LYAPUNOV’S EXPONENTS FOR NONSMOOTH DYNAMICS WITH IMPACTS: STABILITY ANALYSIS OF THE ROCKING BLOCK

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The plane dynamics of a rigid block simply supported on a harmonically moving rigid ground is a problem which still needs investigating, although the matter has been the subject of much research since the last century. Unilateral contacts, Coulomb friction and impacts make the system hybrid as it reveals a mixed continuous and discontinuous nature. Thus, stability analysis requires the extension and adaptation of concepts with regard to variational-perturbative procedures. In particular, discontinuous systems exhibit discontinuities or “saltations” in the fundamental solution matrix which must be analyzed carefully. In this paper, the adaptation of numerical methods that permit us to obtain characteristic multipliers and Lyapunov’s exponents for the rocking mode of the block will be tackled. Analytical methods are used for the linearized equations of motion; the results are compared with those in the scientific literature.

Keywords: Nonsmooth systems; Lyapunov’s exponents; saltation matrix; rocking block dynamics.

1. Introduction
The dynamics of a rigid block, free-standing with friction on a moving ground, has been the object of study in the international scientific community for a long time. The interest in this kind of problem lies in the possibility of understanding the seismic behavior of a variety of man-made structures which can be modeled as free-standing rigid blocks, such as monumental stone structures, concrete radiation shields, water towers, power transformers and any other free-standing equipment, including furniture such as art objects stored in museums.

The first investigations dating back to the end of the 19th century [Milne, 1881, 1885] were motivated by the opportunity and possibility of using information about stability with regard to the overturning of stone blocks, as this may be indicative of seismic intensity in places without seismographs. After more than a century, and an intense period of research in recent times (among others: [Housner, 1963; Spanos & Kok, 1984; Sinopoli, 1987; Hogan, 1989; Yim & Lin, 1991; Augusti & Sinopoli, 1992; Iyengar & Roy, 1996; Virgin et al., 1996; Plaut et al., 1996]), the question of stability with respect to the earthquake of these objects seems nonetheless to be a matter which bears further investigation.

The dynamics of such a system is in fact so complex that it can be taken as emblematic of themes emerging in modern nonlinear dynamics: the unilateral character of the contact link with the ground,
the presence of Coulomb friction in the points maintaining contact, and the velocity discontinuities which occur each time the block impacts the ground have the effect that, during its dynamic evolution, the system is characterized by different kinds of discontinuity that can be manifested at the same instant, or in different instants.

In particular, depending on the values of the geometric and mechanical parameters and the initial conditions, the system can evolve according to diverse mechanisms governed by different systems of differential equations.

To give a schematic idea of the problem, consider the motion of a rigid block (Fig. 1), referred to a moving ground reference system \((O,x,y)\), and let \((x_G,y_G,\theta)\) be the Lagrangian coordinates of the plane motion, where \(G\) is the center of mass and \(\theta\) the rotation angle; in addition, let \(M\) be the mass of the block, \(b\) and \(h\) the base and height length, respectively, and \(a_H = \beta \alpha g \cos(\Omega \tau + \phi)\) a harmonic ground acceleration, with \(\tan \alpha = b/h\) and \(g\) the gravity acceleration.

Since a unilateral constraint does not reduce the degrees of freedom, the differential equations governing the Lagrangian coordinates, during the phases of continuous motion, can be written as

\[
\begin{align*}
M\ddot{x}_G &= \lambda_1 + M\dot{x} \\
M\ddot{y}_G &= \lambda_2 - Mg \\
I_G\ddot{\theta} &= R[\lambda_1 \cos(\alpha - |\theta|) - \text{sgn} \cdot \lambda_2 \sin(\alpha - |\theta|)]
\end{align*}
\]

where \(R\) is the half-diagonal of the block, \(I_G\) the inertia moment, and \(\lambda_1\) and \(\lambda_2\) the horizontal and vertical reactions due to the ground; they are governed by Coulomb’s law and the contact law of impenetrability, respectively.

Even taking into account the case in which the block is initially at rest, the activation, either at the beginning or during the dynamic evolution of one, two or three of the system’s equations (1), according to a given mechanism, depends on the geometric size of the block (namely, \(b\) and \(h\)), on the value of the friction coefficient \(\mu\) and on the instantaneous value of the forcing acceleration. Thus, phases of resting, rocking or sliding-rocking around the base corners, sliding or up-lift are possible, and the structure of the differential equations changes since specific analytical expressions of the ground reactions \(\lambda_1\) and \(\lambda_2\) correspond to each activated mechanism.

The continuous dynamical phases are matched at given instants according to switching conditions, which make the system discontinuous according to three different types of possible discontinuities:

(a) the vector field is continuous, but nonsmooth due to a discontinuous Jacobian. This is the case, for example, in the transition from sliding-rocking to rocking,

(b) differential equations with a discontinuous right-hand side describe the system and the vector field is discontinuous. This is the case of the stick-slip motion governed only by Coulomb’s friction.

![Fig. 1. Positive anti-clockwise rotation angle of the block and positive ground acceleration.](image-url)
The impact instants are, in particular, those in which all three kinds of discontinuities can occur simultaneously; the velocities discontinuity due to the impact is governed by the impact law.

Discontinuous systems of this kind are often called switching or hybrid systems [Leine, 2000], since they reveal a mixed continuous and discrete nature.

The complexity of the dynamics of a free-standing block on a moving ground presents singular aspects which are part of different, specific research fields. There are formal aspects of theoretical completeness of the dynamic formulation: problems of this kind can be considered as a dynamical extension of linear complementarity problems and, in convex analysis, the corresponding dynamic equations can be formulated as differential inclusions [Moreau, 1988; Glocker & Pfeiffer, 1992]. In such cases, the technical effort is mainly concentrated on aspects of formal completeness: once the initial conditions have been fixed, computational techniques are usually adopted to follow the dynamic evolution; however, these do not allow for any prediction about the system behavior as a function of the geometrical and mechanical parameters, if not at the expense of extensive and heavy-going parametric numerical investigations.

On the other hand, there is another vast line of research in the scientific literature that, motivated by practical, engineering aims in connection with the need to evaluate the seismic safety of these systems, has assumed simplifications of the general model by means of an a priori reduction in the degrees of freedom, i.e. by adopting the assumption of friction great enough to allow only the rocking mode, and the use of restitution coefficients to take the variations in velocity due to impacts into account [Housner, 1963; Spanos & Koh, 1984; Hogan, 1989; Yim & Lin, 1991; Iyengar & Roy, 1996; Virgin et al., 1996; Plaut et al., 1996].

These works, starting with the pioneering paper by Housner [1963], have made it possible to obtain information about the features of motion and its stability by means of analytical and numerical investigations. It must, however, be borne in mind that the system behavior is strongly nonlinear in the case of rocking as well, due to the intrinsic nature of the oscillations around the base corners and the impacts [Sinopoli, 1991]. In this respect, the intensive analytical stability analyses performed, namely, the determination of conditions for the existence of symmetric periodic solutions and their stability [Spanos & Koh, 1984; Hogan, 1989; Yim & Lin, 1991], have considered only linearized equations of motion, matched at the impact instants. However, it is precisely the discontinuous and impulsive character of the dynamics at the impact time that make such investigations extremely complex and delicate, and thus material for further study.

In fact, while profound insight into the behavior of dynamic systems can be gained from modern analysis techniques developed in nonlinear dynamics, theoretical knowledge and available methods are generally applied to smooth dynamical systems; the fundamental solution matrix, the eigenvalues of which are the Floquet multipliers, can be obtained in an elegant manner by integrating the so-called variational equation. The extension of these methods to noncontinuous systems is a new field of research. In particular, discontinuous systems exhibit discontinuities or “saltations” in the time evolution of the fundamental solution matrix, and these “saltations” must be accurately analyzed and evaluated if reliable results are to be obtained.

Lastly, there is a third way of tackling the dynamic analysis of a free-standing block with friction on a moving foundation: this has been adopted by the authors who, without renouncing the complexity of the general formulation of the problem, are trying by means of analytical or semi-analytical methods to obtain information on the dynamic behavior of the system as a function of its mechanical and geometric parameters [Sinopoli, 1987, 1991; Augusti & Sinopoli, 1992; Sinopoli, 1993, 1997; Sinopoli & Ageno, 2001]. This is the philosophy that has led to the introduction of a dynamic formulation [Sinopoli, 1997] which makes it possible to examine phases of smooth or nonsmooth dynamics and determine both the activated mechanism and instantaneous acceleration. Such a formulation allows a geometric representation of the configuration space, which permits us to visualize the instantaneous dynamic situation, particularly in those cases in which it seems unclear because of friction, and therefore improve our understanding of the problem.

The first result was the analytical determination of regions of the parameters plane $(\mu, b/h)$...
where a given mechanism is activated, considering the block as initially at rest [Sinopoli, 1997].

Moreover, a numerical code has since been implemented, which includes the numerical adaptation of the geometric representation in the configuration space in order to permit its use during the numerical integration [Sinopoli & Ageno, 2001]. An appropriate selection of the values of the parameters has therefore made it possible to obtain numerically periodic responses. These have been compared with the results obtained in the literature, both for large values of friction, a case in which our general formulation automatically “selects” the rocking mode, and for values of friction which correspond to the activation of sliding-rocking, about which not much has been written [Shenton & Jones, 1991a, 1991b]. It has also been possible to analyze some behaviors of the response as a function of the friction coefficient in the regions close to the boundary which separates the start of the rocking and sliding-rocking mode [Sinopoli & Ageno, 2001].

The stability analysis of these responses, however, requires the extension of concepts relative to variational-perturbative analysis (variational equations of perturbations, eigenvalues of their solutions and Lyapunov’s exponents) to our problem. In this case, it is in fact a question of adapting definitions and procedures originally applicable to continuous problems, to a problem which is continuous in parts. In addition, since the order of the variational differential system is equal to the number of the d.o.f.s (this creates a much greater burden in computing and programming for the sliding-rocking mode compared with the rocking mode), it was decided to experiment and test the instruments for variational analysis first of all by limiting the field to the rocking mode, to avoid complicating an aspect of the research that still needs to be refined theoretically before it can be applied.

The path followed in this paper will be the adaptation of numerical methods in order to obtain characteristic multipliers and Lyapunov’s exponents for the rocking mode. Bearing the general problem in mind, the end purpose is in fact to calculate these stability indicators numerically in any case. In Sec. 2, the exact rocking equations of motion and the impact law will be given; Secs. 2.1–2.3 are devoted to the linearized equations of motion and their corresponding solutions, which can be obtained under the assumptions of slender block and small angles. The basic idea of stability analysis and the concept of the “saltation” matrix will be discussed in Sec. 3. The analytical expression of the “saltation matrix” and its application to the linearized equations of motion will be analyzed in Sec. 4; in this case analytical methods can be used to identify regions where symmetric periodic responses may exist and to compare them with the results obtained in the literature. Numerical integration of the equations of motions and numerical evaluation of Lyapunov’s exponents will be carried out in Sec. 5.

The results obtained seem to be encouraging; this means that our investigations can be widened to include and analyze the more complex sliding-rocking motion in the near future.

2. Rocking Mode and Equations of Motion

Consider the block shown in Fig. 1, simply supported on a rigid ground: it has a base with length $h$ between corners A and B, while $h$ is its height. The block is assumed to be rigid and uniform and the center of gravity $G$ is at the same distance $R$ from A and B. The mass is $M$, the mass moment of inertia about $G$ is denoted by $I_G$ and those about A and B by $I_A = I_B = I_G + M \cdot R^2$; $\alpha$ is the angle included between $R$ and the vertical side of the block, given by $\tan \alpha = b/h$.

If the block is slender and the value of dry friction coefficient is large enough, so that any sliding on the support surface can be overlooked at any time, the motion is an oscillation starting around either base corner edge A or B, depending on the phase of the ground motion.

Let $\theta = \alpha \cdot x$ be the Lagrangian coordinate which describes the rotation of the block and assume that it has either a positive or negative value, if the rocking is either around point A or B; moreover, let $\theta = 0$ when the edge AB of the block is in full contact with the ground.

Assume a horizontal motion of the ground: $x_2(\tau) = x_0 \cdot \cos(\Omega \tau + \phi)$; thus, the block is subjected to a forcing acceleration given by:

$$a_y = g \cdot \sin \phi \cdot (\Omega \tau + \phi)$$

(2)

where $g$ is the gravity acceleration, $\alpha$ the block angle, $\beta$ the nondimensional amplitude, $\Omega$ the frequency, $\tau$ the time and $\phi$ the phase angle; in the following we shall assume $\phi = 0$, but in general with $\tau_0 \neq 0$, where $\tau_0$ is the initial time.

Following the above assumptions, the equations of motion for positive and negative values of the angle, denoted by $\theta_1$ and $\theta_2$ respectively, can be
written as
\[ \dot{\theta}_1 - a_H \ \frac{MR}{I_0} \ \sin(\alpha - \theta_1) = 0 \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (3a) \]
\[ \dot{\theta}_2 - a_H \ \frac{MR}{I_0} \ \sin(\alpha + \theta_2) = 0 \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (3b) \]

Adopt the nondimensional time: \( t = \frac{p}{\gamma} \), where \( p \) is the so-called “eigen-frequency” of the block
\[ \gamma^2 = \frac{MgR}{I_0} - \frac{3g}{4R} \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (4) \]

The nondimensional frequency becomes: \( \omega = \Omega/p \), and the nondimensional equations of motion can be written as
\[ \ddot{x}_1 - \frac{1}{\alpha} \sin[\alpha(1 - x_1)] = 0 \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (5a) \]
\[ \ddot{x}_2 - \frac{1}{\alpha} \sin[\alpha(1 + x_2)] = 0 \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (5b) \]

where the differentiation is with respect to the nondimensional time \( t \).

Equations (5a) and (5b) are matched at the instants \( t_j \), that is, every time the block impacts on the base; in this case, the indicator function \( x_i(t_j) \) is equal to zero
\[ x_i(t_j) = 0 \ \ (i = 1, 2) \ \ \ \ \ \ (6) \]
and the matching conditions follow from the law of impact.

Assume inelastic impact; thus, after a dissipation of energy has occurred with the impact, the motion continues with a rotation around the opposite corner of the block. The angular velocity discontinuity can be evaluated both from the assumption of conservation of angular momentum about the edge corners [Housner, 1963] or from the solution of the corresponding sweeping problem [Sinopoli, 1997]. In the case of slender blocks and relatively high friction coefficients, both led to the same expression for the ratio \( e \) between the angular velocity just after and just before the impact
\[ e = \frac{\ddot{x}_2}{\ddot{x}_1} = 1 - \frac{3}{2} \sin^2 \alpha, \ \ \ \ \ \ (7) \]

Observe that inelastic impact here means that, at the end of the impact, points maintaining contact are characterized by zero relative velocity in the direction normal to the surface of contact (thus, absence of bouncing). Other assumptions regarding the impact law, such as for example elastic behavior, are not considered in this paper, since the practical applications assumed as starting point of our analysis concern stone structures such as ancient stone buildings and columns for which any bouncing effect can be ignored.

It should be noted, however, that often authors have preferred to deal with coefficients of restitution of an experimental nature [Spanos & Koh, 1984; Hogan, 1989]; thus, the restitution coefficient \( e \) has been allowed to vary by assuming different values between \( 0 \) and \( 1 \). In this sense, from now on we shall use different values of the restitution coefficient only for comparing and discussing the results obtained.

According to Eqs. (5a) and (5b), the rocking of the block is governed by nonlinear differential equations matched at the impact instants \( t_j \); observe that no closed form solution exists for each of Eqs. (5), which characterize the phases of continuous motion.

### 2.1. Linearized equations for slender blocks and small angles

Nevertheless, if we consider very slender blocks \( (\alpha \ll 1) \), Eqs. (5) can be reduced to the linear differential equations
\[ \ddot{x}_1 - x_1 = -1 + \beta \cos \omega t \ \ \ \ \ \ (8a) \]
\[ \ddot{x}_2 - x_2 = 1 + \beta \cos \omega t \ \ \ \ \ (8b) \]

which can be solved exactly; note that for very slender blocks the dependence on the geometric parameter \( \alpha \) has disappeared.

Let the velocity \( x_1 \) be denoted by \( y_1 \), and the conditions at the initial time \( t_0 \) by \( x_0 \) and \( y_0 \); thus, Eq. (8a) is satisfied by
\[ y_1(t) = y_1(t, t_0, x_0, y_0, \omega, \gamma) \]
\[ = (x_0 + \gamma y_0 - 1)[\cosh(t - t_0) + (y_0 - \omega \gamma y_0) \sinh(t - t_0) - \gamma \cos \omega t + 1] \]
\[ (9) \]

\[ y_1(t) = y_1(t, t_0, x_0, y_0, \omega, \gamma) \]
\[ = (x_0 + \gamma y_0 - 1) [\sinh(t - t_0) + (y_0 - \omega \gamma y_0) \cosh(t - t_0) + \gamma \omega \sin \omega t] \]
\[ (10) \]
Similarly, if we denote the velocity \(x_2\) by \(y_2\), and the conditions at the time of impact \(t_1\) by \(x_1\) and \(y_1\), Eq. (8b) is satisfied by

\[
x_2(t) = x_2(t, t_1, x_1, y_1, \omega, \gamma)
\]

\[
= (x_1 + \gamma c_1 + 1) \cosh(t - t_1) + (y_1 - \omega \gamma s_1) \sinh(t - t_1) - \gamma \cos \omega t - 1
\]

\[
y_2(t) = y_2(t, t_1, x_1, y_1, \omega, \gamma)
\]

\[
= (x_1 + \gamma c_1 + 1) \sinh(t - t_1) + (y_1 - \omega \gamma s_1) \cosh(t - t_1) + \gamma \omega \sin \omega t
\]

where

\[c_0 = \cos \omega t_0, s_0 = \sin \omega t_0, \gamma = \frac{\beta}{1 + \omega^2}\]

### 2.2. Linearized equations of motion for small angles

Observe that Eqs. (8a) and (8b) have been obtained under the assumption of very slender blocks, that is, for \(\alpha \ll 1\). Thus, since the nondimensional angle \(x\) is equal to \(\theta/\alpha\), the assumption of slender blocks coincides with that of small angles \(\theta\).

Let us consider separately these two assumptions; that is, assume only small oscillation angles: \(x \ll 1\), by allowing the block angle \(\alpha\) to vary. In this case, the linearized equations of motion become

\[
x_1 = \frac{\cos \alpha + \alpha \sin \alpha \cdot \beta \cos \omega t}{\alpha} \cdot x_1 > 0 \quad \text{(13a)}
\]

\[
x_2 = \frac{\sin \alpha + \alpha \cos \alpha \cdot \beta \cos \omega t}{\alpha} \cdot x_2 < 0 \quad \text{(13b)}
\]

Each of Eqs. (13) is a Mathieu equation, that is, a linear differential equation with time periodic coefficients. No closed form solutions exist for Eqs. (13), which now also depend on the block angle \(\alpha\). Therefore, under the assumption of small angles, the motion of the block is governed by the solutions of Eqs. (13), matched at the instants \(t_j\) of impacts.

The periodic perturbative terms in each of Eqs. (13) are infinitesimal terms of second order with respect to the magnitude order of the solution.

It is of course well known that, for smooth systems, such perturbative terms are responsible for the existence or non-existence of periodic solutions; in fact, even if no closed form solutions exist, the Floquet theory allows regions of existence of stable periodic solutions to be identified as a function of \(\beta, \alpha\) and \(\omega\).

In our case, however, the solutions are matched at the impact instants, so that the impact and the corresponding velocity discontinuity behaves as a filter which can change the stability character of the solution. In particular, it is worthwhile to observe that the effects of perturbative terms increase with the block angle \(\alpha\) during the phase of continuous motion, but they are completely absent during impacts, for which \(x_{1,2} = 0\).

Thus, in Eq. (13a), if we ignore the perturbative terms and denote the velocity \(x_1\) by \(y_1\) and the initial conditions at time \(t_0\) by \(x_0\) and \(y_0\), the corresponding solutions are

\[
x_1(t) = \left(x_0 + \gamma_0 s_0 - \frac{\sin \alpha}{\omega \cos \omega t} \cos \omega t - \frac{\sin \alpha}{\alpha \omega^2} \cosh \omega t - t_0\right)
\]

\[
+ \frac{1}{\omega^2} (y_1 - \omega \gamma_0 s_1) \sinh \omega t - t_0
\]

\[
- \frac{\gamma_0 \cos \omega t}{\omega \sin \omega t} \frac{\sin \alpha}{\alpha \omega^2}
\]

\[
y_1(t) = \omega \left(x_0 + \gamma_0 c_0 - \frac{\sin \alpha}{\alpha \omega^2} \sin \omega t - t_0\right)
\]

\[
+ (y_1 - \omega \gamma_0 c_1) \cosh \omega t - t_1 + \gamma_0 \omega \sin \omega t
\]

Similarly, if we denote the velocity \(x_2\) by \(y_2\), and the initial conditions at the time of impact \(t_1\) by \(x_1\) and \(y_1\), then the solutions of Eq. (13b) are

\[
x_2(t) = \left(x_1 + \gamma_0 c_1 - \frac{\sin \alpha}{\alpha \omega^2} \cos \omega t - t_1\right)
\]

\[
+ \frac{1}{\omega^2} (y_1 - \omega \gamma_0 s_1) \sinh \omega t - t_1
\]

\[
- \frac{\gamma_0 \cos \omega t}{\omega \sin \omega t} \frac{\sin \alpha}{\alpha \omega^2}
\]

\[
y_2(t) = \omega \left(x_1 + \gamma_0 c_0 + \frac{\sin \alpha}{\alpha \omega^2} \sin \omega t - t_1\right)
\]

\[
+ (y_1 - \omega \gamma_0 s_1) \cosh \omega t - t_1
\]

\[
+ \gamma_0 \omega \sin \omega t
\]
with
\[ \omega_1^2 = \cos \alpha, \quad c_{0,1} = \cos \omega_{0,1}, \quad s_{0,1} = \sin \omega_{0,1}, \quad \gamma = \frac{\beta}{1 + \omega^2}. \]

2.3. Linearized equations and nonlinear character of the solutions

The linearized equations of motion in conjunction with the nonlinear transition conditions at the impact instants \( t_1 \) make the system discontinuous, so that features similar to those characteristic of nonlinear systems are expected. In addition to the fundamental harmonic response one can expect to find, for instance, subharmonic, super-harmonic, quasi-periodic and chaotic responses.

Thus, the analytical methods allowed by the linearized equations (either Eqs. (8) or (13)), connected by the impact conditions, can guide our understanding of a wide range of system behaviors, while at the same time increasing the usefulness and efficacy of numerical methods applied to the whole nonlinear problem. In particular, the analytical stability analysis of fundamental and subharmonic steady-state responses provides interesting sets of boundaries that are useful for establishing the different behaviors of the block when the parameters of the excitation (amplitude \( \beta \) and frequency \( \omega \)) and those of the system (coefficient of restitution \( c \) and block angle \( \alpha \)) change.

Under the assumption of a very slender block, Spanos and Koh were the first to use this approach [1984], followed by Hogan [1989], who refined the mathematical treatment of the problem, but without changing the basic approach to the central investigation as regards the handling of impact discontinuity.

This paper describes a new technique for the rigorous management of the impact discontinuities of the block dynamics in association with the perturbation method as far as the stability analysis is concerned. This technique may be thought of like a perturbation method as far as the stability analysis is concerned. This technique may be thought of like a perturbation method as far as the stability analysis is concerned. This technique may be thought of like a perturbation method as far as the stability analysis is concerned.

3. Stability Analysis of Periodic Responses

The basic idea here is orbital stability, namely the stability of fixed points of the Poincaré maps associated with the periodic responses. As pointed out by Hogan [1989], the simplest choice is the transverse Poincaré section \( S = \{(x, y, t): x = 0, y > 0\} \). In fact, at every impact with positive angular velocity, the trajectory intersects \( S \). Therefore, a map \( P: S \rightarrow S \) is defined by Eqs. (8) or (13) in conjunction with Eq. (7). In order to analyze the stability of periodic solutions we can examine the stability of the fixed point of the Poincaré map belonging to \( S \), by imposing a small perturbation \( \delta x(t_0) \) of some appropriate quantities at the start of a cycle (at \( t_0 \)).

If after the subsequent intersection of the trajectory with \( S \) at the time \( t_2 \) (with \( t_2 = t_0 + 2n\pi/\omega \) if the motion is a harmonic or subharmonic response), the perturbation \( \delta x(t_2) \) decays, the orbit is stable; otherwise it is unstable. The final and initial variations are related by the equation determined by linearizing \( P \) at the fixed point

\[ \delta x(t_2) = J^* \delta x(t_0), \]

where \( J^* \) is the Jacobian matrix of the Poincaré map. The perturbation decays when the eigenvalues \( \lambda \) of \( J^* \) have modules less than one.

We can further simplify the problem, by introducing other restricting assumptions. In fact, if we consider only rotationally symmetric single impact orbits, \( J^* \) can be restricted to a half period of oscillation and to the first impact after the start of the cycle.

This impact happens at

\[ t_1 = t_0 + \frac{n\pi}{\omega}, \]

where \( n \) indicates the order of the subharmonic symmetric response.

If, for instance, at the start of the cycle, at \( t_0 \), it was \( y_0 > 0 \), at \( t_1 \) we shall have \( y_1 < 0 \), while\(^2\)

\[ x_0 = x_1 = 0. \]

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1. An \((m, n)\) orbit indicates when \(2m\) impacts occur per orbit and when the forcing goes through \( n \) periods in one orbit.

2. Symmetric orbits are only possible for \( n \) odd. This can be seen by considering (14) and (16) and the condition of symmetry:

\[ x_1(t) = -x_2(t + (m\pi)/\omega). \]
Now questions arise about:

(1) the elements of the perturbation vector $\delta x$;

(2) the meaning and calculation of the Jacobian matrix $J$, that also takes into account the effects due to impact.

The two questions are in fact related: the opportunity for a rigorous evaluation of the Jacobian matrix after the impact, allowed by the introduction of an appropriate “saltation” matrix, leads to the indication of the variations in angular displacement and angular velocity (in a way similar to the smooth dynamics analyses) as being advantageous, rather than those in excitation phase and angular velocity, formerly used by Spanos and Koh [1984], Hogan [1989] and other authors. We shall adopt these variations in what follows.

3.1. Fundamental and “Saltation” matrix

Consider the subharmonic response $(1,n)$; if the steady-state motion starts at $t = t_0$, with $\mathbf{x}(t_0) = 0, y(t_0) = y_0$, the time $t_1$ of the first impact is such that

$$t_1 - t_0 = \frac{n \pi}{\omega}$$

$$x_1(t_0) = 0$$

Let us consider some perturbations, $\delta x_0, \delta y_0$, of the initial conditions. The perturbations of position and velocity during the motion are functions of time, and the impact of continuous dynamics can be calculated using the Jacobian $J$

$$\begin{bmatrix} \delta x_2(t) \\ \delta y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial x_2(t)}{\partial y_0} & \frac{\partial x_2(t)}{\partial y_0} \\ \frac{\partial y_2(t)}{\partial y_0} & \frac{\partial y_2(t)}{\partial y_0} \end{bmatrix} \begin{bmatrix} \delta x_0 \\ \delta y_0 \end{bmatrix}$$

$$= J \cdot \delta x(t_0)$$

(21)

Perturbations of position and velocity at the impact instants, on the other hand, require special evaluation since the dynamics is discontinuous and, moreover, the impact instant $t_1$ undergoes a perturbation $\delta t_1$ (which we presume to be positive: $\delta t_1 > 0$; see Fig. 2). If position, velocity and impact instant of the perturbed motion are indicated by an over-line, the variations of position and velocity at the instant $\mathbf{r}_1 = t_1 + \delta t_1$ are

$$\delta x_2(\mathbf{r}_1) = \mathbf{r}_2(\mathbf{r}_1) - x_2(\mathbf{r}_1) = \mathbf{r}_1(\mathbf{r}_1) - x_2(\mathbf{r}_1)$$

$$= 0 - x_2(\mathbf{r}_1)$$

$$\delta y_2(\mathbf{r}_1) = \mathbf{y}_2(\mathbf{r}_1) - y_2(\mathbf{r}_1)$$

(22)

Note that the perturbed motion is continuous at $t_1 (\delta t_1 > 0)$, whereas the unperturbed motion is continuous at $\mathbf{r}_1 = t_1 + \delta t_1$. Thus, by linearizing with respect to time at instant $t_1$ and by taking the impact law into account

$$-x_2(\mathbf{r}_1) = -x_2(t_1) - \frac{\partial x_2}{\partial t} |_{t=t_1} \delta t_1$$

$$= 0 - \frac{\partial x_2}{\partial t} |_{t=t_1} \delta t_1$$

$$= -y_2(t_1) \delta t_1 = -y_2(t_1) \delta t_1$$

$$\mathbf{y}_2(\mathbf{r}_1) = \mathbf{y}_2(\mathbf{r}_1) = e^{\mathbf{J}(t_1)} = e^{\mathbf{J}(t_1) + \delta t_1}$$

(23)

$$= e^{\mathbf{J}(t_1) + \mathbf{y}_1(\mathbf{r}_1) \delta t_1}$$

$$= e^{\mathbf{y}_1(\mathbf{r}_1) + \mathbf{y}_1(\mathbf{r}_1) \delta t_1}$$

Thus, by linearizing

$$-y_2(t_1) = y_2(t_1) + \frac{\partial y_2}{\partial t} |_{t=t_1} \delta t_1$$

$$= y_2(t_1) + \frac{\partial y_2}{\partial t} |_{t=t_1} \delta t_1$$

(24)

so that position and velocity variations at the instant $\mathbf{r}_1 = t_1 + \delta t_1$ can be expressed as

$$\delta x_2(\mathbf{r}_1) = -y_2(t_1) \delta t_1$$

$$\delta y_2(\mathbf{r}_1) = e^{\mathbf{y}_1(t_1)} + \mathbf{y}_1(t_1) - \mathbf{y}_1(t_1) \delta t_1$$

(24)

To obtain relationships (24) as a function of initial perturbations $\delta x_0$ and $\delta y_0$, consider the variation indicatrix function:

$$\mathbf{r}_2(\mathbf{r}_1) = \mathbf{r}_2(\mathbf{r}_1) = \mathbf{r}_1(\mathbf{r}_1) + \mathbf{y}_1(\mathbf{r}_1) \delta t_1$$

(25)

which gives $\delta t_1$ as

$$\delta t_1 = -\frac{\delta x_2(t_1)}{y_1(t_1)}$$

$$= \frac{1}{y_1(t_1)} \left[ \frac{\partial x_1(t_1)}{\partial x_0} \delta x_0 + \frac{\partial x_1(t_1)}{\partial y_0} \delta y_0 \right]$$

(26)
so that

\[ \delta x_2(t_1) = \epsilon \delta x_1(t_1) = \epsilon \left[ \frac{\partial x_1(t_1)}{\partial x_0} \delta x_0 + \frac{\partial x_1(t_1)}{\partial y_0} \delta y_0 \right] \]

\[ \delta y_2(t_1) = -\frac{y_1(t_1) - \delta y_1(t_1)}{y_1(t_1)} \left[ \frac{\partial x_1(t_1)}{\partial x_0} \delta x_0 + \frac{\partial x_1(t_1)}{\partial y_0} \delta y_0 \right] + \epsilon \left[ \frac{\partial y_1(t_1)}{\partial x_0} \delta x_0 + \frac{\partial y_1(t_1)}{\partial y_0} \delta y_0 \right] \]

(27)
If follows that where $S$ and $T$ are the determinant and the trace respectively the determinant and the trace into account.

Let us denote by $D(\lambda)$ and $J(\lambda)$ the Jacobian of the system for the phases of continuous motion; the product matrix $J^*J$ is the Jacobian of the system which takes the velocity discontinuities into account. Therefore, the condition $\lambda_i (i = 1, 2)$ are the eigenvalues of the Jacobian $J^*$, then

$$\lambda_{1,2} = \frac{1}{2} \{ T(J^*) \pm \sqrt{T^2(J^*) - 4D(J^*)} \}.$$  

(29)

Therefore, the condition $|\lambda_i| = 1$ implies that $D(J^*)$ and $T(J^*)$ are related in the equation:

$$t_1 = f(t_0, x_0, y_0) = \frac{1}{\omega} \arccos \left( \frac{x_0 + \gamma y_0 - 1}{(x_0 + \gamma y_0 + \omega \gamma y_0) \sin \frac{n \pi \omega}{\gamma}} \right).$$

(32)

Consider the perturbations $\delta x_0$ and $\delta y_0$ of the initial conditions. According to Eqs. 8(a), 8(b), (9), (10) and (28), the perturbations at the instant $t_1 = t_1 + \delta t_1$ are

$$\begin{bmatrix} \delta x_2(t_1) \\ \delta y_2(t_1) \end{bmatrix} = \begin{bmatrix} e \\ e \end{bmatrix} \begin{bmatrix} 0 \\ e \end{bmatrix} \begin{bmatrix} \frac{n \pi \omega}{\omega} \\ \sin \frac{n \pi \omega}{\omega} \end{bmatrix} \begin{bmatrix} \delta x_0 \\ \delta y_0 \end{bmatrix} = J^* \cdot \delta x(t_0),$$

(33)

If follows that

$$D(J^*) = \lambda_1 \lambda_2 = e^2$$

(34)

$$T(J^*) = 2\varepsilon \cosh \left( \frac{n \pi \omega}{\omega} \right) \left( 1 + \varepsilon + (1 - \varepsilon) \beta \cos \omega t_1 \right) \sin \frac{n \pi \omega}{\omega},$$

(35)

4. Piecewise Linearized Equations and Stability Analysis

Apply the results of the previous section to the linearized equations of motion [Eqs. (8) and (13)] and consider the subharmonic response (1, $n$). Recall that, if the steady-state motion starts at $t = t_c$, with $x_1(t_c) = 0$, $y_1(t_c) = y_c$, the time of the first impact $t_1$ and the corresponding value of $x$ are such that

$$t_1 - t_0 = \frac{n \pi}{\omega},$$

$$x_1(t_1) = 0.$$
To solve Eq. (31), observe that the angular velocity \( y_1(t_1) \) just before impact is the quantity to be evaluated in the expression of \( T(J^a) \) according to Eq. (35), and that \( y_1(t_1) \) depends also on \( \omega t_1 \) as argument of sine and cosine functions. In any case, an expression of \( y_1(t_1) \) that is extremely useful for the next algebraic developments can be obtained from:

(a) Eq. (10) by putting \( x_0 = 0 \);
(b) Eq. (7) which, under the assumption of rotational symmetry, gives \( y_1(t_0) = y_0 = -e \cdot y_1(t_1) \);
(c) Eq. (19), by noting that \( \cos(\omega t_1) = -\cos(\omega t_0) \), and \( \sin(\omega t_1) = -\sin(\omega t_0) \).

The expression of \( y_1(t_1) \) is therefore:

\[
y_1(t_1) = \frac{(-\gamma c_1 - 1)s_1 + (1 + ch)\gamma \cdot \omega \cdot s_1}{1 + e \cdot ch}
\]

(36)

where, for the sake of brevity, \( ch \) and \( sh \) represents, respectively

\[
ch = \cosh(t_1 - t_0) = \cosh\left(\frac{n\pi}{\omega} \right)
\]

\[
sh = \sinh(t_1 - t_0) = \sinh\left(\frac{n\pi}{\omega} \right)
\]

while \( c_1 \) and \( s_1 \) are, as previously stated, \( \cos(\omega t_1) \) and \( \sin(\omega t_1) \).

Note that the quantity \( \omega t_1 \) in (36) is still unknown. It is the solution of the following equation:

\[
X + s_1Y + c_1Z = 0,
\]

(37)

\[
\beta_n(\omega) = (1 + \omega^2) \cdot \sqrt{\frac{\text{sh}^2\left[(1 + e)K_2 + K_1/R_1 + \text{sh}^2(Q_1(1 - e^2) + K_2\omega^2(1 - e^2))]^2}{\left(Q_1(e - 1)\text{sh}^2 + K_1(1 + ch)^2(1 + e)^2R_1\right)^2 + (1 - e^2)(1 - ch^2)}}
\]

(39)

In relationship (39), index \( n \) denotes the \( n \) subharmonic response. We obviously obtain two curves, depending on the fact that either \( K_{1,up} \) or \( K_{1,lo} \) is substituted for \( K_{1,lo} \) will give the upper stability boundary \( \beta_{n,up}(\omega) \), whereas \( K_{1,lo} \) will give the lower stability boundary \( \beta_{n,lo}(\omega) \).

As a first example of the stability boundaries obtained, consider the case with: \( b/h = 1/4 \) and \( e = 0.925 \). By using this value of \( e \), rather than the theoretical one: \( e = 0.912 \) [Housner, 1963; Sinopoli, 1987] just to make comparison with the literature easier [Samous & Koh, 1984; Hogan, 1989], we obtain the boundaries curves for the subharmonic responses until \( n = 7 \) (Figs. 3 and 4).

where

\[
X = (1 - e)(1 - ch)
\]

\[
Y = \gamma \omega \text{sh}(1 - e)
\]

\[
Z = -\gamma (1 + e)(1 + ch)
\]

Equation (37) has been obtained by making the expression of \( y_1(t_1) \) given by (10) equal to the expression of \( y_1(t_1) \) given by (9) at the instant \( t_1 \), through the assumption of rotational symmetry:

\[
y_1(t_0) = -e \cdot y_1(t_1).
\]

The expression of \( \omega t_1 \) is, therefore:

\[
\omega t_1 = a \tan\left(\frac{Y}{X}\right) + \arccos\left(\frac{-X}{Y^2 + Z^2}\right).
\]

(38)

By substituting relationships (36) and (38) in the expression of \( T(J^a) \) at the left side of Eq. (31), it is possible to solve the corresponding algebraic equation and obtain the analytical expression of functions \( \beta_\omega \), for different values of \( n \) and \( e \); they define the stability boundaries of symmetric subharmonic responses for very slender blocks.

Let us put:

\[
K_{1,up} = e^2 + 2 \cdot e \cdot ch
\]

\[
K_{1,lo} = -e^2 - 2 \cdot e \cdot ch
\]

\[
K_2 = 1 + e \cdot ch
\]

\[
Q_1 = K_1 + (1 + \omega^2) \cdot (1 - e) \cdot K_2
\]

\[
R_1 = (e - 1)^2\omega^2\text{sh}^2 + (1 + e)^2(1 + ch)^2
\]

thus, the solutions of Eq. (31) are:

As can be observed from Figs. 3 and 4, by increasing \( \omega \) starting from zero, the values of \( \beta_{n,lo}(\omega) \) first coincide with the values of \( \beta_{n,up}(\omega) \), then they overtake the values of \( \beta_{n,up}(\omega) \) until an intersection point, after which they are finally lower: from this intersection point on, we can consider the actual stability boundaries for the subharmonic responses.

It is of interest to compare these stability boundaries with those obtained by Spanos and Koh [1984] and Hogan [1989]. A comparison between upper and lower boundaries are respectively shown in Figs. 5 and 6, only for the first two subharmonic
Fig. 3. Stability boundaries of subharmonic responses ($n = 1, 3$), for slender block with $h/h = 1/4$ and $c = 0.925$.

Fig. 4. Stability boundaries of subharmonic responses ($n = 5, 7$), for slender block with $h/h = 1/4$ and $c = 0.925$. 

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Fig. 5. Comparison with the upper boundaries \((n = 1, 3)\) obtained by Hogan [1989], for \(b/h = 1/4\) and \(e = 0.925\).

Fig. 6. Comparison with the lower boundaries \((n = 1, 3)\) obtained by Hogan [1989], for \(b/h = 1/4\) and \(e = 0.925\).
responses; pedix H denotes the stability boundaries obtained by Hogan; the upper boundaries are slightly different, whereas the lower boundaries actually coincide.

4.2. Stability analysis for small angles

Now express the Jacobian $J^*$ which takes the velocity discontinuities into account, according to Eqs. (13)–(15) and (28). The determinant $D(J^*)$ does not vary ($D(J') = e^3$), whereas the trace $T(J^*)$ becomes

$$T(J^*) = 2e \cosh \omega_0 \left( \frac{n\pi}{\omega} \right) + \sinh \omega_0 \left( \frac{n\pi}{\omega} \right)$$

and (40), where the difference between the expressions (40) and (35), valid only for very slender blocks, increases with the block angle $\alpha$.

The velocity $y_1(t_1)$ must thus be evaluated according to (15)

$$y_1(t_1) = \frac{\sin \alpha - \gamma(\omega_0\omega_0) \sinh \omega_0 \gamma_0(1 + \cosh)}{1 + e \cosh} (41)$$

so that

$$\beta_2(\alpha) = \left( 1 + \frac{e^2}{\omega_0^2} \right)$$

$$\omega_0 = \cos(\alpha), \quad \omega_0 = \sin(\alpha), \quad \sinh \left( \frac{n\pi}{\omega} \right)$$

where, for brevity, $c_0$ and $s_0$ now represent

$$c_0 = \cosh \omega_0 \left( \frac{n\pi}{\omega} \right)$$

and

$$s_0 = \sinh \omega_0 \left( \frac{n\pi}{\omega} \right)$$

while $c_1$ and $s_1$ are, as already stated, $\cos \omega_1$ and $\sin \omega_1$, respectively. The expression of $\omega_1$ can be evaluated in a manner similar to that of the previous section. Thus $c_1$ and $s_1$ satisfy the following equation

$$X_0 + Y_0 s_1 + Z_0 c_1 = 0 \quad (42)$$

where this time

$$Y_0 = \gamma_0 s_0 (1 - e \frac{\omega_0}{\omega_0})$$

$$Z_0 = -\gamma_0 [c_0 (e + \omega_0^2) + e (e^2 c_0^2 - s_0^2) + 1]$$

$$X_0 = \sin \alpha \gamma_0 c_0 (e - \omega_0) - e (e^2 c_0^2 - s_0^2) + 1$$

so that $\omega_1$ becomes

$$\omega_1 = \tan \left( \frac{Y_0}{Z_0} \right) + \arccos \left( \frac{-X_0}{\sqrt{X_0^2 + Y_0^2}} \right). \quad (43)$$

If we evaluate the stability boundaries of the subharmonic responses by considering (40), (41) and (43), the resulting expression for the lower and upper boundaries is

$$R_2 = c_0 (e + \omega_0^2) + e (e^2 c_0^2 - s_0^2) + 1,$$

$$R_1 = (1 - e)^2 s_0^2 + [c_0 (e + \omega_0^2) + e (e^2 c_0^2 - s_0^2)]$$

The boundaries of the subharmonic responses characterized by $n = 1.3$ and $n = 5.7$ are represented in Figs. 7 and 8, respectively, for $b/h = 1/4$ and a value of the restitution coefficient preserving the angular momentum: $e = 0.012$. The effects of the block angle $\alpha$ on the width of the stability regions can be seen by comparing with
Fig. 7. Small angles assumption. Stable boundaries \((n = 1, 3)\) for \(b/h = 1/4\) and \(e = 0.912\).

Fig. 8. Small angles assumption. Stable boundaries \((n = 5, 7)\) for \(b/h = 1/4\) and \(e = 0.912\).
Fig. 9. Comparison with the results obtained by Hogan [1989] for \( n = 1 \), with \( h/b = 1/4 \) and \( \epsilon = 0.912 \).

Fig. 10. Comparison with the results obtained by Hogan [1989] for \( n = 3 \), with \( h/b = 1/4 \) and \( \epsilon = 0.912 \).
Fig. 11. Regions of stable subharmonic responses \((n = 1, 3)\), for \(b/h = 1/2\) and \(e = 0.7\).

Fig. 12. Regions of stable subharmonic responses \((n = 5, 7)\), for \(b/h = 1/2\) and \(e = 0.7\).
Fig. 13. Comparison with the results obtained by Hogan [1989] for $n = 1$, with $b/h = 1/2$ and $\epsilon = 0.7$.

Fig. 14. Comparison with the results obtained by Hogan [1989] for $n = 3$, with $b/h = 1/2$ and $\epsilon = 0.7$. 
5. Numerical Evaluation of Lyapunov’s Exponents

The method introduced in Sec. 3.1 and used for handling the fundamental matrix during impact discontinuities has been combined with the numerical procedure, discussed in a previous paper [Sinopoli & Ageno, 2001], to solve the system of variational equations during the phases of continuous motion.

Some runs were performed for \( b/h = 1/4 \), taking a constant value of \( \omega = 5 \) and increasing \( \beta \) from 16 to higher values. In fact, we know from the results of the linearized theory that, for small angles, \( \beta_{\text{up} \omega} (5) = 16.033 \) is the upper boundary value for \( \beta \) at \( \omega = 5 \). The same initial conditions were imposed as those which can be analytically calculated in the linearized theory for (1,1) or (1,3) response; these conditions obviously correspond, in the exact theory, to a transient phase, so that it is necessary to wait for some time to obtain steady-state motion and stable mean values of eigenvalues and Lyapunov’s exponents.

Runs were also executed for \( \beta < 16 \) with the aim of checking the switching behavior among different subharmonics.

Eigenvalues and Lyapunov’s exponents associated with the fundamental matrix were evaluated at each instant. Some interesting plots are drafted in the following figures.

The time history of the angular displacement of stable (1,1) response, for \( \beta = 2 \) and \( \omega = 5 \), is shown in Fig. 15(a); the corresponding Poincaré map in the phase-plane is drafted in Fig. 15(b), while the behavior of the real parts of characteristic multipliers is reported in Fig. 15(c).

If we change the initial conditions, again analytically derived according to the linearized theory, in order to switch from (1,1) to the (1,3) response (see Fig. 7), a subharmonic (1,3) motion is in fact found: the time-history of angular displacement is plotted in Fig. 16(a), the Poincaré map is drafted in Fig. 16(b) and the behavior of the first Lyapunov’s exponent is reported in Fig. 16(c).

Overturning was found for a value of \( \beta = 24.5 \), which is greater than that foreseen by the linearized theory; the related plot of angular displacement is in Fig. 17(a); the time-histories of eigenvalues and Lyapunov’s exponents are reported in Figs. 17(b) and 17(c).

Both plots are in accordance with the incipient toppling. In fact, it is worth noting that the first Lyapunov’s exponent reaches zero mean value at about the last impact, just before overturning; note also that the transition to instability is a jump without any other phenomenon.

For values of \( \beta \) lower than 24.5 to about 20, the steady-state of eigenvalues and Lyapunov’s exponents is peculiar. The first eigenvalue shows an oscillation at about a value just below one (the mean value of this oscillation decreases with \( \beta \)), whereas the second eigenvalue decreases by tending to zero very quickly as \( \beta \) increases until \( \beta = 24.5 \). Accordingly, the first Lyapunov’s exponent oscillates at about a value just below zero and the second is often not defined, but is nevertheless negative. For instance, the case with \( \beta = 23.5 \) is plotted in Figs. 18(a)–18(c). The time-history of angular displacement is drafted in Fig. 18(a). The Poincaré map in the phase-plane is illustrated in Fig. 18(b), with the related zoom on the fixed point shown in Fig. 18(c); this behavior ensures that the point is actually an attracting point. Figure 18(d) is the steady state of characteristic multipliers showing the behavior described above. Finally, Fig. 18(e) represents the graph of the first Lyapunov’s exponent; the second Lyapunov’s exponent is not presented since the related characteristic multiplier is
Fig. 15. (a) Time-history of angular displacement; (b) Poincaré map; (c) time-histories of characteristic multipliers (real part) for $n = 1$, with $b/h = 1/4$, $e = 0.932$, $\beta = 2$ and $\omega = 5$. 
Fig. 16. (a) Time-history of angular displacement; (b) Poincaré map; (c) time-history of the first Lyapunov’s exponent for $n = 3$, with $b/h = 1/4$, $e = 0.912$, $\beta = 2$ and $\omega = 5$. 

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Fig. 17. (a) Time-history of angular displacement; (b) time-histories of characteristic multipliers; (c) time-history of Lyapunov’s exponents, for $b/h = 1/4$, $\epsilon = 0.912$ with $\beta = 24.5$ and $\omega = 5$. 
Fig. 18. (a) Time-history of angular displacement; (b) Poincaré map; (c) zoom on Poincaré map; (d) time-histories of characteristic multipliers; (e) time-history of the first Lyapunov’s exponent, for $b/h = 1/4$, $\epsilon = 0.912$ with $\beta = 23.5$ and $\omega = 5$. 

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zero at the steady-state. The asymptotic behavior of the first Lyapunov’s exponent tends to reduce the parts greater than zero in a way which is evident more numerically than graphically as the short time-history plotted in Fig. 18(c) can show; this fact, in addition with the previous consideration about the attracting point of the Poincaré’s map [Fig. 18(c)] should indicate that the point is actually stable.

For values of $\beta$ lower than about 20 the eigenvalues are often complex (whereas above 20 they are always real) and the Lyapunov’s exponents of their real parts are always negative.

Thus, the upper stability boundary evaluated numerically with the exact equations of motion seems to be higher than that obtained with the linearized approaches. This seems true, in particular, above $\beta = 20$ as there was no evidence of a trend towards asymmetric responses, though this may be an open question for further research.

Nevertheless, Lyapunov’s exponents seem: (a) to describe accurately the various states of undergoing stability of the numerically computed motion; (b) to predict the incipient overturning of the block; (c) to be, generally, in agreement with the other features of the dynamic response.

6. Conclusions

In this paper a new technique to evaluate characteristic multipliers and Lyapunov’s exponents for the rocking block — a case where the dynamics exhibits phases of continuous and discontinuous motion — has been proposed; the technique is both an analytical and a numerical application of the general method proposed by Müller [1995].

The basic idea of stability analysis in connection with the concept of “saltation” matrix has been discussed. Moreover, the analytical expression of the “saltation matrix” and its application to linearized equations of motion have allowed to identify the stability regions of symmetric periodic responses as a function of the parameters of the excitation, and to compare them with the results obtained in the literature.

Numerical integration of the equations of motions and numerical evaluation of Lyapunov’s exponents have also been carried out.

Observe that the range of $\beta$ and $\omega$ (excitation parameters) used in the approximated analytical stability boundaries plots, exceeds sometimes values typical of earthquakes. Even though this option may seem to be far from practical applications, it was taken for:

(i) delineating completely the results yielded by the approximated analytical theories;
(ii) developing wide and complete comparisons with the analogous studies found in literature.

Furthermore the analytical study has been useful to validate and guide the results obtained by numerical methods, which can be closer to the practical applications, because they are general and independent from the type of forcing acceleration. This could be also a real seismic ground motion; in fact, future applications of the numerical method should regard other types of forcing ground motion, such as seismic accelerations. Therefore, we shall consider the problem of behavior and role of the stability indicators both for the case of transient motion and for nonperiodic excitation.

Nevertheless, the present results seem to be encouraging: the analytical stability boundaries guide the numerical runs and check the usefulness of the stability indicators obtained by numerical calculations with the new technique introduced for the strong discontinuities of the rocking block dynamics.

References


