Projectors in Nonlinear Evolution Problem: Acoustic Solitons of Bubbly Liquid

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Abstract—The method of one-dimensional disturbances splitting into components of rightward propagating, leftward propagating, and stationary components by projection technique is applied to compressible liquid with bubbles. By such projecting, the fundamental system of equations is transformed to three nonlinear equations of the interacting components. A small parameter is introduced which determines input of nonlinear and dispersive terms. The system is reduced to one of a Korteweg-de Vries type. It is shown that these three-mode evolution equations are approximately reduced to integrable KdV-MKdV equation on a class of initial conditions specified by projecting, soliton solution is presented. © 2000 Elsevier Science Ltd. All rights reserved.

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1. GENERAL REMARKS ABOUT THE PROBLEM OF SEPARATION OF THE WAVES WHICH DIFFER BY DIRECTION OF PROPAGATION

The idea of field presentation as a combination of physically specific components is general in wave theory [1,2]. For weak nonlinear dynamics of essentially one-dimensional evolution, the idea to separate waves according to direction of propagation appears in the pioneering works of Korteweg-de Vries and Burgers, who used a method of slowly varying time. Field disturbances are supposed to be a function of two variables, \( \tau \) (slowly varying time, which is responsible to the form of wave, \( \tau = \mu t, \mu \ll 1 \)) and \( t + r/c \) or \( t - r/c \) for leftward and rightward waves, correspondingly. This way, a problem for nonlinear dynamics has been solved effectively but it permits us to get equations for one concrete mode only. That is a consequence of asymmetric choice of variables: either \( t + r/c, \mu t \) or \( t - r/c, \mu t \) and the consequent procedure based on use of the small parameter \( \mu \), most developed in [3].

It seems to be more effective to separate modes on the level of the basic system of equations. Moreover, it is necessary to include a stationary mode at the same level of description. We do it...
in a vector form considering a complete set of eigenspaces of a certain well-defined operator (see also [4]). A procedure of an overall field separating by matrix projectors which correspond to these subspaces is then mathematically unique and serves for investigation of reciprocal influence of directed and stationary modes in a nonlinear problem [5]. By the projection procedure, the main system is transformed to a system of coupled nonlinear equations. The procedure is algorithmic: one should not take care about presentation of every mode, only to act corresponding projectors on the basic system of equations. The method can easily be applied to a wide range of problems (for example, account of dispersion leads to a system of KdV-Burger equations). Generally, the modes splitting by projectors is often used in quantum mechanics, but this technique is not very habitual and common for gas and liquid dynamics problems [4]. One can also iterate by a small parameter inside the general (nonlinear) evolution operator, resulting in more and more exact equations. Here a hydrodynamic problem of an acoustic pulse propagation in bubbly liquid is taken to illustrate the features of the method proposed.

We present equations which govern compressible liquid with bubbles; matrix projectors for liquid with weak dispersion caused by bubbles presence; the derivation of a coupled KdV system in compressible liquid with bubbles (both for directed and stationary components); further development of the nonsingular perturbation method to construct an evolution equation for single mode interacting with others; the KdV-MKdV equation derivation of such mode; a soliton solution for a KdV-MKdV equation is compared with a solution of a KdV one.

2. BASIC EQUATIONS FOR COMPRESSIBLE LIQUID WITH BUBBLES DYNAMICS

The dynamics of an incompressible bubbly liquid was originally studied by von Wijngaarden, and later, by Prosperetti, who considered the case of energy exchange between gas and liquid [6,7]. They got KdV equations for rightward propagating waves (the boundary regime problem was discussed) in the case of self-action only, since they were bounded by the method they used. Let us consider, in this section, mixtures of compressible liquid and perfect gas. The usual assumptions are made: the bubbles are of the same radius at equilibrium; there are no heat and mass transfer between liquid and gas; characteristic wavelength is much more than the bubble radius, so that the mixture can be treated as a homogeneous continuum.

Values connected with the liquid phase are marked by index ld, with gas—by index g, with mixture—by m. Also, it is supposed that mixture pressure is the same as in the liquid phase [6,7]. Background values are marked by an additional zero. Disturbed values are primed. Use of mass concentration of gas $x$ is preferable to volume $a$ used in [6,7]. The mass concentration is constant in contrast with volume, which needs an additional equation. The only condition we add—that $x$ is constant—is equivalent to the assumption that liquid and gas have the same velocity. The basic hydrodynamic system is

\begin{align}
\frac{\partial \rho{'}_m}{\partial t} + \frac{\partial \rho{'}_m v'}{\partial r} = 0,
\end{align}

\begin{align}
\frac{\partial \rho{'}_ld}{\partial t} - \frac{c^2_l}{\rho_{ld0}} \frac{\partial \rho{'}_ld}{\partial r} - \left( \frac{c^2_l}{\rho_{ld0}} \frac{\rho_{ld}}{\gamma_{ld} - 1} \right) \frac{\rho_{ld} v' \rho_{ld0}}{\partial t} = 0,
\end{align}

\begin{align}
\frac{\partial \rho{'}_m}{\partial t} + \frac{\partial (\rho_m v')}{\partial r} = 0.
\end{align}

Here, $\rho_m$ is the mixture density

\begin{align}
\rho_m = \rho_g \rho_{ld} + (1 - x) \rho_g.
\end{align}

The second equation of (2.1), which expresses the dependence of pressure on density in the adiabatic process $p_{ld} \rho_{ld}^{\gamma_{ld}} = p_{ld0} \rho_{ld0}^{\gamma_{ld}}$, is put on with second-order nonlinear terms only; $p_{ld0}$ means internal pressure of liquid, $\gamma_{ld} = (C_{p,ld}/C_{v,ld})(\partial p_{ld}/\partial p_{ld})_{T=\text{const}}$ $p_{ld0}/p_{ld0}$, $C$ is heat capacity.
Nonlinear Evolution Problem

System (2.1), (2.2) should be completed with some other equations. There are: connection between density and pressure for adiabatic behavior of gas

\[ p_g \rho_g^{\gamma_g - 1} = p_0 \rho_0^{\gamma_g - 1}, \quad (2.3) \]

where \( \gamma_g = C_{p,g}/C_{v,g} \) and a single bubble mass conservation

\[ R^3 \rho_g = R_0^3 \rho_{g0}, \quad (2.4) \]

where \( R \) is the radius of the bubble, and Rayleigh equation for bubble radius dynamics, which was improved by Prosperetti following Keller [8] in assumption of liquid compressibility:

\[ R^3 \frac{\partial^2 R}{\partial t^2} + \frac{3}{2} \left( \frac{\partial R}{\partial t} \right)^2 - \frac{1}{c_{ld}} \left( R^2 \frac{\partial^3 R}{\partial t^3} + 6 R \frac{\partial R}{\partial t} \frac{\partial^2 R}{\partial t^2} + 2 \left( \frac{\partial R}{\partial t} \right)^3 \right) = \frac{p_g - p_0}{\rho_0} \cdot (2.5) \]

The surface tension and viscosity are left out of account. Equations (2.1)-(2.5) form the complete system which allows us to express the second equation of (2.1) in terms of \( \rho_m, p_{ld}, v \). Let us introduce the dimensionless variables \( v^*, p^*, \rho^*, r^*, t^* \) : \( v^* = \frac{\rho_m v}{\rho_m^* \rho_0^*}; \ p_{ld}^* = \frac{\rho_m^2 \rho_0^*}{\rho_{ld}} \); \( \rho_m^* = \epsilon \rho_m \rho_0^*; \ r = \frac{r^* \lambda}{\beta}; \ t = \frac{t^* \lambda}{\beta_{cm}} \), \( \lambda \) means characteristic scale of disturbance, \( c_m \) is a speed of sound in mixture derived from linear equations corresponding to (2.1)-(2.5):

\[ \frac{1}{c_m^2} = \frac{(1 - \alpha_0)^2}{c_{ld}^2} + \frac{\alpha_0 (1 - \alpha_0) \rho_{ld} \gamma_g \rho_{g0}}{\gamma_g \rho_{g0}}. \quad (2.6) \]

We will not write asterisks for dimensionless variables everywhere later. The following two small parameters are introduced: \( \epsilon \) for nonlinear and \( \beta \) for dispersive effects. That is, the final form of the system is

\[ \frac{\partial v'}{\partial t} + \frac{\partial p'}{\partial r} = -\epsilon \left( v' \frac{\partial}{\partial r} v' - \rho' \frac{\partial}{\partial r} p' \right) \]

\[ \frac{\partial p'}{\partial t} + \frac{\partial v'}{\partial r} - \beta^2 \frac{\alpha_0 (1 - \alpha_0) R_{ld}^2 \rho_{ld}^2 \rho_0^2 c_m^4}{3 (\gamma_g \rho_{g0})^2 \lambda^2} \frac{\partial^2 p'}{\partial t^2} = \epsilon (1 - \alpha_0) c_m^2 \left[ \frac{-\gamma_{ld} + \frac{1}{2}}{c_{ld}^2} \rho' \frac{\partial}{\partial r} v' \right. \]

\[ - \frac{c_m^2 \alpha_0 (1 - \alpha_0) \rho_{ld}^2 (\gamma_g + 1)}{(\gamma_g \rho_{g0})^2} p' \frac{\partial}{\partial r} v' \]

\[ - \frac{\gamma_{ld} + \frac{1}{2}}{c_{ld}^2} \frac{\partial^2 p'}{\partial r^2} \frac{\partial v'}{\partial r} \]

\[ + O (\epsilon \beta^2), \quad (2.7) \]

\[ \frac{\partial p'}{\partial t} + \frac{\partial v'}{\partial r} = -\epsilon \left( v' \frac{\partial}{\partial r} p' + \rho' \frac{\partial}{\partial r} v' \right) \]

The third term in the second equation of (2.7) accounts for dispersion in the first approximation. Here, and further, we consider the traditional coupling between the small parameters \( \beta^2 = \epsilon \), that means a possibility of an equilibrium between nonlinearity and dispersion. For convenience of comparison with previous results, mass concentration is expressed through the initial value of volume one: \( x = \alpha_0 \rho_{g0}/\rho_{m0} \). It was supposed that \( \rho_{g0}/\rho_{ld0} \ll 1 \) and \( c_g < c_{ld} \) only. So, \( \rho_{m0} \approx (1 - \alpha_0) \rho_{ld0} \), etc. Naturally, our formulae go to incompressible liquid ones when \( c_{ld} \to \infty \).
3. COUPLED KdV EQUATIONS FOR RIGHTWARD, DOWNWARDS, AND STATIONARY WAVES IN COMPRESSIBLE LIQUID WITH BUBBLES

3.1. Projectors in a Linear Problem with Weak Dispersion

First, we need to find the projectors for the linear problem with weak dispersion. The linear analogue of (2.7) looks like the following:

\[
\frac{\partial}{\partial t} \psi + L \psi = 0, \quad \text{where } \psi = \begin{pmatrix} \psi' \\ p' \\ \rho' \end{pmatrix}
\]

\[
L = \begin{pmatrix} 0 & \frac{\partial}{\partial r} & 0 \\ \frac{\partial}{\partial r} + \frac{\epsilon D}{\partial r^3} & 0 & 0 \\ \frac{\partial}{\partial r} & 0 & 0 \end{pmatrix}, \quad D = \frac{\alpha_0 (1 - \alpha_0) R_k^2 \rho_{l0}^2 c_m^4}{3 (\gamma g p_0)^2 \lambda^2}.
\]

The estimation \( \frac{\partial^2 \psi'}{\partial r^3} = -\frac{\partial^2 \psi'}{\partial r^3} + O(\epsilon) \) was used to exclude time derivatives in \( L \). Let us take the plane waves \( \sim \exp(i \omega t - ikr) \) with amplitudes \( V_k, P_k, R_k \). The eigenvalues of the corresponding system of equations for Fourier-transformed components are \( \omega_{1,2} = \pm k \sqrt{1 - \epsilon D k^2}, \omega_3 = 0 \). These connections serve as dispersion relations for the right- and left-propagating and stationary inputs, \( \psi_{1,2} = (1 \pm (1 - \epsilon D k^2)^{1/2} \pm (1 - \epsilon D k^2)^{-1/2}) V_k, \psi_3 = (0, 0, 1)^T R_k \) are eigenvectors. Projectors are evaluated immediately from these formulae, in \( r \)-presentation they have the form

\[
P_{\pm} = \begin{pmatrix} \frac{1}{2} & \frac{1 - \epsilon \left( \frac{D}{2} \right) \frac{\partial^2}{\partial r^2}}{2} \\ \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1 - \epsilon \left( \frac{D}{2} \right) \frac{\partial^2}{\partial r^2}}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \pm \frac{1 - \epsilon \left( \frac{D}{2} \right) \frac{\partial^2}{\partial r^2}}{2} \\ 0 \end{pmatrix},
\]

\[
P_{\text{stat}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 + \epsilon D \frac{\partial^2}{\partial r^2} & 1 \end{pmatrix}.
\]

Only the first two Taylor terms hold for the problem with weak dispersion. The projecting operators have the common properties of orthogonal projectors: \( P_+ + P_- + P_{\text{stat}} = \mathbb{I}; P_- P_+ = P_+ P_- = \cdots = 0; P_+ = P_- P_-, \cdots, \mathbb{I}, 0 \) are unit and zero matrices. Projectors commute both with \( \frac{\partial}{\partial t} \) and \( L \). In order to obtain the rightward propagating field at any instant, for example, it is sufficient to apply \( P_+ \) to the total field \( \psi_+ = \psi_\text{stat} \); \( \psi_+ \) means rightward input, just the same for other inputs \( P_- \psi = \psi_- \), \( P_{\text{stat}} \psi = \psi_{\text{stat}} \).

3.2. Coupled KdV Equations

Now, with acting projectors on both sides of nonlinear system \( (2.7) \), one can write evolution equations for certain types of variables, where the nonlinear part is presented by superposition of variables products of the second order in agreement with \( (2.7) \). The type of mode (stationary, leftward, or rightward propagating) is defined strictly for linear problems only. We will call these inputs in the same way later, taking into account that only weak nonlinear and dispersion effects are under consideration. Let the modes have numbers 1,2,3 instead of "+", "-", "stat".
correspondingly, that simplify the formulae. For disturbances of pressure, the coupled equations are

\[ \frac{\partial \rho_n'}{\partial t} + c_n \frac{\partial \rho_n'}{\partial r} + \epsilon \sum_{i,m=1}^{3} \Phi_{i,m}^n \frac{\partial}{\partial r} \rho_m' + \epsilon M_n \frac{\partial^3}{\partial r^3} \rho_n' + O(\epsilon^2) = 0, \]

\[ n = 1, 2, 3, \quad c_{1,2} = \pm 1, \quad c_3 = 0, \]

and the coefficients are determined in the following tables:

<table>
<thead>
<tr>
<th>( \Phi_{i,m}^1 )</th>
<th>( \Phi_{i,m}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i \backslash m )</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{K}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{K-4}{2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{T-2}{2} )</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>( \Phi_{i,m}^3 )</th>
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<tbody>
<tr>
<td>( i \backslash m )</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
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\[ K = (1 - \alpha_0)^3 c_m^4 \left( \frac{\gamma_{ld} + 1}{c_{ld}^4} \right) \left( \frac{\alpha_0 p_{ld0}^2 (\gamma_g + 1)}{(1 - \alpha_0) (\gamma_g p_{ld0})^2} \right), \quad T = (1 - \alpha_0) c_m^2 \frac{\gamma_{ld} + 1}{c_{ld}^2}. \]

Also, coupled equations in basic variables \((v_{1,2}', \rho_3')\) or \((v_{1,2}', \rho_3')\) may be obtained, anyway three ones. A similar system of equations for internal gravity waves modes appeared in [9,10].

### 4. Evolution of Single Initially Exited Mode Affected by Others. KDV and MKDV Type Solutions. Quasisolitons

Let us make some remarks concerning general methods of approximate solution of nonlinear systems of equations with dispersion. The ordinary method of the perturbation theory, when solution of (3.1) is found in the form \( \rho_n' = \sum_{i=0}^{\infty} \epsilon^i \rho_n^{(i)} \) cannot be useful, the difference between approximated and exact solutions increase with time essentially over the time interval \( \sim [0, \epsilon^{-1}] \). The so-called nonsingular perturbations method implies iterations inside operators of nonlinear system. We use a development of this approach similar to one from [10,11]. Returning to the system in general form (3.1), the zero approximation \( \rho_n^{(0)}(r,t) \) is taken in the form of solution of the equation

\[ \frac{\partial \rho_n^{(0)}}{\partial t} + c_n \frac{\partial \rho_n^{(0)}}{\partial r} + \epsilon \Phi_{n,n}^{(0)} \frac{\partial \rho_n^{(0)}}{\partial r} + \epsilon M_n \frac{\partial^3 \rho_n^{(0)}}{\partial r^3} = 0, \quad \rho_n^{(0)}(r,0) = \phi_n(r). \]
Then the approximate solution, which accounts for the interaction effects of the first order, is

$$\rho_n^{(1)}(r, t) = \rho_n^{(0)}(r, t) - \epsilon \int_0^t \sum_{m, k \neq n} \Phi_{m,k}^n \rho_m^{(0)} \frac{\partial \rho_k^{(0)}}{\partial r} \left|r_{-c_n(t-r)} \right| \, dr + O(\epsilon^2). \quad (4.2)$$

Suppose that only one mode which has number $i$ is exited initially, $\phi_n(r) = 0$, $n \neq i$. Evolution of all other modes is described by (4.2): $\rho_n(r, t) = (\epsilon \phi_i^{(0)}/2(c_i - c_n)) [(\rho_i^{(0)}(r, t))^2 - \varphi_i^2(r - c_n t)] + O(\epsilon^2)$. The main input which influences on the $i$-mode is the first term, which has the same velocity. We would say that the second wave "goes away" without leaving any sufficient trace, but the influence of the first one is resonant and stored over time. The inverse influence is apparently of the order of $\epsilon^2$. The new evolution equation for the initially excited mode has a form

$$\frac{\partial \rho_i^{(1)}}{\partial t} + c_i \frac{\partial \rho_i^{(1)}}{\partial r} + \epsilon \left( \frac{\Phi_{i,i}^i \rho_i^{(1)} + \epsilon A_i (\rho_i^{(1)})^2}{\partial r} \right) + \epsilon M_i \frac{\partial^3 \rho_i^{(1)}}{\partial r^3} = 0, \quad (4.3)$$

where $A_i = \sum_{m \neq i} (\Phi_{i,m}^i/2 + \Phi_{m,i}^i) \Phi_{ii}^m/(c_i - c_m)$. Equation (4.3) is exactly integrated in the sense of Lax-pare existence [11] and its solitony solution looks as $\rho_i^{(1)} = \rho_i^{(1)}(\xi)$, where $\xi = r - c_i t$ and

$$\rho_i^{(1)}(\xi) = \epsilon^{-1} \left[ \frac{\Phi_{i,i}^i}{6(c_i - c_i)} + \sqrt{\left[ \frac{\Phi_{i,i}^i}{6(c_i - c_i)} \right]^2 + \frac{A_i}{6(c_i - c_i)} \cosh \left( \sqrt{\frac{c_i - c_i}{\epsilon M_i \xi}} \right)} \right]^{-1},$$

$c_i$ is the soliton velocity. To illustrate, let us take an incompressible liquid with bubbles, though formulae are found for a compressible one. Formally, this case follows from (3.1) in the limit $c_{id} \to \infty$. Returning to $(r, t)$-variables, for rightward input $\theta_1 = \theta_{d0} + \theta_{g0}$, one obtains equations like (4.1) and (4.3) with coefficients $c_1 - c_m$, $\epsilon M_1 \to R_{g0}^2 c_m/(6(c_0 - c_0))$, $\epsilon \Phi_{11}^1 \to (\gamma_g + 1)c_m/(2\gamma_g)$, $\epsilon^2 A_1 \to ((\gamma_g + 1)^2/16 + (\gamma_g + 1)/4) c_m/\gamma_g$, in agreement with formulae (3.1). The equation of zero order corresponds to the case of rightward mode self-action and looks just the same as that which was studied by Wijngaarden [6]. The equation of the first order describes inverse influence of the other modes on the initially excited one and is obtained just now. In calculations, we treated values $\alpha_0 = 0.01$, $\rho_{d0} = 10^3 \text{kg/m}^3$, $\rho_{g0} = 10^5 \text{Pa}$, $\gamma_g = 1.4$. Solitary forms $\theta_1^{(0)}(\xi/R_0)$ and $\theta_1^{(1)}(\xi/R_0)$ achieve values of 0.74 and 1.05 correspondingly for $c_1 - c_1 = 25 \text{m/s}$ and 1.46, 1.11 for $c_1 - c_1 = 50 \text{m/s}$ at $\xi = 0$. More $c_1 - c_1$, more difference appears between curves. It is known that any initial disturbance tends to separate to solitons; from this point of view, the more precise description of solitary forms which are described with the method proposed is obviously more desired.

REFERENCES