Singular perturbations of weakly coupled systems of Hamilton–Jacobi equations

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1. Introduction

In [5,10–12] the Ventcel–Freidlin theory for small random perturbations of dynamical systems has been extended to the class of perturbed random evolution processes. These processes are right continuous strong Markov processes \((X^\varepsilon_t, \nu^\varepsilon_t)\) with phase space \(\mathbb{R}^N \times \{1, \ldots, M\}\) whose first component satisfies

\[
\begin{align*}
dX^\varepsilon_t &= b_{\nu^\varepsilon_t}(X^\varepsilon_t)dt + \sqrt{\varepsilon}dW_t \\
\end{align*}
\]

and \(X^\varepsilon_0 = x \in D\), while the second component \(\nu^\varepsilon_t\) is a random process with state space \(\{1, \ldots, M\}\) and transition probabilities

\[
\begin{align*}
\mathbb{P}\{\nu^\varepsilon_{t+\Delta} = j | \nu^\varepsilon_t = i, X^\varepsilon_t = x\} &= d_{ij}(x)\Delta + O(\Delta) \\
\end{align*}
\]

for \(\Delta \to 0, i, j = 1, \ldots, M, i \neq j\). The process \((X^\varepsilon_t, \nu^\varepsilon_t)\) can be seen as a small random perturbation of the random evolution process \((X^0_t, \nu^0_t)\) defined by

\[
\begin{align*}
\frac{dX^0_t}{dt} &= b_{\nu^0_t}(X^0_t) \\
\end{align*}
\]

with \(\nu^0_t\) satisfying (1.2) for \(\varepsilon = 0\).
From a PDE point of view, large deviations estimates for functionals defined on the paths of the process (1.1) can be studied by means of singular perturbation problems for weakly coupled systems of elliptic equations \([10,22]\).

An interesting application of the theory of viscosity solutions is to give simple PDE proofs of large deviations results which require elaborate probabilistic arguments (see, for example, \([13,18]\), see also \([4]\) and \([2]\) for an introduction to the theory of viscosity solutions). A restriction to the applicability of this theory, particularly relevant in the case of Ventcel–Freidlin theory, is the non uniqueness of the viscosity solution to the equation which is obtained in the limit. In \([6,7]\) this difficulty has been (partially) overcome via the application of the so-called weak KAM theory for critical Hamilton–Jacobi equation (see for an introduction \([14,15]\)).

Aim of this paper is to extend the weak KAM approach to the case of perturbed random evolution processes. In particular we will consider the analogous for (1.1) of the singular perturbation problem studied in \([9,21]\):

\[
\begin{cases}
-\varepsilon \Delta u_\varepsilon + \frac{|D u_\varepsilon|^2}{2} - b(x) \cdot D u_\varepsilon - \varepsilon c(x) = 0, & x \in D, \\
u_\varepsilon(x) = 0, & x \in \partial D.
\end{cases}
\tag{1.4}
\]

The solution of (1.4) is related, via the logarithmic transformation \(v_\varepsilon := -\varepsilon \log u_\varepsilon\), to the functional

\[
u_i(x) = \mathbb{E}_x \left( \exp \left\{ -\int_0^{\tau_\varepsilon(x)} c(X_s^\varepsilon) \, ds \right\} \right),
\tag{1.6}
\]

and \(\tau_\varepsilon\) is the first exit time of \(X_\varepsilon^\tau\) from \(D\) (we refer to \([16]\) for the probabilistic interpretation of the functional \(u_\varepsilon\)). Note that \(u_\varepsilon\) solves the linear problem

\[
\begin{cases}
-\varepsilon \Delta u_\varepsilon - b(x) \cdot D u_\varepsilon + c(x) u_\varepsilon = 0, & x \in D, \\
u_\varepsilon(x) = 1, & x \in \partial D.
\end{cases}
\tag{1.5}
\]

In \([6,9,21]\), it is proved that the sequence \(v_\varepsilon\) converges uniformly to the maximal viscosity solution of the limit problem

\[
\begin{cases}
\frac{|D v|^2}{2} - b(x) \cdot D v = 0, & x \in D, \\
v(x) = 0, & x \in \partial D
\end{cases}
\tag{1.5}
\]

if the set where \(c\) is positive contains all the possible \(\omega\)-limits – if there are any – of the dynamical system \(\dot{x}(t) = b(x(t))\) (i.e., all the points \(y \in \overline{D}\) such that \(y = \lim_{t_n \to +\infty} \phi(t_n)\), where \(t_n\) is a diverging subsequence and \(\phi\) is an integral curve of \(b\) with \(\phi(0) \in D\).

It is worth noting that, since (1.5) does not admit in general a unique viscosity solution, the previous result cannot be obtained by means of standard stability results in viscosity solution theory.

Here we study a similar problem for the Markov process \((X_t^\varepsilon, \nu_t^\varepsilon)\) and the corresponding weakly coupled system. On the paths of the process \((X_t^\varepsilon, \nu_t^\varepsilon)\) in (1.1) we consider the functionals

\[
\begin{equation}
\begin{aligned}
u_i^j(x) &= \mathbb{E}_{x,i} \left( \exp \left\{ -\int_0^{\tau_\varepsilon(x)} c_i(X_s^\varepsilon) \, ds \right\} \right),
\end{aligned}
\end{equation}
\tag{1.6}
\]
where $\tau^\varepsilon(x)$ denotes the first exit time from $D$ of the process $X^\varepsilon_t$ which solves (1.1) with initial condition $X^\varepsilon_0 = x, \nu^\varepsilon_0 = i$.

It is possible to show (we refer to [10,17] and references therein) that $u^\varepsilon = (u^1_\varepsilon, \ldots, u^M_\varepsilon)$ solves the weakly coupled system of elliptic equations

$$
\begin{cases}
-\varepsilon^2 \Delta u^i_\varepsilon - b_i(x) \cdot Du^i_\varepsilon - \sum_{j=1}^M d_{ij}(x)(u^j_\varepsilon - u^i_\varepsilon) + c_i(x)u^i_\varepsilon = 0, & x \in D, \\
u^i_\varepsilon(x) = 1, & x \in \partial D,
\end{cases}
$$

where the coupling coefficients $d_{ij}$ are the coefficients appearing in the transition probabilities of the process $\nu^\varepsilon$, see (1.2). By the general theory of large deviations process, it is expected that the functionals (1.6) converge exponentially fast to 0. To determine the corresponding rate of convergence we take the logarithmic transforms $v^\varepsilon_i = -\varepsilon \log u^i_\varepsilon$, $i = 1, \ldots, M$, and we obtain the nonlinear weakly-coupled system

$$
\begin{cases}
-\varepsilon^2 \Delta v^i_\varepsilon + H_i(x, Dv^i_\varepsilon) + \varepsilon \sum_{j=1}^M d_{ij}(x)(e^{(v^j_\varepsilon - v^i_\varepsilon)/\varepsilon} - 1) + \varepsilon c_i(x) = 0, & x \in D, \\
v^i_\varepsilon(x) = 0, & x \in \partial D,
\end{cases}
$$

(1.7)

Our aim is to characterize the limit of $v^i_\varepsilon$ for $\varepsilon \to 0$. To study (1.7), we consider a more general framework and we assume that $H_i(x, p)$ in (1.7) are continuous, convex and coercive Hamiltonians. By the rescaling of the coefficients in $\varepsilon$ in (1.7), it follows that all the $v^i_\varepsilon$ converge to a same function $v$ which solves the Hamilton–Jacobi equation

$$
\begin{cases}
H(x, Dv) = 0, & x \in D, \\
v = 0, & x \in \partial D,
\end{cases}
$$

(1.8)

where $H$ is given by

$$
H(x, p) = \max_{i=1,\ldots,M} H_i(x, p).
$$

The Hamiltonian $H$ is convex and coercive and to study the solutions of (1.8) we follow the weak KAM approach, based on the introduction of an intrinsic length for curves of $\overline{D}$ associated to the Hamiltonian $H$. We define a functional nonsymmetric distance $S : \overline{D} \times \overline{D} \to [0, \infty)$

$$
S(x, y) = \sup \{ u(y) - u(x) \mid u \text{ is a viscosity subsolution to } H(x, Dv) \leq 0 \}
$$

(1.9)

(an equivalent metric definition of $S$ is given in (2.10)). We associate to the function $S$ a closed set $\mathcal{A} \subseteq \overline{D}$ (see Definition 2.3), called the (projected) Aubry set, which is a set where the distance $S$ degenerates. Its crucial property from a PDE point of view, see Proposition 2.4, is that there are subsolutions to $H(x, Du) = 0$ which are strict outside it. Due to this fact the Aubry set plays a crucial role in the analysis of the multiplicity of the solutions to the Dirichlet problem (1.8). Indeed we can identify a
unique solution of (1.8) by prescribing the value not only on the boundary \( \partial D \) but also on the Aubry set (see Proposition 2.6).

We obtain the following characterization of the behavior of the sequence \( v^\varepsilon_i \) of the solutions to (1.7) which is in the spirit of the results in [6,9,21]:

if the Aubry set is contained in the set where all the coefficients \( c_i \) are positive then all the components \( v^\varepsilon_i \) of the solution to the elliptic system (1.7) converge uniformly to the maximal solution of the limit problem (1.8).

The PDE proof of the previous singular perturbation result works as follow. The exponential terms and the particular scaling of the coefficients in \( \varepsilon \) in (1.7) implies that all the components of the solution of the system must converge to a same function \( G \) for \( \varepsilon \to 0 \). By stability results in viscosity solution theory, \( G \) turns out to be a solution of (1.8). The key point is to identify which solution of (1.8) we get with this limit procedure, since in general this equation does not admit a unique solution. In particular, we are interested in the case in which the limit procedure selects the maximal solution to the Dirichlet problem, which is in most cases the relevant one. To do this we use the positivity of the coefficients \( c_i \) on the Aubry set. In fact if \( G \) admits a subtangent \( \psi \) at some point of the Aubry, by a perturbation argument we obtain, for \( \varepsilon \) small, a subtangent \( \psi^\varepsilon \) of some component of the system (1.7) and, using the fact that the \( c_i \)'s are positive in this set, we get to a contradiction. Hence the solution \( G \) cannot have subtangents on the Aubry set and therefore it has to be the maximal solution of (1.8).

Note that to check the assumptions of our main theorem on the asymptotic behavior of \( v^\varepsilon_i \), it is necessary to identify the Aubry set associated to the Hamiltonian (1.8) of the limit problem. In general this can be difficult, therefore we provide in the last two sections some examples in which the intrinsic distance \( S \) and the Aubry set \( A \) can be described more explicitly.

In particular we go back to our model problem, which is the perturbed random evolution process (1.1). The key observation to link the weak KAM theory to the results in [5,10,11] is that, for the large deviation Hamiltonians \( H_i(x,p) = |p|^2/2 - b_i(x) \cdot p \), the distance function \( S \) associated to the Hamilton–Jacobi equation (1.8) coincides with the Eizenberg–Freidlin Quasi-Potential [11] for the random evolution process. Actually for general convex, coercive Hamiltonians the intrinsic distance \( S \) associated to \( H \) replaces the role of the Quasi-Potential. We are interested in giving conditions on the vector fields \( b_i \) which ensure the hypothesis of the previous singular perturbations theorem and therefore the validity of a large deviation result for (1.1). We describe these conditions in Section 4, pointing out also differences and analogies with previous results obtained by Eizenberg and Freidlin in [10–12] and by Bezuidenhout in [5].

The paper is organized as follows. In Section 2 we give a representation formula for the solution of the limit problem (1.8), based on weak KAM approach. In Section 3 we prove the singular perturbation result. Section 4 presents the application of our result to small random perturbations of random evolution equations. Finally, in Section 5 we describe another class of Hamiltonians satisfying our assumptions.

### 2. Representation of the solutions of the limit problem

In this section we study the first-order Hamilton–Jacobi equation which arises in the limit of the singular perturbation problem (1.7), i.e.

\[
\begin{align*}
    H(x, Du) &= 0, \quad x \in D, \\
    u(x) &= g(x), \quad x \in \partial D,
\end{align*}
\]  

(2.1)
where the Hamiltonian $H$ is defined as follows

$$H(x, p) := \max_{i=1, \ldots, M} H_i(x, p).$$

(2.2)

We make that the following basic assumptions:

1. $D \subseteq \mathbb{R}^N$ is an open bounded set with Lipschitz boundary,
   
   $\tag{2.3}$

2. $c_i : D \to \mathbb{R}$ continuous and $c_i \geq 0,$ for any $i = 1, \ldots, M,$
   
   $\tag{2.4}$

3. $d_{i,j} : D \to \mathbb{R}$ continuous and $d_{i,j}(x) > 0, i \neq j, d_{i,i} \equiv 0,$
   
   $\tag{2.5}$

4. $g : \partial D \to \mathbb{R}$ Lipschitz continuous,
   
   $\tag{2.6}$

5. $H_i(x, p)$ continuous, Lipschitz in $x,$ convex and coercive in $p$ and
   
   $\tag{2.7}$

Lemma 2.1. Let $H(x, p)$ be defined as in (2.2).

(i) $H$ satisfies assumption (2.7).

(ii) Set $Z_i(x) = \{p \mid H_i(x, p) \leq 0\}$ and $Z(x) := \{p \mid H(x, p) \leq 0\},$ then

$Z(x) = \bigcap_{i=1}^{M} Z_i(x).$  

(2.8)

(iii) Let $\sigma(x, q) = \sup_{p \in Z(x)} p \cdot q$ be the support function of $Z(x).$ Then $\sigma$ is continuous in $(x, q)$ and nonnegative, convex and positive homogenous in $q$ for any $x \in \overline{D}.$ Moreover,

$\sigma(x, q) \leq \inf_i \sigma_i(x, q),$  

(2.9)

where $\sigma_i(x, q) = \sup_{p \in Z_i(x)} p \cdot q$ are the support functions of $Z_i(x).$

Proof. The proof of (i) and (ii) is immediate. The properties of $\sigma$ follows immediately by the fact it is the support function of the convex, compact, continuous (with respect to the Hausdorff distance) map $Z(\cdot).$ Inequality (2.9) can be easily checked using definition. \hfill $\Box$

We define

$S(x, y) := \inf \left\{ \int_{0}^{1} \sigma(\xi(t), \dot{\xi}(t)) \, dt : \phi \in AC_{x, y, \overline{D}}[0, 1] \right\},$  

(2.10)

where $AC_{x, y, \overline{D}}[0, T] := \{\xi \text{ absolutely continuous s.t. } \xi(0) = x, \xi(T) = y, \xi(t) \in \overline{D} \forall t\}.$

In the next proposition we collect some important properties of the function $S.$

Proposition 2.2. (i) $S(x, y) \geq 0$ for any $x, y \in \overline{D},$ $S(x, x) = 0$ and

$S(x, y) \leq S(x, z) + S(z, y)$ for any $x, y, z \in \overline{D}.$  

(2.11)
Moreover, there exists a constant $M$ such that $S(x, y) \leq Md(x, y)$ for any $x, y \in \overline{D}$ where $d$ is the Euclidean geodesic distance in $\overline{D}$.

(ii) For any $x \in D$, $S(x, \cdot)$ is a viscosity subsolution in $D$ and a viscosity supersolution in $D \setminus \{x\}$ of the equation (2.1).

(iii) $v$ is a viscosity subsolution of (2.1) if and only if $-S(x, y) \leq v(x) - v(y) \leq S(y, x)$ for any $x, y \in D$.

For the proof see [7], Proposition 4.3, and [15], Propositions 4.1 and 4.2.

We now introduce a set where the distance $S$ degenerates. In fact the following definition says that a point belongs to the Aubry set if there is a sequence of cycles through the point with positive Euclidean length and vanishing intrinsic one.

**Definition 2.3.** A point $x$ belongs to the Aubry set $A$ if there exists a sequence $\{\phi_n\} \subset AC_{x,x,\overline{D}}[0,1]$ with $\ell(\phi_n) \geq \delta > 0$ ($\ell$ is the Euclidean length of the curve) s.t.

$$\inf_n \left\{ \int_0^1 \sigma(\phi_n(s), \dot{\phi}_n(s)) \, ds \right\} = 0.$$

An important property of the Aubry set is the existence of a subsolution which is strict out of $A$ (see [15], Proposition 5.3).

**Proposition 2.4.** There exists a $C^1$-subsolution to $H(x,Dv) = 0$ in $D$ which is strict in any open subset $U$ of $D$ with closure disjoint from $A$, i.e.

$$H(x,Dv) \leq -\delta_U \quad \text{in } U$$

in viscosity sense, for some $\delta_U > 0$.

**Remark 2.5.** The previous proposition characterizes the points in the Aubry set, in the sense that $x_0 \notin A$ if and only if there exists a $C^1$-subsolution to $H(x,Dv) = 0$ in $D$ which is strict in a neighborhood of $x_0$ ([15], Proposition 5.8).

The property described in Proposition 2.4 in particular implies that the Aubry set is an uniqueness set for problem (2.1), i.e. an interior hidden boundary on which a datum has to be fixed in order to get a unique solution of the Dirichlet problem (2.1).

The following theorem, see [6], Theorem 2.6, gives a complete characterization of the solutions to (2.1).

**Theorem 2.6.** If $g: \partial D \cup A \to \mathbb{R}$ is a continuous function which satisfies the compatibility condition

$$-S(y,x) \leq g(x) - g(y) \leq S(y,x) \quad \text{for any } x, y \in \partial D \cup A$$

(2.12)

then the unique viscosity solution of (2.1) such that

$$u(x) = g(x) \quad \text{for any } x \in \partial D \cup A$$


is given by
\[ u(x) = \min_{\partial D \cup A} \{ g(y) + S(y, x) \}. \] (2.13)

In particular, the maximal viscosity solution of the Dirichlet problem (2.1) is
\[ G(x) = \min_{y \in \partial D} \{ g(y) + S(y, x) \}. \] (2.14)

We denote by \( \sim \) the equivalence relation
\[ x \sim y \text{ if and only if } S(x, y) = S(y, x) = 0. \] (2.15)

Observe that a class of equivalence which does not reduce to a point is contained in the Aubry set. By the continuity of \( S \), a class of equivalence \( A \) is closed and, by Proposition 2.2 items (ii) and (iii), \( S(x, \cdot) \) and \( S(\cdot, x) \) are constant in \( A \), for any \( x \in D \). We denote by \( S(x, A) \) and, respectively, \( S(A, x) \) these constants.

Proposition 2.7. If \( u \) is a subsolution of \( H(x, Du) = 0 \) in \( D \) and \( A \) is a class of equivalence of \( S \), then \( u \) is constant on \( A \).

Proof. The property follows immediately by Proposition 2.2(iii).

3. The singular perturbation result

We consider the singular perturbation problem
\[
\begin{cases}
-\frac{\varepsilon^2}{2} \Delta v_\varepsilon + H_i(x, Dv_\varepsilon^i) + \varepsilon \sum_{j=1}^M d_{ij}(x)(e^{(v_\varepsilon^i - v_\varepsilon^j)/\varepsilon} - 1) + \varepsilon c_i(x) = 0, & x \in D, \\
v_\varepsilon^i(x) = g(x), & x \in \partial D.
\end{cases}
\] (3.1)

Proposition 3.1. For every \( \varepsilon > 0 \) there exists a unique viscosity solution \( v_\varepsilon = (v_\varepsilon^1, \ldots, v_\varepsilon^M) \) to (3.1). Moreover, there exists a constant \( C \) such that
\[
\| v_\varepsilon^i \|_\infty \leq C \quad \forall \varepsilon > 0, i = 1, \ldots, M,
\] (3.2)

with \( C \) independent of \( \varepsilon \).

Proof. Concerning the existence and uniqueness of a viscosity solution to (3.1) we refer to [17]. To prove (3.2), we observe that for every \( \varepsilon \) the function \( v \) defined as \( v(x) = (K, \ldots, K) \) where \( K := \inf_{\partial D} g \) is a viscosity subsolution to (3.1), because of the assumption \( H_i(x, 0) \leq 0, i = 1, \ldots, M \). Moreover, using the coercivity of the Hamiltonians \( H_i \) and the continuity of \( g \), we can find \( R \in \mathbb{R}^N \) and \( T \) such that the function \( w(x) = (R \cdot x + T, \ldots, R \cdot x + T) \) is a supersolution to (3.1). Then by standard comparison results for viscosity solutions of weakly-coupled systems [17] we get (3.2).
Besides conditions (2.3)–(2.7), which concerns the well-posedness of limit problem (2.1), we will make some additional assumptions for the singular perturbation result. We define

\[ C^+ = \bigcap_{i=1}^{M} \{ x \in D : c_i(x) > 0 \} \]

and we assume that:

\[ A = \bigcup_{i=1}^{m} A_i, \text{ where } A_i \text{ are classes of equivalence for } S \text{ and} \]

\[ S(A_i, A_j) + S(A_j, A_i) > 0 \text{ for } i \neq j; \]  

(3.3)

\[ \text{any } A_i \subseteq A \cap \text{int } D \text{ satisfies } A_i \subset C^+; \]  

(3.4)

\[ \text{for any } Q \subset D, \text{ there exists a constant } C(Q) \text{ independent of } \varepsilon \text{ such that} \]

\[ \| Dv^i_\varepsilon \|_{\infty} \leq C(Q) \forall \varepsilon > 0, i = 1, \ldots, M; \]  

(3.5)

\[ g \text{ satisfies (2.12).} \]  

(3.6)

**Remark 3.2.** Assumption (3.3) says that the set obtained by the quotient of the Aubry set with the equivalence relation (2.15) is totally disconnected, an assumption which is often used in weak KAM theory. We do not know if our result is valid under more general assumptions on the Aubry set. We remark that it is an open problem to give necessary and sufficient conditions on the Hamiltonian which imply that this assumption is satisfied.

Assumption (3.4) corresponds to the assumptions in [6,9,21], recalled in the Introduction. If (3.4) is not satisfied, it has been shown by Eizenberg in [9] for the single equation, see (1.4), that the sequence \( v_\varepsilon \) does not converge to the maximal solution and that, moreover, every solution of (2.1) can be achieved in the limit for an appropriate choice of \( c \geq 0 \). In the case of systems of equations, we are not aware about any similar result. We intend to study this problem in future work. Observe that in the limiting case in which \( c_i \equiv 0 \) for every \( i \) and \( g(x) \equiv k \), then we get \( v^i_\varepsilon \) converges uniformly to the minimal solution of (2.1), which is the constant one.

Assumption (3.5) can be proved for particular coercive Hamiltonians adapting appropriately the weak Bernstein method (see Proposition 4.1 in Section 4, see also [3] and [13]).

Assumption (3.6) is a compatibility condition of the boundary data and it is needed to ensure that the limit problem has a solution (see Theorem 2.6). It is automatically satisfied if \( g \) is constant.

The next theorem is the main result and ensures that, under the previous assumption, the vanishing viscosity limiting procedure in the elliptic system (3.1) selects the maximal viscosity solution to the limit equation (2.1). We will discuss in the next section some examples of systems satisfying the previous assumptions.

**Theorem 3.3.** Assume (3.3)–(3.6), beside the standing assumptions (2.3)–(2.7). Let \( v_\varepsilon \) be the sequence of the solutions to (3.1). Then for every \( i = 1, \ldots, M \), \( v^i_\varepsilon \) converges uniformly to the maximal viscosity solution of (2.1).

**Proof.** We will prove the statement by showing that every convergent subsequence of \( v^i_\varepsilon \) is converging to the maximal solution \( G \) to (2.1) as defined in (2.14). The proof is divided in several steps:
Step 1 (Identification of the limit). We consider a subsequence \( v_{\varepsilon_n} = (v_{\varepsilon_n}^i) \), which converges uniformly to some continuous function \( v = (v_1, \ldots, v_M) \) in \( D \). Such sequences always exist due to Proposition 3.1, condition (3.5) and the Ascoli–Arzelá theorem. We show that actually \( v^i = v^j \) for every \( i, j \); this means that every component \( v_{\varepsilon_n}^i \) of a convergent subsequence \( v_{\varepsilon_n} \) converges uniformly to the same function for every \( i \). We claim that
\[
v^i(x) - v^j(x) \leq 0 \quad \text{for every } j \neq i, \ x \in D. \tag{3.7}
\]

Fixed \( i \), assume by contradiction there exists \( \overline{x} \in D \) such that, for some \( j \neq i \), \( v^i(x) - v^j(x) \geq k_j > 0 \) for \( x \) in an open neighborhood \( B \subset D \) of \( \overline{x} \). Consider a \( C^2 \) function \( \phi \) such that \( v^i - \phi \) has a strict maximum at some \( x_0 \in \overline{B} \). Hence there exists \( x_n \to x_0 \) such that \( v_{\varepsilon_n}^i - \phi \) has a maximum in \( x_n \) and, by the uniform convergence, \( v_{\varepsilon_n}^i(x_n) \to v^i(x_0) \) and \( v_{\varepsilon_n}^j(x_n) \to v^j(x_0) \). Substituting in (3.1), we get
\[
-\frac{\varepsilon_n}{2} \Delta\phi(x_n) + H_i(x_n, D\phi(x_n)) - \varepsilon_n c_i(x_n) + \varepsilon_n \sum_{j=1}^M d_{ij}(x_n)(e^{(v_{\varepsilon_n}^i(x_n) - v_{\varepsilon_n}^j(x_n))/\varepsilon_n} - 1) \leq 0. \tag{3.8}
\]

Moreover, there exists \( n_0 \) such that, for \( n \geq n_0 \), \( v_{\varepsilon_n}^i(x_n) - v_{\varepsilon_n}^j(x_n) > k_j/2 \) and such that \( x_n \in \overline{B} \).

By the regularity of \( \phi, D\phi \) and \( D^2\phi \) are bounded in \( \overline{B} \). By assumptions (2.4) and (2.5), \( c_i \) and \( d_{ij} \) are bounded in \( \overline{B} \) by some constant \( C \) and by assumption (2.7) also \( H(\cdot, D\phi(\cdot)) \) is bounded in \( \overline{B} \). So, recalling that by assumption (2.5) \( d_{ij}(x) > 0 \) for every \( x \in D \), we get, for \( n \) sufficiently large,
\[
-\frac{\varepsilon_n}{2} \Delta\phi(x_n) + H_i(x_n, D\phi(x_n)) - \varepsilon_n c_i(x_n) + \varepsilon_n \sum_{j=1}^M d_{ij}(x_n)(e^{(v_{\varepsilon_n}^i(x_n) - v_{\varepsilon_n}^j(x_n))/\varepsilon_n} - 1) \\
\geq -(\varepsilon_n + 1)K + \varepsilon_n \sum_{l \neq j} d_{il}(x_n)e^{(v_{\varepsilon_n}^i(x_n) - v_{\varepsilon_n}^j(x_n))/\varepsilon_n} + \varepsilon_n d_{ij}(x_n)e^{k_j/(2\varepsilon_n)} \\
\geq -(\varepsilon_n + 1)K + \varepsilon_n d_{ij}(x_n)e^{k_j/(2\varepsilon_n)} \to +\infty \quad \text{as } \varepsilon_n \to 0
\]
and this is in contradiction with (3.8). The claim (3.7) immediately implies that \( v^i = v^j \), for every \( i, j \) and therefore
\[
\lim_{n \to \infty} v_{\varepsilon_n}^i = v \quad \text{uniformly in } D, \text{for any } i = 1, \ldots, M. \tag{3.9}
\]

Step 2 (Subsolution property). We prove that \( v \) is a subsolution of the limit problem (2.1). The argument is analogous to that used in Step 1, so we just sketch it. Assume by contradiction that \( v \) is not a subsolution. Hence there exists \( x_0 \in D \), a smooth function \( \phi, \delta > 0 \) and \( i \in \{1, \ldots, M\} \) such that \( v - \phi \) has a strict maximum at \( x_0 \) and
\[
H(x_0, D\phi(x_0)) = H_i(x_0, D\phi(x_0)) \geq 2\delta > 0.
\]
Since \( v_{\varepsilon_n}^i \) converges to \( v \), see (3.9), there exists \( x_n \to x_0 \) such that \( v_{\varepsilon_n}^i - \phi \) has a maximum in \( x_n \). Hence, since \( v_{\varepsilon_n}^i \) is a subsolution, we get
\[
-\frac{\varepsilon_n}{2} \Delta\phi(x_n) + H_i(x_n, D\phi(x_n)) - \varepsilon_n c_i(x_n) + \varepsilon_n \sum_{j=1}^M d_{ij}(x_n)(e^{(v_{\varepsilon_n}^i(x_n) - v_{\varepsilon_n}^j(x_n))/\varepsilon_n} - 1) \leq 0. \tag{3.10}
\]
On the other hand, since \( H_i(x_n, D\phi(x_n)) \to H_i(x_0, D\phi(x_0)) \), we have for \( n \) large enough \( H_i(x_n, D\phi(x_n)) \geq \delta \). Hence, taking \( n \) sufficiently large (and then \( \varepsilon_n \) sufficiently small) and using the fact that \( e^x - 1 \geq x \), we get

\[
-\frac{\varepsilon_n}{2} \Delta \phi(x_n) + H_i(x_n, D\phi(x_n)) - \varepsilon_n c(x_n) + \varepsilon_n \sum_{j=1}^{M} d_{ij}(x_n)(e^{(v_{\varepsilon_n}^j(x_n) - v_{\varepsilon_n}^j(x_n))/\varepsilon_n} - 1)
\]

\[
\geq -\frac{\varepsilon_n}{2} \Delta \phi(x_n) + \delta - \varepsilon_n c(x_n) + \sum_{j=1}^{M} d_{ij}(x_n)(v_{\varepsilon_n}^i(x_n) - v_{\varepsilon_n}^j(x_n)) \geq \frac{\delta}{2} > 0,
\]

which gives a contradiction to (3.10).

**Step 3** (Supersolution property). We define \( w_{\varepsilon_n}(x) := \min_i v_{\varepsilon_n}^i(x) \) for \( x \in \overline{D} \). Note that for every \( i = 1, \ldots, M \), \( v_{\varepsilon_n}^i \) is a supersolution in \( D \) to

\[
-\frac{\varepsilon_n}{2} \Delta v + \sup_i H_i(x, Dv) - \varepsilon_n \inf_i c_i(x) + \varepsilon_n \sup_i \sum_{j=1}^{M} d_{ij}(x)(e^{(v_{\varepsilon_n}^j(x)/\varepsilon_n)} - 1) = 0.
\]

By stability of viscosity supersolution with respect to infimum, \( w_{\varepsilon_n} \) is a viscosity supersolution of the same equation. Since by definition \( w_{\varepsilon_n} - v_{\varepsilon_n}^j \leq 0 \) in \( D \) for every \( j \), we get that \( w_{\varepsilon_n} \) is also a viscosity supersolution for \( x \in D \) to

\[
-\frac{\varepsilon_n}{2} \Delta v + \sup_i H_i(x, Dv) - \varepsilon_n \inf_i c_i(x) \geq 0. \quad (3.11)
\]

By (3.9) in Step 1 also the sequence \( w_{\varepsilon_n} \) converges uniformly to \( v \) in \( D \). Then, by standard stability results for viscosity solutions w.r.t. uniform convergence (see [2] and [4]), \( v \) is also a viscosity supersolution to the limit problem (2.1).

**Step 4** (Identification \( v = G \)). This is the last step and the central node of the proof. We recall that \( G \) is the maximal viscosity solution to the limit problem (2.1), so, since \( v \) is also a solution to the same problem, \( v(x) \leq G(x) \) for every \( x \in \overline{D} \).

To conclude, because of the representation formula (2.13), it is sufficient to show that \( v(x) = G(x) \) for \( x \in A \). Recall that by Proposition 2.7, both \( v \) and \( G \) are constant on each \( A_j \subseteq A \). Denoted with \( v(A_j) \) and \( G(A_j) \) these values, it is enough to prove that \( v(A_j) = G(A_j) \) for every \( j = 1, \ldots, m \). But this is obvious for the sets \( A_i \) such that there exists \( x \in \partial D \cap A_i \). Eventually relabelling the sets \( A_i \), we can assume that \( A_1, \ldots, A_k \), for some \( k \leq m \) are contained in the interior of \( D \). So we are reduced to prove that \( v(A_j) = G(A_j) \) for every \( j = 1, \ldots, k \). We start showing a weaker result whose proof is postponed at the end of this section.

**Lemma 3.4.** For every \( A_j \subseteq A \cap \text{int} D \) the following condition holds

\[
v(A_j) = \min \left\{ G(A_j), \min_{k \neq j} \{ v(A_k) + S(A_k, A_j) \} \right\}.
\]
To conclude the proof of the theorem we argue by contradiction and we assume without loss of generality that \( v(A_1) < G(A_1) \). Hence, by Lemma 3.4, eventually relabelling the sets \( A_i \), we can assume that

\[
v(A_1) = G(A_2) + S(A_2, A_1) \geq G(A_1)
\]

in contradiction with our assumption. Moreover,

\[
v(A_1) + S(A_1, A_2) = v(A_2) + S(A_2, A_1) + S(A_1, A_2) > v(A_2).
\]

It follows again by Lemma 3.4 that in a neighborhood of \( A_2 \) we have, relabelling the sets \( A_i \),

\[
v(A_2) = v(A_3) + S(A_3, A_2).
\]

Iterating the previous procedure \((k - 1)\)-times, we eventually get \( v(A_{k-1}) = v(A_k) + S(A_k, A_{k-1}) \). Hence we get \( v(A_k) < G(A_k) \) and \( v(A_k) < v(A_h) + S(A_h, A_k) \) for \( h = 1, \ldots, k - 1 \), in contradiction with Lemma 3.4. \(\square\)

**Proof of Lemma 3.4.** Assume by contradiction there exists \( i \in \{1, \ldots, k\} \) such that

\[
v(A_i) < \min\{G(A_i), \min_{h \neq i} \{v(A_h) + S(A_h, A_i)\}\}.
\]

Using the representation formula (2.13), the continuity (2.4) of the functions \( c_i \) and the assumption (3.4) on the positivity of \( c_i \) near \( \mathcal{A} \), we can choose a neighborhood \( A_{\delta} \) of \( A_i \) such that \( A_{\delta} \cap \mathcal{A} = A_i \) and

\[
v(x) = v(A_i) + S(A_i, x),
\]

\[
\inf_i c_i(x) \geq c_0 > 0
\]

for \( x \in A_{\delta} \).

Let \( f \) be a \( C^1 \)-subsolution of \( H(x, Du) = 0 \) which is strict outside \( \mathcal{A} \) (see Proposition 2.4). We denote, see Proposition 2.7, by \( f(A_i) \) the constant value of \( f \) on \( A_i \), moreover, we know that \( f(x) \leq f(A_i) + S(A_i, x) \) for every \( x \in A_{\delta} \). Eventually shrinking the set \( A_{\delta} \) we can assume that \( f(x) < f(A_i) \) for every \( x \in A_{\delta} \setminus A_i \) such that \( S(A_i, x) = 0 \); this is proved in [6], Lemma 3.4.

We define \( \psi^\alpha(x) = v(x_0) + \alpha(f(x) - f(x_0)) \) where \( x_0 \in A_i \) and \( \alpha \in (0, 1) \). The functions \( \psi^\alpha \) and \( v \) are constant on \( A_i \), with value \( \psi^\alpha(x_0) = v(x_0) = v(A_i) \). By Proposition 2.2(iii), we have for \( x \in A_{\delta} \),

\[
\psi^\alpha(x) \leq v(x_0) + \alpha S(x_0, x) \leq v(x_0) + S(x_0, x) = g(A_i) + S(A_i, x) = v(x).
\]

We show that actually \( \psi^\alpha \) is strictly less than \( v \) out of \( A_i \) for any \( \alpha \in (0, 1) \). We distinguish two cases.

If \( S(x_0, x) > 0 \), then, since \( \alpha \in (0, 1) \), we have

\[
\psi^\alpha(x) \leq v(x_0) + \alpha S(x_0, x) < v(x_0) + S(x_0, x) = v(x).
\]

(3.13)
If $S(x_0,x) = 0$, for $x \in A_\delta \setminus A_i$, then we have that $f(x) < f(x_0)$ and therefore we obtain
\[
\psi^\alpha(x) < \psi^\alpha(x_0) = v(x_0) \leq v(x). \tag{3.14}
\]

By (3.13) and (3.14) we get that $v - \psi^\alpha$ has as a local strict minimum at $A_i$ for every $\alpha$.

Let $\rho_{\eta}$ be a standard mollifier, i.e. $\rho_{\eta}(x) = \rho(x/\eta)/\eta^N$ where $\rho : \mathbb{R}^N \to \mathbb{R}$ is a smooth, nonnegative function such that for $x \in A_\delta \sup\{\rho\} \subset B(0,1)$ and $\int_{\mathbb{R}^N} \rho(x) \, dx = 1$. Set $f_{\varepsilon_n} = f * \rho_{\varepsilon_n}$. Then, by the convexity of $H$, we find two constants $C, K > 0$ such that
\[
H(x, Df_{\varepsilon_n}) \leq \int_{\mathbb{R}^N} H(y, Df) \rho_{\varepsilon_n}(x-y) \, dy \leq H(x, Df) + C\varepsilon_n^2, \tag{3.15}
\]
\[
\|D^2 f_{\varepsilon_n}\|_\infty \leq \frac{K}{\varepsilon_n^4}. \tag{3.16}
\]

Set $\psi_{\varepsilon_n}^\alpha(x) = v(x_0) + \alpha(f_{\varepsilon_n}(x) - f_{\varepsilon_n}(x_0))$ and let $w_{\varepsilon_n}$ be defined as in Step 3 of Theorem 3.3. Since $\lim_{\varepsilon_n \to 0} w_{\varepsilon_n}(x') = \psi_{\varepsilon_n}^\alpha(x)$ for every $x \in D$ and $\alpha \in (0,1)$, and since $\psi^\alpha$ is strictly less than $v$ out of $A_i$, there exists a sequence $x_{\varepsilon_n}^\alpha \in \overline{A_\delta}$ of minimum points for $w_{\varepsilon_n} - \psi_{\varepsilon_n}^\alpha$ such that $d(x_{\varepsilon_n}, A_i) \to 0$ for $\varepsilon_n \to 0$. Recalling that $w_{\varepsilon}$ is a supersolution to (3.11) and using (3.15), (3.16) we get for every $\varepsilon_n$ and every $\alpha$

\[
0 \leq -\frac{\varepsilon_n}{2} \Delta \psi_{\varepsilon_n}^\alpha(x_{\varepsilon_n}^\alpha) + \max_i (H_i(x_{\varepsilon_n}^\alpha, D\psi_{\varepsilon_n}^\alpha(x_{\varepsilon_n}^\alpha))) - \varepsilon_n \inf_i c_i(x_{\varepsilon_n}^\alpha)
\]
\[
= -\varepsilon_n \alpha f_{\varepsilon_n}(x_{\varepsilon_n}^\alpha) + H(x_{\varepsilon_n}^\alpha, \alpha Df_{\varepsilon_n}(x_{\varepsilon_n}^\alpha)) - \varepsilon_n \inf_i c_i(x_{\varepsilon_n}^\alpha)
\]
\[
\leq \varepsilon_n \frac{K}{\varepsilon_n^4} + \alpha(H(x_{\varepsilon_n}^\alpha, Df(x_{\varepsilon_n}^\alpha))) + C\varepsilon_n^2 - \varepsilon_n c_0 \leq \alpha \left( \frac{K}{\varepsilon_n^4} + C\varepsilon_n^2 \right) - \varepsilon_n c_0
\]

which gives a contradiction if we fix $\varepsilon_n$ and choose $\alpha$ sufficiently small. \qed

4. A model problem: Large deviations of random evolution equations

In this section we discuss the model problem we presented in the Introduction, i.e. small random perturbations of the random evolution process (1.3). The Hamiltonians $H_i$ associated to large deviations estimates are of the form
\[
H_i(x, p) = \frac{|p|^2}{2} - b_i(x) \cdot p \tag{4.1}
\]
and $b_i$ are assumed to be Lipschitz. We will check in the following under which conditions on the vector fields $b_i$ the assumptions (3.3)–(3.6) of the singular perturbation result, Theorem 3.3, are satisfied.

First of all, observe that condition (3.6) is immediate for the system (1.7), since $g$ is assumed to be constant. As for condition (3.5), we have the following result.
Proposition 4.1. Let \( v_\varepsilon = (v^1_\varepsilon, \ldots, v^M_\varepsilon) \) be the unique viscosity solution to (3.1) with \( H_i \) as in (4.1). Then for any open set \( Q \subset D \), there exists a constant \( C(Q) \) independent of \( \varepsilon \) such that

\[
\|Dv^i_\varepsilon\|_{\infty} \leq C(Q) \quad \forall \varepsilon > 0, i = 1, \ldots, M.
\]

Proof. This result relies on a quite standard argument in viscosity solution theory, which is the weak Bernstein method. Then we will give here just a brief sketch of the argument and we refer to [13], Lemma 3.1, for a detailed proof in a similar case.

Since the system (3.1) is uniformly elliptic for every \( \varepsilon > 0 \), we get \( v^i_\varepsilon \in C^2(D) \cap C(\overline{D}) \), for every \( i \) (see [17]). Moreover, we recall that, by Proposition 3.1, \( \|v^i_\varepsilon\|_{\infty} \leq K \), with \( K \) independent of \( \varepsilon \). We introduce the auxiliary functions

\[
\psi_i(x) := \zeta^2(x)|Dv^i_\varepsilon(x)|^2 - \lambda v^i_\varepsilon(x),
\]

where \( \lambda \geq 0 \) will be fixed below and \( \zeta \) is a smooth cutoff function with compact support in \( Q' \) with \( Q \subset Q' \subset D \) and \( \zeta \equiv 1 \) on \( Q \). We fix \( k \) and \( x_0 \) such that

\[
\psi_k(x_0) = \max_{i=1,\ldots,M} \max_{Q'} \psi_i(x).
\]

Then \( (\psi_k)_x(x_0) = 0 \) and \( \Delta \psi_k(x_0) \leq 0 \). Using these relations and the fact that \( v^k_\varepsilon \) is a smooth solution to the system (3.1) (see the computations at pp. 243–244 in [13]) we obtain that

\[
\lambda |Dv^k_\varepsilon(x_0)|^2 \leq C\lambda + C\zeta(x_0)|Dv^k_\varepsilon(x_0)|^3 + C|Dv^k_\varepsilon(x_0)|^2
\]

for some positive constant \( C \), which depends on the coefficients of the system (3.1) and can be chosen independent of \( \varepsilon \). Now we choose \( \lambda = \mu(\zeta(x_0)|Dv^k_\varepsilon(x_0)| + 1) \) with \( \mu \) sufficiently large so that the previous inequality gives \( |Dv^k_\varepsilon(x_0)|^2 \leq C \), where \( C \) is independent of \( \varepsilon \). We observe that

\[
\max_{i=1,\ldots,M} \max_{Q'} \zeta^2|Dv^i_\varepsilon|^2 \leq 2K\lambda + \zeta^2|Dv^k_\varepsilon(x_0)|^2 \leq 2K\lambda + C \leq M,
\]

with \( M \) independent of \( \varepsilon \). Then we conclude recalling that \( \zeta \equiv 1 \) on \( Q \). \( \Box \)

We will now study the Aubry set related to the Hamiltonian (2.2) obtained from to Hamiltonians (4.1). This is an essential point in our approach and the remaining part of the section is devoted to this point. We consider in particular two opposite situations: in the first case all the vector fields \( b_i \) are transversal to the set \( D \), i.e. the integral curves of each \( b_i \) starting in \( D \) exit \( D \) in finite time, in the second one the set \( \overline{D} \) is an invariant set for all vector fields \( b_i \), i.e. the integral curves of each \( b_i \) starting in \( \overline{D} \) remain forever in \( \overline{D} \). These cases have been considered also in [5,10,11].

Let \( Z_i \) and \( \sigma_i \) be defined as in Proposition 2.1. An easy computation gives

\[
Z_i(x) = B(b_i(x), |b_i(x)|)
\]

\[
\sigma_i(x, q) = |b(x)||q| + b_i(x) \cdot q.
\]
We denote by $S_i$ the corresponding distance defined as in (2.10) and by $A_i$ the Aubry set of $S_i$. We also consider the Quasi-Potential associated to the vector fields $b_i$ (see [16])

$$V_i(x, y) := \inf \left\{ \int_0^T \frac{1}{2} \left| \dot{\phi}(s) - b_i(\phi(s)) \right|^2 ds : \phi \in AC_{x,y,D}[0,T], T > 0 \right\}.$$ 

As in the case of a singular vector field, we define an Action Functional associated to the random evolution process (1.3)

$$U(x, y) := \inf \left\{ \frac{1}{2} \int_0^T \inf_{i=1,\ldots,n} \left| \dot{\xi}(t) - b_i(\xi(t)) \right|^2 dt : \psi \in AC_{x,y,D}[0,T], T > 0 \right\}.$$ 

We set

$$R(x, p) = \inf_{b(x) \in \text{co}(b_1(x),\ldots,b_n(x))} \frac{1}{2} |q - b(x)|^2 = \inf_{\alpha_i \geq 0, \sum_i \alpha_i = 1} \frac{1}{2} |q - \sum_i \alpha_i b_i(x)|^2$$

and we define the Quasi-Potential

$$V(x, y) := \inf \left\{ \int_0^T R(\xi(t), \dot{\xi}(t)) dt : \xi \in AC_{x,y,D}[0,T], T > 0 \right\}. \quad (4.3)$$

Note that

$$V(x, y) \leq U(x, y) \quad \text{for any } x, y \in D. \quad (4.4)$$

In [11], it is proved that $V$, which is obtained by taking the lower semicontinuous envelope of the functional defining $U$, is a rate function for the large deviation estimate proved in that paper.

We recall that in the case of a single dynamical system it was proved in [6], Proposition 4.1, [7], Proposition 4.2, that

$$S_i(x, y) = V_i(y, x) \quad \text{for any } x, y \in D. \quad (4.5)$$

The same result holds also for random evolution systems, in fact the distance function introduced in (2.10) coincides with the Quasi-Potential (4.3).

**Proposition 4.2.** We have for every $x, y \in D$

$$S(x, y) = V(y, x) \leq U(y, x) \leq V_i(y, x) = S_i(x, y) \quad (4.6)$$

for any $x, y \in D, i = 1, \ldots, M$.

**Proof.** We prove only that $S(x, y) = V(y, x)$ since the other inequalities are obvious (see (4.5) and (4.4)). It is easy to show that

$$H(x, p) = \max_i \left( \frac{|p|^2}{2} - b_i(x) \cdot p \right) = \sup_{b(x) \in \text{co}(b_1(x),\ldots,b_M(x))} \left( \frac{|p|^2}{2} - b(x) \cdot p \right).$$
so if \( p \in Z(x) \) then \( \frac{|p|^2}{2} - b(x) \cdot p \leq 0 \) for any \( b(x) \in \text{co}(b_1(x),\ldots,b_M(x)) \). This means that \( p \in Z(x) \) belongs to \( \mathbb{B}(b(x),|b(x)|) \) for any \( b(x) \in \text{co}(b_1(x),\ldots,b_M(x)) \). Hence, for any \( b(x) \in \text{co}(b_1(x),\ldots,b_M(x)) \), we get

\[
\sigma(x,q) \leq b(x) \cdot q + |b(x)||q| \leq \frac{1}{2}|q + b(x)|^2
\]

and then

\[
\sigma(x,q) \leq R(x,-q).
\]

From this, using the definitions of \( V \) and \( S \), we obtain \( S(y,x) \leq V(x,y) \). Now we claim that for every \( y \in D \) the function \( V(\cdot,y) \) is a viscosity subsolution to (2.1) in \( D \). In fact, if the claim is true, by Proposition 2.2(ii), we obtain \( -S(y,x) \leq V(y,y) - V(x,y) \leq S(x,y) \), which permits to conclude recalling that \( V(y,y) = 0 \). To prove the claim, first of all we observe that, by standard dynamic programming principle (see [2]), \( V(\cdot,y) \) is a viscosity subsolution to \( H_1(x,DV) = 0 \) in \( D \) where

\[
H_1(x,p) = \sup_{a \in \mathbb{R}^N} \left\{ -p \cdot a - \inf_{b(x) \in \text{co}(b_1(x),\ldots,b_M(x))} \frac{1}{2}|a - b(x)|^2 \right\} \\
= \sup_{a \in \mathbb{R}^N} \sup_{b(x) \in \text{co}(b_1(x),\ldots,b_M(x))} \left\{ -p \cdot a - \frac{1}{2}|a - b(x)|^2 \right\}.
\]

It is a simple computation to see that \( H \) defined in (2.2) coincides with

\[
\sup_{b(x) \in \text{co}(b_1(x),\ldots,b_M(x))} \left( \frac{|p|^2}{2} - b(x) \cdot p \right) = \sup_{b(x) \in \text{co}(b_1(x),\ldots,b_M(x))} \sup_{a \in \mathbb{R}^N} \left\{ -p \cdot a - \frac{1}{2}|a - b(x)|^2 \right\}.
\]

Therefore, \( H_1(x,p) = H(x,p) \) for every \( x \in D \), and our claim is proved. \( \Box \)

4.1. Transversal vector fields

For a deterministic system \( b \) a classical assumption which gives large deviations principles [18] is the transversality of the vector field \( b \) with respect to the domain \( D \), i.e.

\[
\exists T > 0 \text{ such that for any integral curve of } \dot{x} = b(x(t)) \text{ with } x(0) \in D, \text{ there exists } s < T \text{ for which } x(s) \notin \mathcal{D}. \tag{4.7}
\]

It corresponds to ask that integral curves of \( b \) exit in an uniformly bounded time from the domain \( D \). The dynamical condition (4.7), which is known in the probabilistic literature as the Levinson’s condition, is equivalent, see [18], to the following PDE condition:

there exists a \( C^1 \) function \( \psi \) such that \( \frac{|D\psi|^2}{2} - b(x) \cdot D\psi < 0 \) for \( x \in D \).

The Levinson’s condition for random evolution systems has been described in [10] and reads as follows

there exists \( T < \infty \) such that \( \mathbb{P}_{x,i}(\tau^0 \leq T) = 1 \forall x \in D \) and \( i = 1,\ldots,M \) \( \tag{4.8} \).
Proof. The equivalence between (ii) and (iii) is a general fact (see Proposition 2.4).

In [10] several conditions on the dynamical systems \(b_i\) are formulated assuring (4.8). In particular it is assumed (see Condition (H3*)) that

\[
\text{for any vector field } \alpha(x) = (\alpha_1(x), \ldots, \alpha_M(x)) \text{ such that } \\
\alpha_i(x) \geq 0, \sum_{i=1}^M \alpha_i(x) = 1 \text{ for } x \in D \\
\text{the vector field } b(x) = \sum_{i=1}^M \alpha_i(x)b_i(x) \text{ satisfies (4.7).}
\]

(4.9)

For our result we consider a slightly weaker condition:

\[
\text{for any measurable } \lambda(\cdot) = (\lambda_1(\cdot), \ldots, \lambda_M(\cdot)) \text{ with } \lambda_i \geq 0, \sum_{i=1}^M \lambda_i = 1 \\
\text{the vector field } \bar{b}(x(\cdot)) = \sum_{i=1}^M \lambda_i(\cdot)b_i(x(\cdot)) \text{ is regular in the sense of (4.7).}
\]

(4.10)

We will show that, in the setting described by condition (4.10), our singular perturbation result holds for any choice of \(c_i \geq 0\).

The following proposition gives a PDE interpretation of (4.10) (see also [8], Proposition 4.2).

**Proposition 4.3.** They are equivalent:

(i) condition (4.10);

(ii) the Aubry set \(A\) is empty;

(iii) there exists a \(C^1\) function \(\psi\) such that \(H_i(x, D\psi) < 0\) in \(D\), for any \(i = 1, \ldots, M\).

Proof. The equivalence between (ii) and (iii) is a general fact (see Proposition 2.4).

If (iii) holds, then, choosing appropriately \(k > 0\), the function \(\phi = -\psi/k\) is a \(C^1\) subsolution to \(b \cdot D\phi < -1\) for any \(b \in \text{co}(b_1, \ldots, b_M)\). Such a condition is equivalent to (i), in particular it implies the regularity of the vector fields \(b_i\) in (4.10), as proved in [19], Lemma 2.8, [4], Lemma 6.1.

We show now that (i) implies (ii). By contradiction assume that (i) holds and that \(A\) is not empty. Let \(\bar{x} \in A\). We claim that there exist some measurable \(\bar{\lambda}_i\) with \(\sum_i \bar{\lambda}_i = 1\) such that the integral curve of \(\dot{x} = \sum_i \bar{\lambda}_i(\cdot) b_i(x(\cdot))\) issuing from \(\bar{x}\) stays in \(\overline{D}\) for the whole time. If the claim is true, this gives a contradiction with (i).

If \(\bar{x} \in A\) is an equilibrium, i.e. there exists \(\bar{\lambda}_i \in [0, 1], \sum_i \bar{\lambda}_i = 1\), with \(\sum_i \bar{\lambda}_i b_i(\bar{x}) = 0\), the claim trivially holds true.

Assume now that \(\bar{x}\) is not an equilibrium. We can find a sequence of cycles \(\xi_n\) parameterized by the arc-length, with \(T_n := \ell(\xi_n)\) bounded by below by a positive constant, such that

\[
\xi_n(0) = \xi_n(T_n) = \bar{x} \quad \text{for any } n, \quad \xi_n(t) \in \overline{D} \quad \forall t, \quad \lim_n \ell_b(\xi_n) = 0.
\]

Fix \(0 < T < \inf_n T_n\): eventually passing to a subsequence we get that \(\xi_n(T) \to y\) as \(n \to \infty\), with \(y \neq \bar{x}\). Moreover, by its definition we obtain \(S(\bar{x}, y) = 0 = S(y, \bar{x}) = V(\bar{x}, y)\). So we can find a sequence of curves \(\phi_n\) such that \(\phi_n(0) = \bar{x}, \phi_n(T_n) = y, \phi_n(t) \in \overline{D}\) and

\[
\int_0^{T_n} R(\phi_n(t), \dot{\phi}_n(t)) \, dt \leq \frac{1}{n}
\]
for every \( n \). Fix \( 0 < T \leq \inf n T_n \): eventually passing to a subsequence we get \( \phi_n(T) \to z \) as \( n \to \infty \). Note that, by its definition, \( V(z, y) = 0 = S(y, z) \), since

\[
V(z, y) \leq V(z, \phi_n(T)) + V(\phi_n(T), y) = S(\phi_n(T), z) + V(\phi_n(T), y) \leq K|\phi_n(T) - z| + 1/n
\]

for every \( n \). Moreover, we get for every \( n \)

\[
\int_0^T R(\phi_n(t), \dot{\phi}_n(t)) \, dt \leq 1
\]

which gives in particular (see formula (2.29) in [12])

\[
\frac{1}{2} \int_0^T |\dot{\phi}_n(t)|^2 \, dt \leq K(T, \|b_1\|_\infty, \ldots, \|b_M\|_\infty)
\]

and so, as shown in [12] or in [16], Lemma III 2.1b, the sequence \( \phi_n \) is equi-\( \text{Holder} \) continuous of order \( 1/2 \) in \([0, T]\). Therefore, by Ascoli–Arzelá arguments, we get that, eventually passing to a subsequence, \( \phi_n \) converges uniformly in \([0, T]\) to an absolutely continuous curve \( \phi \) such that \( \phi(0) = x, \phi(T) = z, \)

\( \phi(t) \in D \). Moreover, by lower semicontinuity of the functional \( I(\xi) = \int_0^T R(\xi(t), \dot{\xi}(t)) \, dt \), we conclude

\[
\int_0^T R(\phi(t), \dot{\phi}(t)) \, dt = 0.
\]

Recalling the definition of \( R(x, v) \), this implies that \( \inf_{\alpha_i} |\dot{\phi}(t) - \sum_i \alpha_i b_i(\phi(t))|^2 = 0 \) for almost every \( t \in [0, T] \). Moreover, it is easy to see that actually \( \min_{\alpha_i} |\dot{\phi}(t) - \sum_i \alpha_i b_i(\phi(t))|^2 = 0 \) for a.e. \( t \in [0, T] \).

We consider now a set valued function

\[
S : [0, T] \to B := \left\{ \lambda \in \mathbb{R}^M \mid 0 \leq \lambda_i \leq 1 \text{ for every } i = 1, \ldots, M \text{ and } \sum_{i=1}^M \lambda_i = 1 \right\}
\]

defined as follows

\[
S(t) = \left\{ (\lambda_1, \ldots, \lambda_M) \text{ such that } |\dot{\phi}(t) - \sum_i \lambda_i b_i(\phi(t))|^2 = \min_{\alpha_i} |\dot{\phi}(t) - \sum_i \alpha_i b_i(\phi(t))|^2 \right\}.
\]

\( S \) is a measurable map taking values in compact convex subsets of \( B \). Then a result of set valued analysis gives that there exists a measurable selection of \( S \) (see [1], Theorem I.14.1). This means in particular that there exist measurable functions \( \lambda_i(\cdot) \), with \( 0 \leq \lambda_i(t) \leq 1, \sum_i \lambda_i(t) = 1 \) for every \( t \in [0, T] \), such that

\[
|\dot{\phi}(t) - \sum_i \lambda_i b_i(\phi(t))|^2 = \min_{\alpha_i} |\dot{\phi}(t) - \sum_i \alpha_i b_i(\phi(t))|^2 = 0 \text{ for a.e. } t \in [0, T].
\]

In particular we get that \( \dot{\phi}(t) = \sum_{i=1}^M \lambda_i(t) b_i(\phi(t)) \) for \( t \in [0, T] \).
Recalling that $\phi(t) \in \overline{D}$ for every $t \in [0, T]$, we proved our claim for an integral curve $\phi$ of the system $\dot{x}(t) = \sum_{i=1}^{M} \lambda_i(t) b_i(x(t))$ in some interval $[0, T]$ and then in some maximal closed interval $[0, T_0]$. Assume by contradiction that $T_0 \neq +\infty$. Then, arguing as above, we see that there exists an integral curve $\phi'$ of $\dot{\phi}'(t) = \sum_{i=1}^{M} \lambda'_i(t) b_i(\phi'(t))$, such that $\phi'(t) \in \overline{D}$ for every $t \in [T_0, T_1]$ for some $T_1 > T_0$. So, defining a curve in $[0, T_1]$ by juxtaposition of $\phi$, $\phi'$, we obtain that the claim holds in $[0, T_1]$, in contrast with the maximality of $T_0$. □

Therefore, due to Proposition 4.3, in case of transversal vector fields verifying (4.10), the assumption (3.4) is always satisfied for any choice of $c_i(x)$.

4.2. Exponentially stable equilibrium point

We consider now, in this and in the following subsection, the case in which the domain $D$ is invariant with respect to the vector fields $b_i$. We start with a very special case which can be treated in a rather simple way. We assume that for any $i = 1, \ldots, M$,

$$b_i(0) = 0 \quad \text{and} \quad b_i(x) \cdot x \leq -\delta|x|^2 < 0 \quad \text{for any } x \neq 0.$$  \hspace{1cm} (4.11)

Hence the origin is an exponentially attractive equilibrium point for all the vector fields $b_i$. In this setting our singular perturbation result holds if

$$c_i(0) > 0 \quad \forall i.$$  

Remark 4.4. The same condition (4.11) was considered in [5], in a different setting (see the following Remark 4.6).

By a simple computation we check that the function $\phi(x) := -\delta|x|^2/2$ is a smooth subsolution to $H(x, Dv) \leq 0$, strict out of $x = 0$. Then we get that $A_i = \mathbb{A} = \{0\}$ for every $i$. So (3.4) is satisfied if $c_i(0) > 0$ for any $i = 1, \ldots, M$.

4.3. Invariant domain

We finally consider the case in which the domain $D$ is invariant with respect to the vector fields $b_i$. We assume in particular the following conditions:

$$n(x) \cdot b_i(x) \leq 0 \quad \text{for } x \in \partial D, i = 1, \ldots, M,$$  \hspace{1cm} (4.12)

where $n(x)$ is the exterior normal to $\partial D$ and

there exist $m$ classes of equivalence $K_1, \ldots, K_m \subset \overline{D}$ for $V$ such that the set $\bigcup_{i=1}^{m} K_i$ contains all the $\omega$-limit points of $\dot{x}(t) = \sum_{i=1}^{M} \lambda_i(t) b_i(x(t))$,

$$x(0) = x \in D,$$  \hspace{1cm} (4.13)

where $\lambda_i(\cdot)$ are measurable functions such that $0 \leq \lambda_i(t) \leq 1$ and $\sum_{i=1}^{M} \lambda_i(t) = 1$.

We will show that our singular perturbation result holds in this setting if the following condition holds

if $K_i \subset \text{int } D$ then $K_i \subset C^+$.  

Remark 4.5. Condition (4.12) is equivalent, under suitable regularity assumption on $\partial D$, to say that $D$ is an invariant domain for the vector fields $b_i$.

Remark 4.6. The case of invariant domain has also been considered in [5] under assumption (4.11) and in [11] under the assumption that there exists a unique compact set $K_0$ which is a class of equivalence for $V$ and which contains all the $\omega$-limit points of $\dot{x} = b_i(x)$ for $i = 1, \ldots, M$.

However the problem studied in [5] and [11] was slightly different. Indeed there the authors were interested in the behavior, as $\varepsilon \to 0$, of the exit place from $D$ of the perturbed random evolution equation (1.1). This problem can be studied in terms of the limit behavior of functionals $u^i_\varepsilon(x) = \mathbb{E}_{x,i}(X^\varepsilon_t)$, where $(X^\varepsilon_t, \nu^\varepsilon_t)$ satisfies (1.1) with initial data $X(0) = x$, $\nu(0) = i$, $\psi \in C(\partial D)$ and $\tau^\varepsilon$ is the first exit time of $X^\varepsilon$ from $D$.

Our setting is different since we consider functionals as defined in (1.6) in the Introduction, whose limit behavior is related to large deviations estimates of the expected exit time from $D$ of the process $X^\varepsilon$.

Remark 4.7. The reachability sets (and consequently the set of $\omega$-limit points) for systems as in (4.13) can be described in terms of controllability properties of the family of vector fields $(b_1, \ldots, b_M)$, in particular by computing the Lie algebra generated by these vector fields, see [20].

We analyze in the following the connections between $\bigcup_i K_i$ and the Aubry set $A$, in order to get a condition assuring (3.4). We give now a result on the invariance of the Aubry set with respect to the dynamical system described in (4.13) (actually weak invariance or viability, see Remark 4.9).

Proposition 4.8. For every $\pi \in A$ there exist measurable functions $\overline{\lambda}_i(\cdot)$, with $0 \leq \overline{\lambda}_i(t) \leq 1$, $\sum_i \overline{\lambda}_i(t) = 1$ for every $t \geq 0$, such that the integral curve $\eta$ of the dynamic $\dot{x}(t) = \sum_{i=1}^M \overline{\lambda}_i(t)b_i(x(t))$ starting at $\pi$ and defined in $[0, +\infty)$ satisfies:

(i) $\eta(t) \in A$ for every $t \geq 0$;
(ii) $V(\eta(t), \eta(s)) = 0$ for every $t, s \geq 0$.

Note that the interesting point in item (ii) is that $V(\eta(t), \eta(s)) = 0$ holds true for $t > s$, when $t < s$, this relation immediately comes from the definition of Quasi-Potential.

Remark 4.9. Proposition 4.8(i) affirms in particular that the Aubry set $A$ is forward weakly invariant or viable for the family of dynamics

$$\dot{x}(t) = \sum_{i=1}^M \lambda_i(t)b_i(x(t)), \lambda_i(\cdot) \text{ measurable s.t. } 0 \leq \lambda_i(t) \leq 1, \sum_i \lambda_i(t) = 1 \forall t \geq 0.$$ 

In control theory, a closed set $K$ is called viable or, equivalently, weakly invariant (see [11]) w.r.t. a controlled dynamical system $\dot{x} = f(x,u)$ if for every point $x \in K$ there exists at least one measurable function $\overline{u}(\cdot) \text{ (an admissible open loop control)}$ such that the integral curve of $\dot{x}(t) = f(x(t), \overline{u}(t))$ starting at $x$ satisfies $x(t) \in K$ for every $t \geq 0$.

Proof. If $\pi$ is an equilibrium, that is there exist some constants $\lambda_i$ such that $0 \leq \lambda_i \leq 1$, $\sum_i \lambda_i = 1$ and $\sum_i \lambda_i b_i(\pi) = 0$, there is nothing to prove.
Assume then \( \overline{\varphi} \) to be in \( \mathcal{A} \) but not an equilibrium. We can find a sequence of cycles \( \xi_n \) parameterized by the arc-length, with \( T_n := \ell(\xi_n) \) bounded by below by a positive constant, such that

\[
\xi_n(0) = \xi_n(T_n) = \overline{\varphi} \quad \text{for any } n, \quad \lim_{n \to \infty} \ell_b(\xi_n) = 0.
\]

Fix \( 0 < T < \inf_n T_n \): eventually passing to a subsequence we get that \( \xi_n(T) \to y \) as \( n \to \infty \), with \( y \neq \overline{\varphi} \). Moreover, by its definition we obtain \( S(\overline{\varphi}, y) = 0 = S(y, \overline{\varphi}) = V(\overline{\varphi}, y) \). So we can find a sequence of curves \( \phi_n \) such that \( \phi_n(0) = \overline{\varphi} \), \( \phi_n(T_n) = y \) and

\[
\int_0^{T_n} R(\phi_n(t), \dot{\phi}_n(t)) \, dt \leq \frac{1}{n}
\]

for every \( n \). Fix \( 0 < T \leq \inf_n T_n \). Arguing exactly as in the proof of Proposition 4.3, we get that, eventually passing to a subsequence, \( \phi_n \) converges uniformly in \([0,1]\) to an absolutely continuous curve \( \phi \) such that \( \phi(0) = x \), \( \phi(T) = z = \lim_{n \to \infty} \phi_n(T) \) and \( \dot{\phi}(t) = \sum_{i=1}^M \lambda_i(t)b_i(\phi(t)) \) for some measurable functions \( \lambda_i(\cdot) \), with \( 0 \leq \lambda_i(t) \leq 1 \), \( \sum_i \lambda_i(t) = 1 \) for every \( t \in [0,1] \). This permits to conclude that every point \( \phi(t) \) for \( t \in [0,T] \) belongs to \( \mathcal{A} \), since \( \phi(t) \) is in the same equivalence class of \( \overline{\varphi} \) (see the discussion before Proposition 2.7). Indeed \( S(\phi(t), x_0) = V(x_0, \phi(t)) = 0 \) and

\[
S(x_0, \phi(t)) \leq S(x_0, y) + S(y, z) + S(z, \phi(t)) = S(z, \phi(t)) = V(\phi(t), z) = 0.
\]

Moreover, by the same argument we get \( V(\phi(t), \phi(s)) = 0 \) for every \( t, s \in [0,T] \).

We have proved that the items (i), (ii) hold true, for an integral curve \( \phi \), at least in some interval \([0,T]\), for \( T > 0 \), and so in some maximal closed interval \( I \). We assume by contradiction that \( I = [0, T_0] \neq [0, +\infty) \). Arguing as above, we see that there exists an integral curve \( \phi' \) of the vector field \( \phi'(t) = \sum_{i=1}^M \overline{\lambda}_i(t)b_i(\phi(t)) \)

\[
V(\phi'(t), \phi'(s)) = 0 \quad \text{for any } t, s \in [T_0, T_1],
\]

for some \( T_1 > T_0 \). For \( t \in [T_0, T_1], s \in [0, T_0] \), we find

\[
0 \leq V(\phi'(t), \phi(s)) \leq V(\phi'(t), \phi'(T_0)) + V(\phi(T_0), \phi(s)) = 0,
\]

which shows that (ii) holds in \([0, T_1]\), in contrast with the maximality of \( T_0 \), defining a curve in \([0, T_1]\) by juxtaposition of \( \phi, \phi' \).  \( \square \)

**Corollary 4.10.**

\[
\bigcup_{i=1}^m K_i = \mathcal{A}. \tag{4.14}
\]

**Proof.** For every \( i \), \( K_i \) is an equivalence class for \( V \) and then also for \( S \). We show that for every fixed \( i \), \( K_i \subset \mathcal{A} \).

First assume \( K_i = \{ x_0 \} \). Then it is easy to show, by definition of \( K_i \), that \( x_0 \) is an equilibrium for some system in the form of (4.13). This means that there exist \( \lambda_1, \ldots, \lambda_M \in [0, 1] \) such that \( \sum_i \lambda_i = 1 \).
and \( \sum_i \lambda_i b_i(x_0) = 0 \). This implies that \( Z(x_0) = \{0\} \) and we can conclude, by a simple argument which uses the upper semicontinuity of \( Z \) (see [15], Lemma 5.2, that \( x_0 \in A \).

Assume now that \( K_i \) contains more that one point. Then, as we observe in Section 2 (before Proposition 2.7), \( K_i \subseteq A \).

To prove the reverse inclusion, Proposition 4.8 implies in particular that for any \( x \in A \), there exists some \( K_i \) such that \( S(x, y) = 0 = S(y, x) \) for \( y \in K_i \). In fact it is sufficient to choose \( K_i \) containing the limit points of the integral curve \( \eta \) as in Proposition 4.8 starting at \( x \in A \).

Finally we conclude, by (4.14), that a sufficient condition for assumption (3.4) to hold is

\[
\text{if } K_i \subseteq \text{int } D \text{ then } K_i \subset C^+.
\]

**Remark 4.11.** We observe that our result gives an explicit representation formula for the rate function of the functionals (1.6) in terms of the distance associated to (1.8). In fact, as consequence of the singular perturbation result, we have

\[
\varepsilon \log \left( \mathbb{E}_{x, i} \left( \exp \left\{ -\int_0^{\tau^\varepsilon(x)} c_i(X_s^\varepsilon) \, ds \right\} \right) \right) \rightarrow \min_{y \in \partial D} S(y, x) \text{ uniformly as } \varepsilon \rightarrow 0.
\]

**5. Singular perturbations of eikonal Hamiltonians**

In this last section we present another class of Hamiltonians to which we can apply the singular perturbation result in Section 3.

We consider Hamiltonians \( H_i \) of eikonal type

\[
H_i(x, p) = F_i(x, p) - f_i(x)
\]

with \( F_i \) continuous in \((x, p)\), convex in \( p \) and satisfying \( F_i(x, 0) = 0, F_i(x, p) > 0 \) for \(|p| \neq 0 \) for any \( x \in D \) (for example, \( F(x, p) = p^t A(x) p \) with \( A(x) \) a positive definite matrix). The functions \( f_i : \overline{D} \rightarrow \mathbb{R} \) are continuous and nonnegative.

Setting \( Z_i(x) = \{ p \mid H_i(x, p) \leq 0 \} \), \( \sigma_i(x, q) = \sup_{p \in Z_i(x)} p \cdot q \), we can introduce a distance function \( S_i \) as in (2.10) with \( \sigma_i \) in place of \( \sigma \) and define the corresponding Aubry set \( A_i \), see Definition 2.3. It is not difficult to see that \( A_i \) coincides with \( \{ x \in D : f_i(x) = 0 \} \), since out of this set the function \( \psi \equiv 0 \) is a strict subsolution of the equation \( H_i(x, Du) = 0 \).

The Aubry set \( A \) of the Hamiltonian \( H(x, p) = \max_i H_i(x, p) \) is given by the set

\[
A = \left\{ x \in \overline{D} : \min_i f_i(x) = 0 \right\},
\]

since \( \psi \equiv 0 \) is a strict subsolution to \( H(x, Du) = 0 \) out of \( A \). In this case we have that \( A = \bigcup_{i=1}^M A_i \) and assumption (3.4) reads

\[
\{ x \mid f_i(x) = 0 \text{ for some } i \in \{1, \ldots, M\} \} \subseteq C^+.
\]
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References