach to solving unconstrained minimization problems. Bertsekas [5] then developed a generalized Armijo scheme for constrained optimization problems.

In the literature, two promising line search schemes, namely the quadratic interpolation scheme by Maher [33] and the self-regulated averaging (SRA) scheme by Liu et al. [31], were recently proposed to enhance the computational performance of determining a suitable stepsize. Below we further discuss these two promising line search schemes as well as the Armijo scheme.

1. **Self-regulated averaging scheme**: The SRA scheme was recently developed by Liu et al. [31] for solving the MNL SUE problem in the link domain. This scheme determines a suitable stepsize as follows:
\[
\alpha(n) = \frac{1}{\beta(n)}
\]
\[
\beta(n) = \begin{cases} 
\beta(n-1) + \lambda_1 & \text{if } ||\mathbf{h}(n) - f(n)|| \geq ||\mathbf{h}(n-1) - f(n-1)||, \\
\beta(n-1) + \lambda_2 & \text{otherwise}
\end{cases}
\]
where \( \lambda_1 > 1 \) and \( 0 < \lambda_2 < 1 \). Similar to the original MSA scheme, the following conditions should be satisfied in order to guarantee convergence [10,31,41]:
\[
\alpha(n) > 0, \quad \sum_{n=1}^{\infty} \alpha(n) = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \alpha(n) = 0 \quad \text{or} \quad \sum_{n=1}^{\infty} (\alpha(n))^2 < \infty
\]

Fig. 2 provides an illustration of the SRA scheme. From Eq. (24), we can observe that similar to the original MSA scheme, the stepsize sequence from the SRA scheme is still strictly decreasing. However, the decreasing speed is more efficient since the next stepsize is determined according to the residual error (i.e., the deviation between the current solution and its auxiliary solution) relationship of two consecutive iterations. When the current residual error is increased compared to the previous iteration (i.e., tends to diverge), parameter \( \lambda_1 > 1 \) is used to make the stepsize reduction more aggressive (e.g., at iterations 19 and 31). In contrast, when the residual error is decreased (i.e., tends to converge), parameter \( 0 < \lambda_2 < 1 \) is used to make the stepsize reduction more conservative. Hence, the stepsize sequences from the SRA scheme indeed satisfy the above conditions.

2. **Quadratic interpolation**: The quadratic interpolation scheme was first suggested by Maher [33] also for solving the MNL...
SUE problem in the link domain. This scheme calculates an approximate stepsize as follows:

$$\alpha(n) = -\frac{-\nabla_a Z(\alpha)|_{a=0}}{-\nabla_a Z(\alpha)|_{a=0} + \nabla_a Z(\alpha)|_{a=1}},$$

(26)

where $\nabla_a Z(\alpha)$ is the derivative of objective function with respect to stepsize. In this scheme, $\nabla_a Z(\alpha)$ is evaluated twice per iteration, one at $\alpha=0$ and another at $\alpha=1$, to determine an approximate stepsize. However, this scheme can only be considered as a practical method as its convergence cannot be guaranteed in general. To demonstrate the quadratic interpolation scheme, consider the following example in Fig. 3. The objective function with respect to stepsize is $Z(\alpha) = 2(1-\alpha)^2 + 5\alpha^2$. It is easy to see that the derivative of objective function with respect to stepsize is $\nabla_a Z(\alpha) = 14\alpha - 4$, $\nabla_a Z(\alpha)|_{a=0} = -4$, and $\nabla_a Z(\alpha)|_{a=1} = 10$. Hence, an approximate stepsize is $\frac{-(-4)}{|-(-4) + 10|} = 0.286$, which is the same as the exact stepsize for this example since $Z(\alpha)$ is a quadratic function.

3. Armijo: for completeness, the generalized Armijo strategy is illustrated in Fig. 4. The basic idea is as follows: Given that $f(n)$ is not an optimal solution, we set

$$\alpha(n) = \beta^{m(n)}s,$$

(27)

where $m(n)$ is the first nonnegative integer $m$ such that

$$Z(f(n)) - Z(f(n)(\beta^m s)) \geq \sigma \nabla Z(f(n))^T(f(n) - f(n)(\beta^m s)),$$

(28)

where $Z(\cdot)$ and $\nabla Z(\cdot)$ are the objective function and its gradient; $\sigma \in (0, 1)$, $\beta \in (0, 1)$, and $s > 0$ are fixed scalars. From Eq. (28), one can see that in order to obtain an appropriate stepsize, only some evaluations of the objective function and its gradient are required. Therefore, we can avoid solving the computationally expensive exact line search of the one-dimensional minimization problem.

Remarks.

1. For the line search step, it is impractical to compute an exact line search for the PCL SUE objective function. The reasons are twofold: (1) the complex decision variables, $f_{rs(k)}$, need to be stored in order to calculate $Z_2$ and $Z_3$, and (2) it is computationally expensive to calculate all three terms in the PCL SUE objective function.

2. Note that Chen et al. [14] proposed two practical schemes to determine a suitable stepsize: (a) optimize the stepsize just according to $Z_1$ since it is the dominant part of the whole objective function, and (b) optimize the stepsize according to Fisk’s objective function: $Z_1 + Z_2^{\text{Fisk}}$, where $Z_2^{\text{Fisk}} = 1/\theta \sum_k f_{rs}^T \log f_{rs}^T$. These two practical schemes may be more computationally efficient compared to optimizing the whole objective function. However, convergence cannot be assured, as there is no guarantee that the stepsize obtained from the approximation will decrease (or at least not worsen) the objective value.

3. The line search schemes are compared in Table 2 to highlight the computation with respect to the objective function and its derivative evaluations. Since the objective function in Eq. (5) is quite complex, evaluating the objective function and its derivatives constitutes the main part of the whole computation. Compared to other schemes (e.g., bisection, quadratic interpolation, and Armijo), the SRA scheme is easy to implement. It does not need to evaluate the complex objective function or its derivatives (with respect to solution variable or stepsize).
4. Computational results in a real network

In this section, a set of numerical experiments using a real network in the city of Winnipeg, Canada, is conducted to examine the efficiency of the path-based partial linearization algorithm with two recent advanced line search schemes (SRA and quadratic interpolation) and three traditional line search schemes (MSA, bisection, and Armijo) for solving the PCL SUE model. The Winnipeg network, shown in Fig. 5, consists of 154 zones, 1067 nodes, 2535 links, and 4345 O–D pairs. The network structure, O–D trip table, and link performance parameters are from the Emme/2 software [28]. To set up a fair comparison of different line search schemes, we use a working route set generated by Bekhor et al. [3]. In this working route set, there are totally 174,491 routes with an average of 40.1 routes for each O–D pair, and the maximum number of routes generated for any O–D pair is 50. Note that this working route set has been adopted to examine the length-based and congestion-based C-logit SUE models in Zhou et al. [46] and Xu et al. [45], the cross-nested logit (CNL) SUE model in Bekhor et al. [4], various logit-based SUE model in Chen et al. [15], and the nonadditive traffic equilibrium model in Chen et al. [17].

The convergence criterion used in the numerical comparison is based on the mean square error (MSE) of the generalized route costs and the minimum generalized O–D cost given in Eq. (29). Note that the MSE is a direct measure of the equilibrium conditions given by the KKT conditions [2]

$$
MSE = \frac{1}{N} \sum_{rs} \left( \sum_{k} \left( g_{rs}^{ct} - \pi_{rs} \right)^2 \right) \frac{1}{\theta} \left( \ln \left( f_{rs}^{ct} + \sum_{j} \left( f_{rs}^{ct} \frac{1}{p_{rs}^{ct}} \right) \right) \right)
$$

(29)

where $g_{rs}^{ct}$ is the generalized cost on route $k$ between O–D pair $rs$; $\pi_{rs}$ is the minimum O–D cost between O–D pair $rs$; and $N$ is the number of routes. The tolerance error of MSE is set at 1E–10; the dispersion parameter ($\theta$) is set at 0.1; the initial stepsize of MSA, SRA, and Armijo schemes are all set at 1.0; and the stopping error of bisection method is 1E–4. Other parameters are set as: $\lambda_1 = 1.9$, and $\lambda_2 = 0.1$ for the SRA scheme; $\beta = 0.6$, $s = 1.0$, and $\sigma = 0.5$ for the Armijo rule. The path-based partial linearization algorithm is coded in Intel Visual FORTRAN 9.1 and run on a 3.80 GHz processor and 2.00 GB of RAM.

4.1. Convergence characteristics

The convergence characteristics of the solution algorithm with various line search schemes are shown in Fig. 6 and Table 3. From Fig. 6, the SRA, Armijo, quadratic interpolation, and bisection schemes can promise convergence, while the MSA scheme has difficulty in converging to the desired accuracy level within the maximum amount of CPU time (600 s) allowed. The inexact line
search schemes (SRA, Armijo, and quadratic interpolation) use about half of the CPU time compared to the exact line search scheme (bisection) when reaching a MSE of $10^{-10}$. The computational efforts required by each line search scheme are shown in Table 3 in terms of the number of iterations, number of objective function evaluations, number of derivative evaluations, and CPU time. Note that from Fig. 6, we can observe that using the SRA, quadratic interpolation, and Armijo schemes have more computational benefits compared to the bisection and MSA schemes. Despite the low number of iterations required by the bisection scheme to reach convergence, majority of the computational efforts are spent on evaluating the complex derivatives of the objective function in Eq. (5). For further investigation, Fig. 7 plots the stepsize trajectories of these three inexact line search schemes.

Fig. 7(a) plots the stepsize trajectory in log scale for the MSA and SRA schemes. One can see that the stepsize sequences of both schemes are strictly decreasing. In this experiment, $\theta_2$ in the SRA scheme (see Eq. (24)) is active for all iterations. This means that the flow difference $||h(n) - f(n)||$ keeps decreasing during the iteration process and the iterative solutions are getting closer to the optimal solution. For the exact and inexact line search schemes shown in Fig. 7(b), we can observe that the step sizes are not strictly decreasing, but all three step sizes are within an acceptable range. The step sizes approximated by the quadratic interpolation scheme exhibit similar pattern as the exact stepsize determined by the bisection scheme, while the step sizes generated by the Armijo scheme in Eq. (28) remains constant after the first few iterations.

### 4.2. Sensitivity analysis

The above analysis indicates that the SRA, quadratic interpolation, and Armijo schemes outperform the bisection and MSA schemes in terms of computational efficiency. In the following analyses, we examine the robustness and sensitivity of the path-based partial linearization algorithm with these three promising line search schemes. Specifically, the effects of travel demand level and dispersion parameter on the computational performance are investigated in Fig. 8. First, we analyze the impact of different demand levels ($\pm 40\%$ of the base demand level) on the computational performance. From Fig. 8(a), we can observe that CPU time increases when the demand level increases. It seems that the SRA and quadratic interpolation schemes exhibit a linear trend, while the Armijo scheme exhibits a nonlinear trend for the same convergence tolerance (i.e., MSE $= 10^{-10}$ in Eq. (29)). Regardless of the demand level, the Armijo scheme requires more computational efforts compared to the SRA and quadratic interpolation schemes. This is because the Armijo scheme requires calculating both the objective function and its derivatives to determine an acceptable step size with a sufficient decrease in objective value (not necessarily the optimum) as indicated in Eq. (28) and Table 3. Fig. 8(b) examines the impact of dispersion parameter value ($\theta$) on the performance of three line search schemes. In this experiment, we vary $\theta$ from 0.01 to 10 (note that the base dispersion parameter value is 0.1 used in Section 4.1). When $\theta$ tends to be zero, the objective function in Eq. (5) is dominated by the two entropy terms ($Z_1$ and $Z_2$). Flow allocations tend to spread to more routes (i.e., stochasticity effect) with consideration of route overlapping (i.e., similarity effect) irrespective of the route costs (i.e., congestion effect). On the other hand, flow allocations tend to concentrate on the minimum cost routes when $\theta$ is large (i.e., the objective function in Eq. (5) is dominated by $Z_1$). For all three line search schemes, CPU time increases as $\theta$ increases.

### 5. Practical implications of the PCL SUE model

In this section, we examine the practical implications of implementing the PCL SUE model in the Winnipeg network. To this end, we investigate how the PCL SUE model handles the effects of
congestion, stochasticity, and similarity in comparison with the multinomial logit (MNL) SUE model and the deterministic user equilibrium (DUE) model. Figs. 9 and 10 provide the flow allocation comparison between DUE and PCL SUE models and MNL SUE and PCL SUE models, respectively. Each figure consists of three sets of results: link flow allocation (left panel), link flow difference between two models (middle panel), and route flow allocation (right panel).

Specifically, the link flow allocation comparison is given in terms of link volume to capacity (V/C) ratio distribution for each model. The link flow difference between two models is depicted in a GIS map to facilitate visualization. Links are coded in color and thickness to highlight the link flow differences (e.g., darker color in Fig. 10 indicates that MNL SUE link flows are higher than PCL SUE link flows in the city center). The route flow allocations are compared based on the percentage of O–D demand assigned to the used routes between the DUE and PCL SUE models in Fig. 9. In addition, Fig. 10 uses the mean absolute percentage error (MAPE) to measure route flow difference between the MNL SUE and PCL SUE models according to the average similarity index (ASI) of each route (i.e., $\text{ASI}_{rs} = 1 / |K_{rs}| \left( \sum_{k} |r_{rs}^{k} - r_{rs}^{PCL}| / r_{rs}^{PCL} \right)$, where $|K_{rs}|$ is the number of routes in O–D pair $rs$). Note that

$$\text{MAPE}_{m} = \frac{1}{|\text{ASI}_m|} \sum_{rs} \sum_{k \in \text{ASI}_m} \left( \frac{|r_{rs}^{k} - r_{rs}^{PCL}|}{r_{rs}^{PCL}} \right) \times 100\%,$$

where $|\text{ASI}_m|$ is the number of routes with an ASI in category $m$. $r_{rs}^{k}$ and $r_{rs}^{PCL}$ are the flows on route $k$ of O–D pair $rs$ assigned by the PCL and MNL SUE models, respectively.
In terms of the link V/C ratio distributions, there is a clear difference between the DUE and PCL SUE models (see left panel of Fig. 9), while the difference between the MNL SUE and PCL SUE models is not significant (see left panel of Fig. 10). The reason for the difference is that the DUE model only considers the congestion effect (i.e., Z₁ in Eq. (5)), while the two SUE models consider both congestion and stochasticity effects (i.e., Z₁ and Z₂ in Eq. (5)). In other words, the DUE model only assigns flows to the routes that minimize congestion according to Wardrop’s first principle: all used routes have route costs equal to the minimum O–D cost, while unused routes have route costs higher than or equal to the minimum O–D cost. As for the SUE models, all routes are used. However, lower cost routes are preferred. This result is illustrated in the right panel in Fig. 9 that there is a significant difference on the percentage of demand allocated to the used routes by the DUE and PCL SUE models. In the middle panel of Fig. 9, it shows the DUE model assigns significantly less flows to the links around the city center than the PCL SUE model does as indicated by the lighter color.

For this particular network, it seems that the assigned link V/C distributions in the left panel of Fig. 10 are not very different between MLN SUE and PCL SUE models. However, a closer look at the right panel reveals that the MAPE of route flows is indeed significantly different as the ASI increases (i.e., more overlapping between routes). This is attributed to the similarity effect (i.e., Z₂ in Eq. (5)) of the PCL SUE model. The middle panel further illustrates the link flow difference between the two SUE models, particularly the link flows in the central areas where roads are much denser with more overlapping between routes. The MNL SUE model assigns more flows to routes passing through the central areas as indicated by the links in darker color. The PCL SUE model allocates less flows on these routes compared to the MNL SUE model, because similarity among routes is explicitly considered in the PCL SUE model.

6. Concluding remarks

In this paper, we presented a path-based partial linearization algorithm for solving the paired combinatorial logit (PCL) stochastic user equilibrium (SUE) formulation with recent advances in line search strategies to improve the computational efforts. Specifically, the self-regulated averaging (SRA) and quadratic interpolation schemes were examined along with the three traditional line search methods (i.e., the predetermined method of successive averages (MSA) scheme, the exact bisection scheme, and the inexact Armijo scheme). Numerical results using a real-size Winnipeg network revealed the SRA and quadratic interpolation schemes were more efficient than the traditional schemes. The sensitivity analysis with respect to various demand levels and dispersion parameter values also supported the findings that the SRA and quadratic interpolation schemes are indeed more robust compared to the traditional line search methods. The computational efficiency and robustness of the SRA and quadratic interpolation schemes are attributed to their smart stepsize determination mechanism. They substantially reduce or even avoid evaluating the complex PCL SUE objective function and its derivatives.

In addition, numerical results were also conducted to examine how the PCL SUE model handles the effects of congestion, stochasticity, and similarity in comparison with the multinomial logit (MNL) SUE model and the deterministic user equilibrium (DUE) model. The numerical results indeed demonstrated that the PCL SUE model is capable of handling the route overlapping problem through the similarity index and the hierarchical logit structure through its extended objective function, which includes a user equilibrium term and two entropy terms. This alleviates the inability of the MNL SUE model by accounting for the correlation in the similarity terms.

Numerical results with the SRA and quadratic interpolation schemes were conducted using a real-size Winnipeg network with a number of scenarios. The results revealed the SRA and quadratic interpolation schemes were more efficient than the traditional schemes. The sensitivity analysis with respect to various demand levels and dispersion parameter values also supported the findings that the SRA and quadratic interpolation schemes are indeed more robust compared to the traditional line search methods. The computational efficiency and robustness of the SRA and quadratic interpolation schemes are attributed to their smart stepsize determination mechanism. They substantially reduce or even avoid evaluating the complex PCL SUE objective function and its derivatives.

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