Modeling urban housing market dynamics: can the socio-spatial segregation preserve some social diversity?

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Abstract

This paper is concerned with issues related to social diversity in urban environments. We introduce a model of real estate transactions between agents which are heterogeneous in their willingness to pay. A key feature of the model is the assumption that agents preferences for a location depend both on an intrinsic attractiveness of the location, and on the social characteristics of its neighborhood. Focusing on the case of a monocentric city, the stationary state is analytically characterized and gives the distribution of income over space. The model is studied through numerical simulations as well. The analytical and numerical analysis reveal that, even if socio-spatial segregation occurs, some social diversity is preserved at most locations. Comparing with empirical data on transaction prices in Paris, the results are shown to nicely fit some stylized facts.

Keywords: segregation model, housing market and prices formation, social diversity, agent-based model

1 Introduction

The place where people live and the way they are distributed across cities matter, from both a social and an economic point of view. This article deals with the dynamics of price formation in a urban housing market and seeks to explain how individuals with different willingness to pay are distributed over a city.

Housing price formation can entail segregation when people are heterogeneous. This involves the theory of hedonic prices and raises the question of whom people want to live with. Its dynamics indicates how prices diffuse through a heterogeneous space.

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Since the pioneering work of Schelling (1971), a lot of articles have been concerned with measuring the extent and impact of segregation in education, housing, and the labor market. The economic literature on this topic, largely focused on US cities, tends to measure segregation in the same way as Echenique & Fryer (2007), who relate the level of segregation to social interactions, or more recently Alesina & Zhuravskaya (2010 forthcoming), who seek to explain the development of segregation (see, for instance, Cutler et al. (1999)) or to evaluate the effect of segregation on the socioeconomic performance of minorities (Jenks & Meyer (1990) or Cutler et al. (2008)).

The segregation we consider here is specifically related to income distributions. This is done via the modeling of transactions on the real estate market, with a numerical and analytical analysis of the resulting dynamics of prices and of the equilibrium state. Following Von Thünen’s model and Alonso’s study (see Alonso (1964)) of land use, we start from the assumption that, in the context of a monocentric city, the center is the most convenient place to live. We then show how competition between people affects the spatial distribution of agents who are heterogeneous in their willingness to pay.

A wide literature has been produced on the formation of prices in the real estate and housing markets. Following the path opened by Rosen (1974), most studies have focused on explaining prices through hedonic estimations, showing how the price per square meter can be influenced both by variables intrinsic to the flat or house such as size or level of comfort and by extrinsic variables concerning the surrounding area and its amenities. Space plays an important role in price formation, as shown by Baltagi & Bresson (2010). The role of the quality and density of the neighborhood, the reputation of nearby schools and the level of security are particularly emphasized by Ioannides & Zabel (2003), Figlio & Lucas (2004), Bono et al. (2007) or Seo & Simons (2009). Following Tiebout (1956), these authors point out that the decision of where to live is based on families’ preferences for the quality of public services and amenities, particularly education. Prices on the real estate market vary with the quality of a bundle of public services provided by a jurisdiction, since better public service creates demand and willingness to pay, which are capitalized into housing prices. Brueckner et al. (1999) explain that the relative location of different income social groups depends on the spatial pattern of amenities in a city.

The question of how prices diffuse has mainly been studied in a regional context. According to Clapp et al. (1995), many reasons, especially informational factors, may cause or strengthen the position of leading economic regions. Empirical analyses have shown how housing prices diffuse from a main economic region to peripheral areas, as Meen (1996) studying the case of the UK market, Berg (2002) for Sweden or Oikarinen (2005) for Finland. More recently, Holly et al. (2010) show how prices diffuse from the dominant London area to surrounding regions, which in turn and with some delay, impact other regions. In a nutshell, those studies show that housing price movements diffuse from economic centers to the surrounding regions. The explanation for the leading role of main economic centers has usually been based on the assumption that business cycles affect economic centers first and more peripheral areas later. Within a metropolitan area, however, it is difficult to state clearly whether the center is likely to lead surrounding areas or vice versa.

Seeking to explain how prices are distributed over space calls for an assumption about people’s preferences. Do they prefer to live with people who are richer than they are, or poorer? The literature on well-being tends to argue that people feel better when those around them
are poorer (for a detailed survey see Luttmer (2005)). Clark & Oswald (2002) show empirically that unemployed people are less unhappy when they live with other unemployed people. Goyal & Ghiglino (2010) explore the role of shifting social interactions: they use examples to illustrate how poorer individuals lose while richer ones gain as we move from a society which is segregated (by economic class) towards an integrated society. In this article, we assume that individuals prefer to live with people who are richer than they are. This is more in line with the literature showing that when people decide where to live, one of their main criteria is the quality of their environment, as demonstrated in the literature on hedonic prices. We note however that the model allows for many generalizations, and in particular different hypothesis on social preferences could be studied within the same framework.

The model we propose is inspired by Short et al. (2008) and Berestycki & Nadal (2010), who model the evolution of the spatial distribution of crime in a city, attributing to each location an attractiveness for illegal activity (the more attractive a location the higher the probability of a burglary). In the present context of housing markets, the originality of our model is precisely that each agent attributes to each location a specific level of attractiveness. This attractiveness results from a combination of an intrinsic or objective part, and of a subjective part. To simplify the demonstrations, we consider an intrinsic attractiveness stemming directly from the location of the site: this is in line with Alonso and von Thünen (the closer to the center, the higher the level of intrinsic attractiveness). The endogenous (subjective) attractiveness results from the intrinsic individuals’ social preferences (the closer my fellows, the higher the level of subjective attractiveness). Then the main assumptions of the model are that, (i) people make decisions according to both their willingness to pay (WTP) and their individual evaluation of the level of attractiveness of the different locations; (ii) buyers, heterogeneous in their WTP, base their search for housing on the level of attractiveness of the location of the dwelling; (iii) agents are both buyers and sellers; (iv) the intrinsic attractiveness depends on the distance from the geographical center (maximum at the center and decreasing with the distance). This last assumption describes a monocentric city. Nevertheless, other cases, in particular polycentric cities, can be easily modeled by considering other forms of the intrinsic attractiveness.

We give an analytic demonstration of the main theoretical results, then extend our understanding of the dynamics through the simulation of our agent-based model. We then empirically verify the pertinence of our results, through the analysis of a database of real-estate transactions in Paris. The analytical resolution of the model underlines the existence of a critical endogeneous income threshold. Above this threshold, people can freely choose their location and the price of their housing does not depend on their capacity to pay but on the attractiveness of the location (which depends on both an intrinsic and a subjective value). For people below the income threshold, their housing possibilities depend directly on their willingness to pay and on a critical distance. The simulation reveals that, for a large distribution of income, the space is composed of segregated areas but also of social diverse areas: this result is robust to parameter variations. Finally, the simulated results are compared with empirical data on transaction prices in Paris. We show that they nicely fit some stylized facts concerning the dependency of the mean and of the standard deviation of the transaction prices in the distance to the center.

The paper is organized as follows. Section 2 presents the model. Section 3 deals with the theoretical analysis. Section 4 presents the numerical simulations. Section 5 compares the
simulated data with the empirical data. The conclusion follows.

2 Theoretical Model: the assumptions

We propose here a model of residential location where people take their decision according both to their revenues and their individual evaluation of the level of attractiveness of the different locations. Buyers, heterogeneous by their willingness to pay are looking for flats, according to the level of attractiveness of the place where the flat is located. The assumptions of the model are presented below.

A0: Space

A finite number of goods (apartments for sale) are located on a discrete set $\Omega$ of locations $X$ uniformly distributed in a bounded open set $\tilde{\Omega}$ in $\mathbb{R}^2$. The total number of locations is $\text{Card}(\Omega) = L^2$, and the space is of linear size (diameter) $aL$, where $a$ gives the typical distance between two neighboring locations. There is a total number $N$ of apartments with identical intrinsic characteristics at each location $X$ in $\Omega$.

We consider Cartesian coordinates on the space $\tilde{\Omega}$, the origin being considered at the geographical center of this set. The distance $D(X)$ to the center of a location $X$ of coordinates $(x, y)$ is thus

$$D(X) = \sqrt{x^2 + y^2},$$

with $D(X) \leq aL/2$.

A1: Agents

Time is discrete and indexed by $t$. The time increment is $\delta t$. The horizon is infinite. At each period (given time $t$), there is a finite number of agents in the economy, who can be in one of the three following states: (1) buyer, (2) seller, (3) housed. We assume an infinite “reservoir” of agents outside the city. From this reservoir, at each time $t$ a constant number $\gamma L^2$ of agents arrive on the market. They are new buyers who add to the buyers who did not succeed in the previous period. Housed agents become sellers at a homogeneous rate $\alpha$. When they succeed in selling their good, they live the market (going back to the external reservoir). Note that the total number of agents of each type, buyers, sellers and housed, are dynamical variables since they depend on the success rate of the previous period, and on the inflow and outflow rates. In what follows, the term ”insiders” designs people who are already housed, while ”outsiders” designs people looking for a flat.

A2: Demand prices

Agents are characterized by their willingness to pay (or sell) (hereafter WTP or WTS), which determines the maximum price the agent is ready to pay for an asset. For simplicity, we consider a finite number $K$ of WTP. Agents with the same willingness are designed by $k$-agents, $k \in \{0, ..., K - 1\}$, and have WTP $P_k$, ordered by increasing values, $P_0 < P_1 < ... < P_{K-1}$. When the agent is acting as a buyer, his demand price is $P^d_k = P_k$.

We assume that these $K$ values are uniformly distributed among the agents in the external reservoir – hence among the $\gamma L^2$ new agents at any time $t$, a fraction $1/K$ has WTP equal to $P_k$, $k \in \{0, ..., K - 1\}$. This is not a restrictive hypothesis, as shown in the Appendix, section A.4.
A3: Attractiveness

The attractiveness of each location depends both on intrinsic objective characteristics (areas amenities) and on subjective characteristics which depend on the social neighborhood. At each location $X$ corresponds an attractiveness which has two components:

1. an intrinsic attractiveness, $A^0(X)$, idiosyncratic to the position considered, that may be, for example, linked to the presence of amenities. The more central is $X$, the higher $A^0(X)$ is. It is assumed to be independent of time in a first approach. If we look at a neighborhood on a sufficiently short timescale, its intrinsic attractiveness does not change. As discussed in the Introduction, in the present work we restrict the analysis to an intrinsic attractiveness which decreases from the center, modeling here the preferences of the agents for the center. For illustrative purpose, whenever a specific expression of $A^0(X)$ will be needed, and in all the numerical simulations, we will express it as a two-dimensional Gaussian function:

$$A^0(X) = A_{\text{max}}^0 \exp \left(-\frac{D(X)^2}{R^2}\right)$$

(2)

where $A_{\text{max}}^0$ is the maximal intrinsic attractiveness, $D(X)$ is the Euclidean distance, and $R$ determines at which distance from the center the intrinsic attractiveness is still significant. Generalizations are discussed in the Appendix.

2. a subjective attractiveness, whose value depends on the WTP of the agent that looks at the location $X$. It is all the higher from the point of a view of a $k$-agent as agents of same and higher WTP are buying a good on $X$.

The total attractiveness at time $t$ on a location $X$ seen by a $k$-agent is updated at each step of the dynamics according to:

$$A_k(X,t+\delta t) = A_k(X,t) + \omega \delta t (A^0(X) - A_k(X,t)) + \epsilon \delta t v_{k\geq}(X,t)$$

(3)

with

$$v_{k\geq}(X,t) = \sum_{k'\geq k} v_{k'}(X,t)$$

(4)

where $v_k(X,t)$ is the density of $k$-buyers on location $X$ which realize a transaction at time $t$. In other terms, we can say that the level of attractiveness depends on the intensity of the demand: the higher the demand, the higher the level of attractiveness. Henceforth, what we mean by density is a number over the number of space locations, $L^2$. The above equation (3) means that, (i) when there is no transaction at a given location for a certain amount of time, the attractiveness relaxes towards its intrinsic value $A^0(X)$; and, (ii) agents prefer to have neighbors with at least equivalent WTP. Note that other hypothesis on agents’preferences can be modeled by the appropriate modification of this attractiveness dynamics.

A4: Offer prices

Each seller is characterized by an offer price which depends on his willingness to sell and on the level of the mean attractiveness of the location. Let us recall here that, at each location, the attractiveness directly depends on the intensity of the demand. A $k$-agent, when acting as a seller, has his willingness to sell determined by his WTP $P_k$. The resulting offer price is given by

$$P^o_k(X) = P^0 + (1 - \exp(-\xi \bar{A}(X,t))) P_k$$

(5)
where \( P^0 \) is the minimum price of an offer, 

\( \xi \) is a parameter, and \( \bar{A}(X, t) \) is the average attractiveness of location \( X \) at time \( t \) defined by:

\[
\bar{A}(X, t) \equiv \frac{1}{K} \sum_{k=0}^{K-1} A_k(X, t).
\] (6)

For a good at location \( X \), that one calls a \( X \)-good, which has not yet been exchanged, the offer price is given by

\[
P^o(X) = P^0 + (1 - \exp(-\xi \bar{A}(X, t))) \, P^1
\] (7)

where \( P^1 \) is the maximum increment of price with respect to \( P^0 \).

For \( \xi \) small enough, the above offer prices become, respectively,

\[
P^o_k(X) = P^0 + \xi \bar{A}(X, t) \, P_k
\] (8)

\[
P^o_k(X) = P^0 + \xi \bar{A}(X, t) \, P^1
\] (9)

**A5: The matching**

At each time step (between times \( t \) and \( t + \delta t \)):

1. The density \( \rho_k \) of \( k \)-buyers is the sum of a constant proportion \( \frac{\gamma}{K} \) of new buyers, and of the density of buyers who have not yet succeed. Each one of these \( \rho_k L^2 \) agents has a probability \( \pi_k(X) \) to visit a given location \( X \), that depends on the attractiveness of the location:

\[
\pi_k(X, t) = \frac{1 - \exp(-\lambda A_k(X, t))}{\sum_{X' \in \Omega} 1 - \exp(-\lambda A_k(X', t))}
\] (10)

For a small value of \( \lambda \) (that is for \( \lambda \max_k, X A_k(X, t) << 1 \)), this is equivalent to:

\[
\pi_k(X, t) = \frac{A_k(X, t)}{\sum_{X' \in \Omega} A_k(X', t)}
\] (11)

2. At location \( X \), a transaction between a \( k \)-buyer and a \( k' \)-seller with offer price \( P^o_k(X) = P^0 + (1 - \exp(-\xi \bar{A}(X, t))) \, P^1 \), can be realized if \( P_k > P^o_k(X) \). The transaction price is assumed to be a linear combination of offer and demand prices:

\[
P_{tr} = (1 - \beta) P^o_k(X) + \beta P^d_k
\] (12)

where \( \beta \) is a constant coefficient.

Similarly, for a good exchanged for the first time, the transaction price is

\[
P_{tr} = (1 - \beta) P^o(X) + \beta P_k
\] (13)

**3 Theoretical model: The equilibrium**

**3.1 Continuous time dynamics**

The evolution of the system is further formalized through partial differential equations, taking the continuous time limit (limit \( \delta t \to 0 \)).
The total density of insiders with the willingness to pay $P_k$, $u_k(X,t)$ satisfies the following differential equation:

$$(1 - \alpha) \partial_t u_k(X,t) = v_k(X,t) - \alpha u_k(X,t) \tag{14}$$

At each period, there is, on the market, a proportion $\alpha$ of people leaving the insider side. Some of them, purely leave the market (they move from the city to an other for example) while some others become outsiders (they sell their dwelling to buy an other one).

The updating rule of the attractiveness of a location $X$ seen by a $k$-agent gives in the continuous limit:

$$\partial_t A_k(X,t) = \omega (A^0(X) - A_k(X,t)) + \epsilon v_k(X,t) \tag{15}$$

The density of outsiders $\rho_k(X,t)$ is written as

$$\rho_k(X,t) = v_k(X,t) + \bar{v}_k(X,t), \tag{16}$$

with $\bar{v}_k(X,t)$ pointing out the density of agents who do not succeed in buying a $X$-good at time $t$. Given the rules of the dynamics, the evolution of the outsiders density $\rho_k(X,t)$ can be written as

$$\partial_t \rho_k(X,t) = -\rho_k(X,t) + \gamma K \pi_k(X,t) + \sum_{X' \in \Omega} \bar{v}_k(X',t) \tag{17}$$

In the above equation, $\gamma K \pi_k(X,t)$ is the density of new buyers that visit $X$ at time $t$. The last term in the right hand side of (17) is the fraction of the agents, among those who have not succeeded in buying a good at time $t$, who decided to search for a flat at location $X$.

### 3.2 Equilibrium

#### 3.2.1 Stationary state: the general case

We define the equilibrium as the stationary state (whenever it exists) of the dynamics, i.e. the variables of the model $\rho_k(X,t)$, $A_k(X,t)$, $u_k(X,t)$, $v_k(X,t)$, become constant. All the variables in the stationary state will be written with a star $^*$.

Writing $\partial_t u_k = 0$, $\partial_t A_k = 0$ and $\partial_t \rho_k = 0$, in, respectively, equations (14), (15) and (17), one gets for the stationary state:

$$v_k^*(X) = \alpha u_k^*(X), \tag{18}$$

$$A_k^*(X) = A^0(X) + \frac{\epsilon}{\omega} v_k^*(X), \tag{19}$$

$$\rho_k^*(X) = \pi_k^*(X) \sum_{X' \in \Omega} \rho_k^*(X'). \tag{20}$$

with $\pi_k^*(X) = \frac{1 - \exp(-\lambda A_k^*(X))}{\sum_{X' \in \Omega} 1 - \exp(-\lambda A_k^*(X'))}$ (or, in the small $\lambda$ limit, $\pi_k^*(X) = \frac{A_k^*(X)}{\sum_{X' \in \Omega} A_k^*(X')}$).

Summing equation (17) on the whole space, one concludes that:

$$\sum_{X \in \Omega} v_k^*(X) = \alpha \sum_{X \in \Omega} u_k^*(X) = \frac{\gamma K}{\pi} \tag{21}$$

Consequently, in the stationary state, the total density of $k$-housed agents on the whole space is equal to $\frac{\gamma K}{\pi}$. The total density of $k$-transactions during a step is $\frac{\gamma K}{\pi}$. Thus, the total number of housed agents is the same for all the $k$ values.

In the following, we restrict the analysis (existence and characterization) of equilibria in a
non-saturated regime, as defined below.

**Definition: non-saturated equilibrium.** A non-saturated equilibrium is defined as an equilibrium where, for any given \( k \in \{0, \ldots, K-1\} \), at any location \( X \in \Omega \), either the WTP of the \( k \)-agents are too low, so that none of them can afford to buy a good at this location, hence \( \bar{v}_k^*(X) = 0 \), or the \( k \)-agents can afford to buy a good at this location, and in such case any \( k \)-demand is satisfied, that is

\[
\bar{v}_k^*(X) = 0. \tag{22}
\]

The uniqueness of such a non-saturated equilibrium depends on some of the initial conditions. Because there is no rationing on the market we consider (the inward flux is larger than the outward flux and than the total number \( N \) of goods at a given location), the only possible regime is everybody housed or nobody housed. As we will see section 4, the above definition defines an equilibrium which is a fairly good approximation of what is observed in the numerical simulations. Things would be different on a rationed market (saturated case) as it will be explained below.

The distribution of the housed agents will obviously depend on their willingness to pay: agents with the highest WTP will succeed to buy a good in the locations wanted however it will not systematically be the case for the agents with the lowest WTP. We will show below that, in the stationary state, there exists a WTP threshold \( P_c^* \) such that agents with a WTP larger than \( P_c^* \) can buy a good anywhere, whereas agents with a lower WTP cannot afford to buy a good at location closer to the center than some critical distance which depends on the agent’s WTP. We will denote by \( k \) the marginal \( k \) for which the WTP is equal to \( P_c^* \) (hence \( P_k = P_c^* \)). This WTP threshold results from the dynamics, depending on the magnitude of the subjective contribution of the attractiveness, depending itself on the inhabitants on a location). We will see that in the large \( K \) limit, \( P_c^* \) and \( k/K \) have a well defined limit.

To summarize, the agents are divided in two categories, those whose the WTP allows to buy the goods located in the place they want and those whose the WTP is not high enough to buy the more attractive good located in the center. The computation of the critical distance will be detailed in the section 3.2.4. We first analyze the case of \( k \geq \tilde{k} \).

### 3.2.2 Above the threshold

Let us first assume \( P_c^* \leq P_{K-1} \), hence \( \tilde{k} \leq K-1 \). We consider the \( k \)-agents with \( k \geq \tilde{k} \): they have high enough revenues to insure a flat wherever they want to buy. We work under the hypothesis of a non saturated market, hence:

\[
\forall k \geq \tilde{k}, \forall X \in \Omega, \bar{v}_k^*(X) = 0. \tag{23}
\]

We consider here the small \( \lambda \) limit: in Assumption A5 we take (11) for the probability to choose a location, that is \( \pi_k^*(X) = \frac{A_k^0(X)}{\sum_{X' \in \Omega} A_k^0(X')} \).

**Proposition 1:** In a non-saturated equilibrium, the density of housed \( k \)-agents with \( k \geq \tilde{k} \) does not depend on the level of individual WTP but only on the intrinsic attractiveness of the location, according to:

\[
\forall k \geq \tilde{k}, \forall X \in \Omega, u_k^*(X) = \frac{\gamma}{K \alpha} \frac{A_k^0(X)}{Z^0} \tag{24}
\]

with

\[
Z^0 \equiv \sum_{X \in \Omega} A_0(X). \tag{25}
\]
Proof (by recurrence on $k$): At the equilibrium, $u^*_k(X) = \frac{v^*_k(X)}{\alpha} = \frac{\rho^*_k(X)}{\alpha}$ which gives

$$u^*_k(X) = \frac{\gamma}{K\alpha} \frac{A^*_k(X)}{Z_k}$$

(26)

with

$$Z_k \equiv \sum_{X \in \Omega} A^*_k(X)$$

(27)

From the fixed point equations, one gets (and this is true for any $k \in \{0, ..., K - 1\}$),

$$Z_k = Z^0 + \frac{\epsilon}{\omega} \sum_{X \in \Omega} \sum_{k' \geq k} v^*_k(X) = Z^0 + \frac{\epsilon\gamma}{\omega K} (K - k)$$

(28)

In particular, for the housed $(K - 1)$-agents (highest WTP):

$$u^*_{K-1}(X) = \frac{\gamma}{K\alpha} \frac{A^0(X) + \frac{\epsilon\alpha}{\omega} u^*_k(X)}{Z^0 + \frac{\epsilon\gamma}{\omega K} (K - k)}$$

(29)

Solving this equation leads to:

$$u^*_{K-1}(X) = \frac{\gamma}{K\alpha} \frac{A^0(X)}{Z^0}$$

(30)

Now assume that (24) is true for $k + 1 \leq k' \leq K - 1$ with $k \geq \bar{k}$. Then one has

$$u^*_k(X) = \frac{\gamma}{K\alpha} \frac{A^0(X) + \frac{\epsilon\alpha}{\omega} u^*_k(X) + (K - k - 1) \frac{\epsilon\gamma}{\omega K} \frac{A^0(X)}{Z^0}}{Z^0 + \frac{\epsilon\gamma}{\omega K} (K - k)}$$

(31)

which gives $u^*_k(X) = \frac{\gamma}{K\alpha} \frac{A^0(X)}{Z^0}$. Hence the proposition is proved by recurrence for all $k \geq \bar{k}$.

3.2.3 Below the threshold

In this part, the budget constraint matters. On the real estate market, agents with high willingness to pay and agents with low willingness to pay are in competition. The question now is to determine where people can localize. The presence of agents with high willingness to pay creates an area of high attractiveness and high prices. A minimum condition for transactions is that demand prices of the $k$-agents are higher than the offer prices of the $k$-agents. From equation 5 one gets the condition:

$$P_k \geq P^0 + (1 - \exp(-\xi \bar{A}^*(X))) P_k$$

(32)

This enables us to look for the minimal distance from the center at which a $k$-agent can buy a good. At this distance, the $k$-agents have thus by definition the lowest WTP among the agents present. Let us assume that all the WTP superior or equal to $P_k$ have reached the density expected in the stationary state. We can then analytically deduce a critical distance of settlement $d^*_c(k)$ above which the $k$-agents can settle if the system is not saturated. As one would expect, we find that the lower the WTP, the higher the distance.

We recall that we consider the small $\lambda$ limit, i.e., in the stationary regime, $\pi^*_k(X) = A^*_k(X)/Z_k$. 
**Proposition 2:** In a non-saturated equilibrium, for each level \( k \) of individual WTP there is a critical distance \( d^*_c(k) \) above which the \( k \)-agents can afford to buy the goods present on the corresponding locations. By definition of \( \bar{k} \), \( d^*_c(k) = 0 \) for \( k \geq \bar{k} \). For \( k \leq \bar{k} \), the density of housed \( k \)-agents depends on the level of individual revenues and on the intrinsic attractiveness of the location according to:

\[
\begin{align*}
  u^*_k(X) &= \frac{\gamma}{K \alpha} \frac{A^0(X)}{\sum_{X' \in \Omega} A^0(X')} \text{ if } D(X) \geq d^*_c(k) \\
  & = 0 \text{ otherwise}
\end{align*}
\]

(33)

where

\[
\Omega_k \equiv \{ X \in \Omega | D(X) \geq d^*_c(k) \}
\]

(35)

**Proof:** From (20), the density of potential buyers on a location \( X \) (outsiders) is given by:

\[
\rho^*_k(X) = \frac{A^*_k(X)}{\sum_{X' \in \Omega} A^*_k(X')} \sum_{X' \in \Omega} \rho^*_k(X')
\]

(36)

For \( k < \bar{k} \), \( v^*_k(X) = 0 \) for \( D(X) < d^*_c(k) \), and \( \bar{v}^*_k(X) = 0 \) for \( D(X) \geq d^*_c(k) \). Hence, since \( \rho^*_k(X) = v^*_k(X) + \bar{v}^*_k(X) \), one has

\[
\sum_{X \in \Omega_k} \rho^*_k(X) = \sum_{X \in \Omega_k} v^*_k(X) = \sum_{X \in \Omega_k} v^*_k(X)
\]

(37)

As we have seen, in the stationary state the total number of \( k \)-agents on the lattice is \( \frac{\gamma}{\alpha K} \), and \( v^*_k(X) = \alpha u^*_k(X) \), so that finally

\[
\sum_{X \in \Omega_k} \rho^*_k(X) = \frac{\gamma}{K}
\]

(38)

Then, summing Eq. (36) over \( X \) in \( \Omega_k \), one gets:

\[
\rho^*_k(X) = \frac{\gamma}{K} \frac{A^*_k(X)}{\sum_{X' \in \Omega_k} A^*_k(X')}
\]

(39)

Let us now consider the highest WTP for which the critical distance \( d_c(k) \) is not zero \( (k = \bar{k} - 1) \). If \( D(X) \) is greater than \( d_c(k) \), then

\[
\begin{align*}
  \rho^*_k(X) &= v^*_k(X) = \alpha u^*_k(X) = \frac{\gamma}{K} \frac{A^*_k(X)}{\sum_{X' \in \Omega_k} A^*_k(X')}
\end{align*}
\]

(40)

\[
\begin{align*}
  u^*_k(X) &= \frac{\gamma}{K \alpha} \frac{A^0(X)}{\sum_{X' \in \Omega_k} A^0(X')} [A^0(X) + \sum_{X' \in \Omega_k} A^0(X')] + \frac{\gamma}{K} \frac{\sum_{X' \in \Omega_k} A^0(X')}{\sum_{X' \in \Omega_k} A^0(X')} + \frac{\gamma}{K}
\end{align*}
\]

(41)

The equation [41] leads to:

\[
\begin{align*}
  u^*_k(X) &= \frac{\gamma}{K \alpha} \frac{A^0(X)}{\sum_{X' \in \Omega_k} A^0(X')}
\end{align*}
\]

(42)

By recurrence, one can generalize to any \( k < \bar{k} \):

\[
\begin{align*}
  v^*_k(X) &= \alpha u^*_k(X) = \frac{\gamma}{K} \frac{A^0(X)}{\sum_{X' \in \Omega_k} A^0(X')} \text{ if } D(X) \geq d^*_c(k)
\end{align*}
\]

(43)

\[
\begin{align*}
  & = 0 \text{ otherwise}
\end{align*}
\]

(44)
3.2.4 WTP Threshold

In this section, to simplify the analysis we assume that the WTP are uniformly distributed between two extreme values. More exactly, the WTP are characterized by reservation prices

\[ P_k = P_0 + \Delta \frac{k}{K-1}, \quad k = 0, ..., K-1, \quad (45) \]

where \( P_0 \) is the minimum reservation price and \( \frac{\Delta}{K-1} \) is the increment between two consecutive prices. The same gap is kept between each WTP. Clearly other distributions of reserve prices could be considered: this particular choice has the advantage to allow for an analytical characterization, as presented below.

As shown in the Appendix, section A.1, one finds that the critical distance \( d^*_c(k) \) to the center from which the \( k \)-agents satisfies:

\[ d^*_c(k) \geq R \sqrt{-\ln \left[ \frac{1}{\xi A^0_{\text{max}}} \ln \frac{f_k}{P_0} \right] \left[ 1 + \frac{\epsilon \gamma}{2 \xi Z^0} \frac{K+1}{K-\frac{(k+1)K}{K^2}} \right]} \quad (46) \]

where, as previously defined, \( Z^0 = \sum_{X \in \Omega} A^0(X) \). The above equation is valid whenever the argument of the logarithm is smaller or equal to 1, and otherwise \( d^*_c(k) = 0 \). Note that \( A^0_{\text{max}} \), the intrinsic attractiveness at the center, is the maximal intrinsic attractiveness on the whole space. The above formulae is specific to the choice of the dependency in the distance to the center of the intrinsic attractiveness \( A^0 \) (generalizations are discussed in the Appendix, section A.4).

The WTP threshold, \( P^*_c \), is given by the smallest \( k \) value (which we have denoted \( \tilde{k} \)) for which \( d^*_c(k) = 0 \). By taking the infinite limit for \( K \), one obtains \( P^*_c \) as solution of

\[ P^*_c = \Lambda \exp \eta \left( 1 - \left( \frac{P^*_c - P_0}{\Delta} \right)^2 \right) \quad (47) \]

with

\[ \eta \equiv \frac{\epsilon \gamma}{2 \xi Z^0} \xi A^0_{\text{max}} \quad (48) \]

and

\[ \Lambda \equiv P^0 e^{\xi A^0_{\text{max}}} \quad (49) \]

If \( P_0 < \Lambda \exp \eta \), and \( \Lambda < P_0 + \Delta \), there exists a WTP threshold \( P^*_c \) in \( [P_0, P_0 + \Delta] \). If \( P_0 > \Lambda \exp \eta \), no agent can be housed a good in the center.

As one can check on Eq. (47), the threshold \( P^*_c \) is an increasing function of \( \eta \): this is reasonable, since it means that \( P^*_c \) increases if the incoming flux increases, or if the social factor \( \epsilon \) increases; whereas \( P^*_c \) decreases if the relaxation of the attractiveness towards the intrinsic value \( A^0 \) is faster (larger \( \omega \)), or if the domain of strongly attractive locations increases (larger \( Z^0 \)). The numerical determination of \( P^*_c \) from equation 47 allows to plot the curve of \( P^*_c \) as a function of \( \eta \), as illustrated on Figure 1.

3.3 Price transaction distribution

In the same way that there exists a critical distance to the center \( d^*_c(k) \) from which the \( k \)-agents are housed, there also exists a minimum WTP for an agent to be present on each location \( X \). This minimum WTP will be indexed by \( k_0(X) \) and the associated WTP will be
noted $P_c(X)$. The mean transaction price on a location $X$ will obviously depends on this minimal WTP. To deduce the expression of $k_0$, one uses the same type of argument than for the calculation of $d^*_c(k)$ (see the appendix).

To obtain the average of the transaction prices on a location $X$, we use the fact that the distribution of the transaction matches the distribution of the housed agents.

Using this, we can show that the averaged transaction prices on a site $X$, denoted $\langle P_{tr}(X) \rangle$, may be written as follows (see the Appendix, section A.3, for details):

$$\langle P_{tr}(X) \rangle = (1 - \beta)P^0 + [1 - (1 - \beta)\frac{P^0}{P_0 + \Delta\frac{k_0}{K-1}}][P_0 + \frac{\Delta}{2}(1 + \frac{k_0}{K-1})]$$

(50)

Let us now summarize the main results of this section and think about the consequences in terms of space organization (social mix or segregation). Considering the stationary state, we have first examine the case where the outsiders have a willingness to pay higher than a certain critical value $P_c$. We have demonstrate that, in this situation, the distribution of income in the space (measured by the density of housed $k$-agents with $k \geq \bar{k}$), does not depend anymore on the level of individual revenues but only on the intrinsic attractiveness of the location. In other terms, above a certain threshold income, there exists a certain social mix, with very rich people neighbors of less rich others. Concerning the agents poorer than the critical threshold, their opportunity of being housed will depend on a critical distance $d^*_c(k)$. This critical distance is clearly related to their level of WTP.

4 Numerical simulations

In what follows, we simulate the model presented above. Numerical simulations have been performed on a square lattice of linear size $L = 100$, hence with $L \times L = 10000$ sites. Each site $X$ is characterized by its Cartesian coordinates $(x, y)$ in the frame whose origin is the center of the square lattice. In this section, we propose to study a particular case of the model, for a first approach of the real estate market with this model, we consider a monocentric model. Let us note that the model is generic enough to test several hypotheses concerning the attractiveness.

To model the preferences of the agents for the center, the intrinsic attractiveness $A^0(X)$ is chosen to decrease with the distance to the center.
We recall that we choose to express it as a two-dimensional Gaussian function:

\[ A^0(X) = A^0_{\text{max}} \exp\left(-\frac{x^2 + y^2}{R^2}\right) \]  

(51)

where \( A^0_{\text{max}} \) is the maximal intrinsic attractiveness and \( R \) determines at which distance from the center the intrinsic attractiveness is still significant. Since at \( t = 0 \) the simulated city is empty, the initial attractiveness of a location \( X \) seen by each \( k \)-agent is equal to the intrinsic attractiveness.

The total number of offers \( N \) is chosen identical on each of one of the 10000 sites.

The dynamics used for the numerical simulations is schematized on Figure 2.

![Figure 2: Model dynamics.](image)

Numerical simulations have been performed with the following values of the parameters:
\[ \lambda = 10^{-2}, \; \xi = 0.1, \; A^0_{\text{max}} = 1, \; \epsilon = 0.022L^2, \; R = 10, \; \Delta = 225000, \; \omega = \frac{1}{15}, \; P^1 = 200000, \; \frac{\gamma}{K} = \frac{1000}{L^2}, \; \beta = 0.1, \; N = 200. \]

The initial number of offers is voluntarily high enough to prevent the system from saturating, i.e. when the number of available assets becomes 0 on some sites.

4.1 Socio-spatial segregation and income mix

The dynamics schematized in the figure 2 is applied with the conditions pre-cited. Two methods of selection of the offers to be sold in the first place at a given location \( X \) are proposed: either (a) the offers are sold by growing offer price, or (b) they are randomly chosen. In both cases, the system reaches a stationary state with a constant number of housed agents present on the lattice. There are three types of agents in the game. Agents already housed (constant number), newcomers looking for a flat and agents who have not found a flat in the previous period. We focus on the state and space distribution of housed people. The stationary state is particularly analyzed rather than the evolution to this state.
4.1.1 Simulation results

**Population distribution** At the earliest steps of the dynamics (no matter the choice of the offers to be sold, (a) or (b)), the exponential chosen to represent the central attractiveness is conspicuous in the curve giving the number of agents according to the distance to the center (as shown in Figure 3).

According to the figure 3 (illustrating the case (a) for the choice of the offers to be sold), the $k$-agents distribute themselves in the same proportions in each location (the shape of the intrinsic attractiveness is conspicuous in the curves) apart from the lowest WTP agents. Then, the dynamics leads to a non-uniform distribution of the $k$-agents on each location. The agents are housed in a way that is different from the one directly linked to the exponential function describing the initial attractiveness. The figures 4 show the occupancy ratio for the different WTP in the stationary state with respect to the distance to the center.

![Figure 3: Occupancy ratio for the different WTP with respect to the distance to the center at the early steps of the dynamics.](image)

![Figure 4: Occupancy ratio per WTP versus the distance to the center in the stationary state. $K = 10$ revenue are present on the lattice. The distance to the center is in arbitrary unit. The offers to be sold are ordered by growing prices (case (a), left panel), and randomly chosen (case (b), right panel).](image)
In the stationary state, people are located both at the interior of the zone of significance of the intrinsic attractiveness and far at the exterior. In other terms, the agents are housed in an area with a radius larger than the radius of significance of the intrinsic attractiveness ($R$). The k-agents are housed at some critical distance of the centre: in case (a), the occupancy ratio of the k-agents for some given $k$ increases rapidly - but continuously - from 0 to its highest value, whereas in case (b), the occupancy ratio of the k-agents jumps discontinuously at a critical distance from the center.

Let us focus on the case (a). According to Fig.4, the simulated city displays three distinguishable zones around the center:

1. The central zone (group of points at a distance to the center close to 0) where the main inhabitants are the agents from superior revenue, i.e. with high reservation prices.
2. The peripheral zone where the main inhabitants are the agents from inferior WTP, i.e. with low reserve prices.
3. The intermediary zone (distance to the center close to 20) where the agents from all WTP are present in comparable proportions, what one calls social mix in what follows.

The presence of this last zone of social mix was not conspicuous, since it could have been expected that agents would be segregated by WTP on the whole lattice.

**Social mix index** In order to characterize more quantitatively the social mix, we introduce a measure of the segregation derived from the dissimilarity index proposed by Duncan and Duncan (1955):

$$ID(X) = \sum_{k=0}^{K-1} |\nu_k(X) - \frac{1}{K}|.$$  \hspace{1cm} (52)

where $\nu_k(X)$ is the relative density of $k$-agents:

$$\nu_k(X) \equiv \frac{u_k(X)}{\sum_{k=0}^{K-1} u_k(X)}.$$ \hspace{1cm} (53)

This social mix index $ID$ is defined for each location. This consists of the difference between the uniform distribution and the effective distributions (obtained by simulation) of the $k$-agents. The greater this index is, the greater the segregation between agents.

An alternative choice is the mixing entropy

$$H(X) = - \sum_{k=0}^{K-1} \nu_k(X) \log \nu_k(X).$$ \hspace{1cm} (54)

One computes these quantities for the stationary residential distributions in the case (a) (ordered offers) for two different rates of newcomers $\gamma_1 = \gamma$ and $\gamma_2 = 3\gamma$ with $\gamma_1 < \gamma_2$ with respect to the center (cf. Fig.5, left and Fig.5, right).

A global analysis of the behavior of the curves allows to highlight three spatial zones: a central zone (distance to the center near 0) and a peripheral zone with a relative high index (low entropy), and an intermediate zone with a weak index (high entropy). It confirms the presence of an area of social mix. When the rate of newcomers increases, for example $\gamma_1$ switched to $\gamma_2$, the index (resp. entropy) is on the whole higher (lower) meaning a general decline of the social mix.
Figure 5: On the left: Dissimilarity index for two rates of newcomers $\gamma_1 = \gamma$ and $\gamma_2 = 3\gamma$. On the right: Entropy for two rates of newcomers $\gamma_1 = \gamma$ and $\gamma_2 = 3\gamma$.

It means that if we keep constant the initial number of offers and increase the number of newcomers, the domain of social mix tends to vanish. The level of segregation by WTP is increased: on some sites, agents with some given WTP are not present in the same proportions, that is some revenues are over represented.

**Price distribution** Once a stationary regime is reached, the transaction prices and the associated variances are computed and averaged. Theses averages are represented.

Figure 6: Averaged transaction prices with respect to the distance to the center: the dashed curve represents the case where the assets bought are randomly chosen and the solid curve represents the case where the assets are bought by growing price order.

Prices are higher in the center and decrease with the distance to the center: this result was already expected, directly induced by the fundamental hypotheses of the model. Something more surprising is to observe that prices present a larger variance in the central area than at the periphery. We can thus summarize the distribution of transaction prices in two parts: in the central part, a large distribution of the prices show an active dynamics, whereas, in the near peripheral area, a trend to homogenization is observed in the prices. This effect will be more visible in the plot of the variance of the prices. In the previous simulations, when the agents prospect on a site, they buy at first the less expensive offers. As we previously
precised, instead of considering increasing price orders, a variant consists in choosing randomly
the offers at a given site. For this case, the plot of the averaged transaction prices at the
stationary state is displayed on Fig.6 with a dashed light curve, showing very little differences
with the preceding case. This shows that if we are only interested in the transaction price
distribution, we can indifferently study either cases, (a) or (b).

4.1.2 Comparison between analytical and simulation results

In what follows, simulated and theoretical spatial distributions of the agents per WTP are
compared. The theoretical results have been obtained by considering a non-saturated regime
in which there only exists two states of occupation on a site: either all the potential buyers
from a given WTP manage to be housed, or no agent from a given WTP can be housed. This is what is observed in the simulations for the case (b). A slightly different stationary
regime is observed in the case (a), where the less expensive offers are sold first: only part of
the potential buyers with a given WTP at a location $X$ may be housed. That is why, for
what concerns the income spatial distribution, comparison between theoretical and numerical
results we only be done for the case (b).

Above the threshold Figure 7 represents the occupancy number obtained with $K = 10$
WTP for $k = 9$ (the higher WTP) with the simulations and with the analytical formula
(Eq.(33)). According to these results, the density obtained analytically (Eq.(33)) turns out
to be a good approximation for the occupation ratio of the agents that can settle at the center
(i.e. $d^*_c(k) = 0$).

![Figure 7: Occupancy number with respect to the distance to the center for the revenue $k = 9$
($K = 10$): the solid curve is from the simulations whereas the dotted one is from the analysis.]

Critical distance The plot of the critical distance with respect to the index of the WTP,
$k$, on Fig.8 shows a good agreement between the simulations and the analytical formula for
the low values of $k$.

Below the threshold An example of occupancy number with respect to the distance
to the center for $k$-agents non housed in the center ($d^*_c(k) > 0$) is illustrated on Fig.9. The
solid curve corresponds to the occupancy number obtained from the simulations whereas the
Figure 8: Evolution of the critical distance with respect to $k$ ($K = 50$): the points are generated by the simulations, the solid curve is the analytical bound (the right hand side of (46)) obtained in the limit of small $\lambda$ (in the simulations, $\lambda = 0.01$).

Figure 9: Occupancy number with respect to the distance to the center for the revenue $k = 2$ ($K = 10$): the solid curve is from the simulations whereas the dotted one is from the analysis, Eq.33, with $d^*_c(k)$ replaced by its lower bound (the right hand side of (46)).
dotted curve is the plot of the analytical expression (Eq. 33). A good concordance between simulation and analysis is observed.

**Transaction price** As shown in Fig. 10, the analysis performed, despite using approximations, gives a coherent estimate of the averaged transaction prices generated by the simulations.

![Figure 10: Averaged transaction prices with respect to the distance to the center (K = 50): the solid curve comes from the simulations whereas the dashed one comes from the analysis.](image)

To conclude, in the case where the offers are randomly selected (not sold by growing order of prices), the numerical results are in good agreement with the non-saturated equilibrium defined in the analytical part: at some distance specific to the agent’s WTP, one has a sharp transition from no housing \( (v(x) = 0) \) to complete housing (all potential buyers become housed, \( \bar{v}(x) = 0 \)).

In the case where the less expensive assets are sold first, there is a domain of locations where both \( v(x) \) and \( \bar{v}(x) \) are non-null. The analytical study of the spatial distribution of the agents is thus more complicated, and will not be discussed here. We recall, however, that for what concern the transaction prices distribution, the two schemes give the same results.

## 5 Empirical evidence

The main information source on the real estate prices in Paris is the B.I.E.N. database, organized by the “Chambre des Notaires de Paris” which registers real estate transactions for Paris and Île-de-France. This database concerns all the categories of real estate (flats, private houses, parking lots, fields, commercial or industrial buildings, and whole buildings). For each transaction, more than one hundred anonymous different information are extracted from legal documents associated with each real estate transaction. This information includes: the location of the good (city, quarter, arrondissement in Paris, street and number and geocode), its intrinsic characteristics (purpose, size, age of construction, number of stairs in case of a building and the amenities), the intrinsic characteristics of the transaction (type of moving, state of occupation, new or old building, price, credit, fiscal information, the origins of the property), the socio-economic profile of the seller and the buyer (status, socio-professional categories, geographic origins, age). According to the Chambre des Notaires, the B.I.E.N. base
presents a high level of reliability. In 2000, 90% of the Parisian transactions were registered. Data are available on the period 1990-2003. We focus on the exploitation of certain variables, more pertinent for our study. These variables concern the good exchanged and the traders. Concerning the good, we only use the location and transaction prices.

5.1 A particular Parisian dynamic

The empirical analysis of the dynamic of price formation in Paris suggests that in this city, prices are moving according to a double dynamic, which could fit our modeling of attractiveness (extrinsic and intrinsic). The evolution of average prices per arrondissement exhibits a particular spatiotemporal dynamic. In all the arrondissement, prices are going down from 1990 until 1995 then up. but on the increasing part of the dynamic, the ranking of the arrondissement (from the highest price to the lowest one) has changed, as shown in Fig. 11. We consider a sub-sample of the whole database, limiting our analysis to medium size dwellings (surface area between 20 and 40 square meters)

Figure 11: Average prices per square meter and arrondissement (in thousands of euros) on the period 1990-2003.

Looking at the map of the prices for the year 2003 on figure 5.1 a first evidence is that prices are higher in the center but the ”Rive Gauche” effect is important. At a similar distance, prices are higher on the left bank than on the right one.

Figure 12: Transaction prices of the 20 to 40 squared meters’ goods in Paris in 2003 (Left) with respect to the location, (Right) with respect to the distance to the geographical center.
5.2 A comparison between empirical and simulated data

We now focus on the year 2003, in the aim to compare simulated and empirical static trend. During this period, 9735 Parisian transactions were registered. We take into account 4299 transactions corresponding to transactions concerning goods between $20m^2$ and $40m^2$. The Fig.13 respectively show the mean transaction prices from the data base of the notaries and from the simulations. On the data one sees that, as a first approximation, the prices are indeed decreasing from the center. In addition, there is a local strong increase of the prices at the periphery, corresponding to two “hot-spots” of high prices, the 12 and 16 Paris arrondissements. In this study, focusing on the main feature which is the decrease of prices form the center, we do not take into account these parts for the comparison between model and data. In the Conclusion we will discuss the extensions of the model needed for a finer modeling of the specific case of Paris data. On Figure 13 one thus sees that the shape of the decrease in prices is qualitatively reproduced by the model, with a plateau in the low price domain.

The standard deviation of the transaction prices are plotted on Fig.14. Note that, in the data, the standard deviation increases linearly with the mean price - but is not proportional to it: the relationship is affine. A similar behavior is obtained with the model, with however two regimes. One at low prices, and one at price values larger than 80000 euros, the later being in good correspondence with the data: the slope in both cases is of order 0.1.

The global trends observed in the data are in essence reproduced by the model. The choice of a Gaussian intrinsic attractiveness to model the significance of the geographic center appears as satisfying for a first approach for this particular case of Paris.

6 Conclusion

This paper has studied a particular housing market model of dynamic prices formation and agents localization, which actually introduces a new framework for studying such markets and the resulting socio-spatial segregation. The model allows to specify both the intrinsic attractiveness of a location, resulting, e.g., from the amenities, and a subjective part, depending on the social preferences of the agents - and thus evolving in time together with the social
Figure 14: Standard deviation of the averaged transaction prices with the distance to the center (left: data, right: model).

Figure 15: Standard deviation with respect to the transaction prices in thousands of euros (left: data, right: model). The straight lines are linear regressions: (left) slope 0.098, intercept $-8.054$, correlation coeff. $-0.99$; (right, low prices) slope 0.9, intercept $-62.000$ correlation coeff. $-1.0$; (right, large prices) slope 0.1, intercept $-2.000$, correlation coeff. $-0.97$. 
characteristics of the neighborhoods.

For the particular choices made in the present study - a monocentric city and a preference for living with richer people –, the analysis yields three insights. First, as expected, a socio-spatial segregation appears, with richer people living near the center and poorer people at the periphery. Second, whatever the parameters of the simulation, it exists an area of social mix, between the two segregated zones. The fact that we consider agents heterogeneous in their willingness to pay allow us to go beyond the simple Schelling case with binary status. Third, the variance at the center, where the higher prices are localized is more important than at the periphery which suggests that the center presents a certain social diversity, with people paying very different prices. Everybody is rich at the center but certain are richer than others. Looking at the empirical distribution of prices in Paris for the year 2003, we observe that the simple Alonso model (prices higher at the center) is, as a first order approximation, a good description of Paris structure: higher prices are indeed concentrated around Notre Dame cathedral.

However, a finer characterization of the housing market in Paris gives that this monocentric structure is only correct in average – e.g. there are also higher prices on the left bank side than on the right bank one –, and provided one leaves aside some small domains of very high prices far from the center. A better characterization and modeling of the Paris market (as well as of any other specific urban housing market) will require a more precise choice of the intrinsic attractiveness, together with the analysis of the conditions on the dynamics of the subjective attractiveness for the emergence of “hot spots” - local domains of high prices unrelated with the level of the intrinsic attractiveness.

Future works will explore alternative hypothesis concerning the social preferences, such as agents preferring to live in more popular area, as well as concerning the structure of the intrinsic attractiveness, such as shapes corresponding to polycentric cities. Extensions of the model will also be studied. Notably, in the line of Short et al. (2008) and Berestycki & Nadal (2010), it is natural to add a diffusion term in the attractiveness dynamics: a high attractiveness of a location is expected to have some positive influence on the attractiveness of nearby locations. It is such extension which is likely to lead to dynamical instabilities, with the emergence of hot spots of high prices.

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References


A Appendix

A.1 Critical distance

In this section we derive the expression [46], which is a lower bound on the critical distance in the case of a non-saturated equilibrium with $\lambda$ small.

An necessary condition for a $k$-agent to be housed in a location $X$ in the stationary state is that:

$$\xi \bar{A}^*(X) \leq \ln \frac{P_k}{P_0}$$  \quad (A-1)

Let thus determine the minimal distance to the center from which a $k$-agent can be housed. At this distance, the $k$-agents have the lowest WTP among the agents present. For the sake of simplicity, let us assume that all the $k'$-agents with a WTP greater than $P_k$ or equal to $P_k$ have reached the density foreseen in the stationary state for the agents whose the WTP is above the threshold (cf. equation (24)). By rewriting the inequality (A-1) replacing the occupancy ratio by the lower bound $\bar{\chi}_{\omega K}^2$, one gets:

$$A^0(X) + \frac{\bar{\chi}_{\omega K}^2}{\sum_{X' \in \Omega} A^0(X')} \left[ \frac{(K+1)K} {2} - \frac{(k+1)k} {2} \right] \frac{A^0(X)} {\sum_{X' \in \Omega} A^0(X')} \leq \frac{1}{\xi} \ln \frac{P_k}{P_0}$$  \quad (A-2)

$$A^0(X) \leq \frac{\frac{1}{\xi} \ln \frac{P_k}{P_0}} {1 + \frac{\bar{\chi}_{\omega K}^2}{\sum_{X' \in \Omega} A^0(X')}}$$  \quad (A-3)
A lower bound for the critical distance is thus:

\[ d^*_c(k) = [A_0]^{-1} \left( \frac{1}{\xi} \ln \frac{P_0}{P_0^2} \right) \]

(A-4)

In particular, with \( A^0(X) = A^0_{\max} \exp(-\frac{D(X)^2}{R^2}) \), the critical distance satisfies:

\[ d^*_c(k) \geq R \sqrt{\ln \left[ \frac{1}{\xi A^0_{\max}} \ln \frac{P_0}{P_0^2} \right]} \]

(A-5)

The distance \( d^*_c(k) \) determined here is the distance to the center from which the \( k \)-agents and the agents with a WTP above \( P_c \) could be housed with a density given by the equation (24).

It turns out that the lower bound (A-4) is a very good approximation of the critical distance of housing observed in the simulations (cf. section 5 on the comparison between simulations and theoretical results).

### A.2 Minimum WTP

Let us determine the \( k \)-agents with the lowest WTP present on a location \( X \). If the number of WTP tends to infinity (i.e., \( K \) goes to infinity), for a given location, one can find an integer \( k_0 \) such that, in the large \( K \) limit the condition (A-1) becomes an equality, together with a well defined limit for \( z_0 \equiv k_0/K \).

In the regime where \( \lambda A^*(X) \) is small compared to 1, the condition (A-1) written as an equality for the integer \( k_0 \):

\[ P^0 \xi A^0(X) = k_0 \frac{\Delta}{K-1} + P_0 - P^0 \]  

(A-6)

For the sake of simplicity, one makes the hypothesis that the agents are present with the same proportions and a density given by the equation (30). That leads to:

\[ P^0 \xi A^0(X) \left[ 1 + \frac{\epsilon \gamma}{\omega K^2} \sum_{X' \in \Omega} A^0(X') \right] \left( \frac{(K+1)K}{2} - \frac{(k_0+1)k_0}{2} \right) = k_0 \frac{\Delta}{K-1} + P_0 - P^0 \]  

(A-7)

(A-8)

The equation (A-8) gives a second order equation for \( k_0 \):

\[ \frac{k_0^2}{K^2} \eta + \frac{k_0}{K} \left( \frac{\Delta}{P_0^0} \frac{K}{K-1} \frac{A^0_{\max}}{A^0(X)} + \frac{\eta}{K} \right) - \left[ \xi A^0_{\max} + \eta \frac{K+1}{K} + \frac{P^0 - P_0}{P_0} \frac{A^0_{\max}}{A^0(X)} \right] = 0 \]  

(A-9)

where, we recall,

\[ \eta = \frac{\epsilon \gamma \xi A^0_{\max}}{2 \omega \sum_{X' \in \Omega} A^0(X')} \]  

(A-10)

Finally, the \( k_0 \) agents with the lowest WTP are such as:

\[ \frac{k_0}{K} = \frac{1}{2} \left[ -\frac{1}{K} - \frac{\Delta K}{P_0^0(K-1) A^0(X)} \right] + \left( \frac{1}{K} + \frac{\Delta K}{\eta P_0^0(K-1) A^0(X)} \right)^2 + 4 \left[ \frac{\xi A^0_{\max}}{\eta} + \frac{K+1}{K} + \frac{(P_0 - P_0^0) A^0_{\max}}{\eta P_0^0} \right] \]  

(A-11)
In the infinite $K$ limit, this gives

$$z_0 = -\frac{1}{2} \frac{\Delta}{\eta P_0} A_{0\max}^0 + \frac{1}{2} \left[ \frac{\Delta}{\eta P_0} A_{0\max}^0 \right]^2 + 4 \left[ \frac{\xi A_{0\max}^0}{\eta} + 1 + \frac{(P_0 - P_0) A_{0\max}^0}{\eta P_0(X)} \right]$$

(A-12)

### A.3 Transaction prices

To obtain the average of the transaction prices at a location $X$, one needs to consider the distributions of the agents by WTP on each of these locations. Indeed, the $k$-sellers follow the same distribution than the $k$-housed agents. Let first express the average of the prices of the offers hold by $k$-agents with $k \geq k_0$.

$$P_{ok}(X) - P_0 = \sum_{k_0}^{K-1} P_{rk}(X)(1 - \exp(-\xi \bar{A}^*(X)))$$

(A-13)

$$= [P_0 + \frac{1}{K - k_0} (\frac{(K - 1)K}{2} - \frac{(k_0 - 1)k_0}{2})] \frac{\Delta}{K - 1}(1 - \exp(-\xi \bar{A}^*(X)))$$

$$= [P_0 + \frac{\Delta}{2} (1 + \frac{k_0}{K - 1})][1 - \exp(-\xi \bar{A}^*(X))]$$

$$= [P_0 + \frac{\Delta}{2} (1 + \frac{k_0}{K - 1})][P_0 - P_0 k_0 \frac{\Delta}{K - 1}]$$

$$= [P_0 + \frac{\Delta}{2} (1 + \frac{k_0}{K - 1})][1 - \frac{P_0}{P_0 + k_0 \frac{\Delta}{K - 1}}]$$

The average of the demand prices on all the WTP present on a site $X$ is:

$$\bar{P}_{dk}(X) = P_0 + \frac{\Delta}{2} (1 + \frac{k_0}{K - 1})$$

(A-14)

Finally, the transaction price averaged on the WTP of the sellers and of the buyers is written in the following way:

$$\bar{P}_{tr}(X) = (1 - \beta)P_0 + [(1 - \beta)(1 - \frac{P_0}{P_0 + k_0 \frac{\Delta}{K - 1}}) + \beta][P_0 + \frac{\Delta}{2} (1 + \frac{k_0}{K - 1})]$$

(A-15)

$$= (1 - \beta)P_0 + [-(1 - \beta)\frac{P_0}{P_0 + k_0 \frac{\Delta}{K - 1}} + 1][P_0 + \frac{\Delta}{2} (1 + \frac{k_0}{K - 1})]$$

(A-16)

### A.4 WTP distribution

In the model we have assumed that the $K$ values of WTP are uniformly distributed among the agents in the external reservoir. One may want to start with a particular probability distribution function (pdf) of the reservation price. Let $Q(P)$ be this pdf, with support on $[P_0, P_0 + \Delta]$. One can construct the discrete set of $K$ levels from $Q(P)$ by asking for having the same density of agents in any one of these levels. The $\{P_k, k = 0, ..., K - 1\}$ are thus defined by

$$\int_{P_0}^{P_k} Q(p)dp = \frac{k}{K}$$

(A-17)
for $k = 1, ..., K$ (with $P_K = P_0 + \Delta$). Of particular interest is the large $K$ limit. In such case, one has $P_k = P(k/K)$, the function $P(z)$ on $z \in [0, 1]$ being given as solution of

$$\int_{P_0}^{P(z)} Q(p)dp = z,$$

or, equivalently by

$$Q(P(z)) P'(z) = 1$$

(A-18)

where $P'(z) = dP/dz$, with $P(0) = P_0$ and $P(1) = P_0 + \Delta$.

Formally, all the analysis done in this paper for the uniform case, (45), can be done for an arbitrary distribution. In particular, the critical WTP threshold is here given by $P^*_c = P(z^*_c)$, $z^*_c$ being obtained from solving the equation

$$P(z^*_c) = \Lambda \exp \eta \left(1 - (z^*_c)^2\right).$$

(A-19)