On the lattices of NP-subspaces of a polynomial time vector space over a finite field

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Abstract

In this paper, we study the lower semilattice of NP-subspaces of both the standard polynomial time representation and the tally polynomial time representation of a countably infinite dimensional vector space $V_{\infty}$ over a finite field $F$. We show that for both the standard and tally representation of $V_{\infty}$, there exists polynomial time subspaces $U$ and $W$ such that $U + V$ is not recursive. We also study the NP analogues of simple and maximal subspaces. We show that the existence of P-simple and NP-maximal subspaces is oracle dependent in both the tally and standard representations of $V_{\infty}$. This contrasts with the case of sets, where the existence of NP-simple sets is oracle dependent but NP-maximal sets do not exist. We also extend many results of Nerode and Remmel (1990) concerning the relationship of P bases and NP-subspaces in the tally representation of $V_{\infty}$ to the standard representation of $V_{\infty}$.

1. Introduction

In 1975 Metakides and Nerode [26] initiated the systematic study of recursion theoretic algebra. The motivation was to establish the recursive content of mathematical constructions. The novelty was the use of the finite injury priority method from recursion theory as a uniform tool to meet algebraic requirements. Prior to that time the priority method has been limited primarily to internal applications within recursion

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theory in the theory of recursively enumerable sets and in the theory of degrees of unsolvability and their generalizations.

Recursion theoretic algebra has been developed since, in depth, by many authors in such subjects as commutative fields, vector spaces, orderings, and boolean algebras (see [15] for references and a cross-section of results before 1980). Recursion theoretic algebra yielded as a byproduct a theory of recursively enumerable substructures (see the survey article [28] for references).

Simultaneously, in computer science there was a vast development of P and NP problems in complexity theory. This subject started out as a tool for measuring the relative difficulties of classes of computational problems (see [13, 14, 20]). Many papers in this area have dealt with coding a given problem $M$ into a calibrated problem to find an upper bound on the complexity of $M$, and coding a calibrated problem into a given problem $M$ to find a lower bound on the complexity of $M$ (see [19, 22]). Due to the intractability of the fundamental problem $P = NP$, Baker-Gill-Solovay [2] began a line of inquiry using diagonal arguments to produce sets ("oracles") $R_1, R_2$ such that $P^{R_1} = NP^{R_1}, P^{R_2} \neq NP^{R_2}$. Typical of recent work in this direction is the construction by Yao [44] of oracles relative to which none of the polynomial time hierarchy collapses, and the result of Cai [6] that this holds for oracles with probability 1. The Baker-Gill-Solovay, Yao, and Cai results are fundamental, but they do not use the priority method which was used systematically with success in recursion theoretic algebra.

Those studying recursion theoretic algebra have wondered whether a more sophisticated cousin of recursion theoretic algebra, complexity theoretic algebra, might be feasible also using the priority method as a fundamental tool. Priority arguments have been used by many authors in the study of $P^A$ and $NP^A$ sets for recursive or recursively enumerable oracles $A$. For example, Homer and Maass [21], used priority arguments to investigate the lattice of $NP^A$ sets. Shinota and Slaman [42] and Shore and Slaman [43] have used priority argument to study the structure of the polynomial time Turing degrees relative to a recursive oracle. Downey and Fellows [17] used priority arguments to study the density of their fixed parameter complexity classes.

We showed that the priority method also plays a fundamental role in the study of complexity theoretic algebra. For example, in [31] we studied the lower semilattice of $NP$ ideals of polynomial time presentations of the free Boolean algebra. Unlike the situation for recursive Boolean algebras where there is a unique recursive presentation of the free Boolean algebra up to recursive isomorphism, there are many inequivalent polynomial time representations of the free Boolean algebra up to polynomial time isomorphisms. Thus in [31], we concentrated on two very natural polynomial time representations of the free Boolean algebra, namely the standard representation where the underlying universe is the set of binary representations of the natural numbers and tally representation where the underlying universe is the set of tally representations of the natural numbers. We examined several $NP$ analogues of results in the lattice of r.e. ideals of a recursive presentation of the free Boolean algebra. For example, every recursive ideal can be extended to a recursive maximal ideal. The natural analogue of this result, i.e. that every polynomial time ideal can be extended to a polynomial
time maximal ideal, was proved for both the standard polynomial time representation and tally polynomial time representations of the free Boolean algebra in [31]. Similarly, it is known that there exists a recursively enumerable (r.e.) ideal which is not extendable to an r.e. maximal ideal. The analogue of this latter result, i.e. that there exists an NP ideal which is not contained in any NP maximal ideal, was shown to be oracle dependent in [31]. In [35], we studied the semilattice of NP subspaces of both the standard and tally representation of a countable, infinite dimensional vector space \( V_\infty \) over a polynomial time field. In that paper, we studied a polynomial time analogue of a simple result of Dekker [16], namely that every r.e. subspace of \( V_\infty \) has a basis in \( P \). In [35], we showed that if the underlying field is infinite, then, for either the standard or tally representation of \( V_\infty \), every r.e. subspace has a basis in \( P \) which actually improves Dekker’s result. However, if the underlying field is infinite, then in the tally representation of \( V_\infty \), we showed that the question of whether every NP subspace has a basis in \( P \) is oracle dependent. We note also that Cenzer and Remmel, in a series of papers [7–12, 40], have developed a rich polynomial time model theory where the priority method also plays a role.

Thus complexity theoretic algebra in this sense is feasible. The proofs in recursive algebra are effective but one does not pay attention to resource bounds. The proofs in complexity theoretic algebra are more intricate than those of recursion theoretic algebra because one is forced to pay careful attention to resource bounds. Recursion theoretic algebra uses unbounded computational resources, complexity theoretic algebra cannot. Sometimes, one gets the opposite result in complexity theoretic algebra for an analogous result in recursion theoretic algebra, as in the case of the analogue of Dekker’s result mentioned above, because the result in recursion theoretic algebra uses the lack of bounds on resources in an essential way. Sometimes, one gets the same result in both subjects, as is the case that every polynomial time ideal of the free Boolean algebra can be extended to a maximal polynomial time ideal, but by a harder resource-bounded argument.

In this paper, we shall study the semilattice of NP subspaces of polynomial time representations of a countable, infinite dimensional vector space \( V_\infty \) over a finite field \( F \). We say semilattice in this case since unlike the situation of r.e. subspaces of \( V_\infty \) where the set of r.e. subspaces is closed under intersection and under sum which is the natural join in the lattice of r.e. subspaces of \( V_\infty \), the set of \( \text{NP}^4 \) subspaces of our polynomial time representations of \( V_\infty \) is closed under intersection but not under sum. That is, we shall show that there exist two polynomial time subspaces \( U \) and \( W \) such that \( U + W \) is not even recursive much less in \( \text{NP} \) or \( P \). We investigate the lower semilattice of \( \text{NP}^4 \) subspaces because vector spaces are well understood structures which pervade mathematics and in which all the phenomena of the type discussed above occur. Further, the corresponding recursion theoretic algebra is already well developed (see [26, 28]), making comparisons easy. This case is a good paradigm for investigations of semilattices of NP substructures of more complex structures. Our
work here can be viewed as an extension of the study of the lattice of NP sets to algebraic structures in the spirit of Homer and Maass [21].

The space $V_\infty$ can be coded into the natural numbers $N = \{0, 1, 2, \ldots\}$ as a polynomial time vector space in many ways. We shall think of $V_\infty$ as the set of finitely nonzero sequences from $F$ with the operations taken componentwise. We refer to $e_1, e_2, \ldots$ as the standard basis of $V_\infty$ where $e_n$ is the sequence of length $n$, $(0, \ldots, 0, 1)$ with $n - 1$ zeros and 1 denotes the unit of $F$. We say that the zero vector $0$ has height 0 and that a vector of the form $(a_1, \ldots, a_n)$, where $a_i \in F$ for all $i$ and $a_n \neq 0$ has height $n$. Now the question of whether $V_\infty$ is polynomial time, recursive, etc., depends on how we code the sequences $(a_1, \ldots, a_n)$. Following [30, 35], we will distinguish two specific polynomial time representations of $V_\infty$ which we call the tally and standard representations of $V_\infty$. For a given $n \in N$, let $\text{bin}(n)$ denote the binary representation of $n$ and $\text{tal}(n) = 0^n$ denote the tally representation of $n$. Let $\text{Bin}(N) = \{\text{bin}(n) : n \in N\}$ and $\text{Tal}(N) = \{\text{tal}(n) : n \in N\} = \{0\}^*$. If $F$ is a field with $k$ elements, there is no loss of generality in assuming that the domain of $F$ is the set $(0, 1, \ldots, k - 1)$ where 0 is the zero of $F$ and 1 is the multiplicative identity of $F$. Of course, addition and multiplication in $F$ can be carried out by a table look up. We then identify each vector $v \in V_\infty$ with a natural number $R(v)$ as follows:

$$R(0) = 0$$

$$R((a_1, \ldots, a_n)) = a_1 + a_2 k + \cdots + a_n k^{n-1} \quad \text{if} \quad a_n \neq 0. \quad (1)$$

Next we define maps

$$st : V_\infty \to \text{Bin}(N) \quad \text{by} \quad st(v) = \text{bin}(R(v)), \quad (2)$$

$$\text{tal} : V_\infty \to \text{Tal}(N) \quad \text{by} \quad \text{tal}(v) = \text{tal}(R(v)). \quad (3)$$

It then follows that if $st(V_\infty)$ consists of the set $U_{st} = \{st(v) : v \in V_\infty\}$ with the operations of vector addition $+_st$ and scalar multiplication $\lambda_{st}$ for each $\lambda \in F$ induced by the corresponding operations from $V_\infty$, $U_{st} = \text{Bin}(N)$ is a polynomial time subset of $(0, 1)^*$ and the operations $+_st$ and $\lambda_{st}$ are the restrictions to $U_{st}$ of polynomial time functions. Similarly, we let $\text{tal}(V_\infty)$ consists of the set $U_{tal} = \{\text{tal}(v) : v \in V_\infty\}$ with the operations of vector addition $+_tal$ and scalar multiplication $\lambda_{tal}$ for each $\lambda \in F$ induced by the corresponding operations from $V_\infty$. Thus $U_{tal} = \text{Tal}(N)$ is a polynomial time subset of $(0, 1)^*$ and the operations $+_tal$ and $\lambda_{tal}$ are the restrictions to $U_{tal}$ of polynomial time functions. Moreover, it was shown in [35], that both $st(V_\infty)$ and $\text{tal}(V_\infty)$ possess polynomial time dependence algorithms. That is, given any set of vectors $v_1, \ldots, v_n$ in $V_\infty$, there are polynomial time algorithms which will determine whether $\{st(v_1), \ldots, st(v_n)\}$ is dependent in $st(V_\infty)$ and whether $\{\text{tal}(v_1), \ldots, \text{tal}(v_n)\}$ is dependent in $\text{tal}(V_\infty)$.

In the lattice of recursively enumerable (r.e.) sets, $\mathcal{E}$, of the natural numbers $N$, an r.e. set $S$ is simple if $N - S$ is infinite and for any infinite r.e. set $W$, $W \cap S \neq \emptyset$. An r.e. set $M$ is maximal if $N - M$ is infinite and for any r.e. set $W \supseteq M$, either $N - W$ or $W - M$ is finite. The analogues of these notions in the lattice of NP$^d$ sets, $\mathcal{E}_{\text{NP}^d}$,
any oracle $A$ are the following. Let $A$ be an oracle, then an $NP^A$ set $S \subseteq \{0,1\}^*$ is $NP^A$-

simple if $\{0,1\}^* - S$ is infinite and for any infinite $NP^A$ set $W \subseteq \{0,1\}^*$, $W \cap S \neq \emptyset$. An $NP^A$ set $M \subseteq \{0,1\}^*$ is $NP^A$-maximal if $\{0,1\}^* - M$ is infinite and for any $NP^A$

set $W \supseteq M$, either $\{0,1\}^* - W$ or $W - M$ is finite. It was shown by Homer and Maass [21], that there exists oracles $A$ and $B$ such that $NP^A \neq P^A$ and no $NP^A$-simple set exists and there exists $NP^B$-simple sets. It follows from a result of Briedbart [5] that there are no $NP^A$-maximal sets for any $A$.

In the lattice of r.e. subspaces of a recursively presented copy of $V_{\infty}$, $\mathcal{L}(V_{\infty})$, an r.e. subspace $S$ of $V_{\infty}$ is simple if the dimension of the quotient space $V_{\infty}/S$ is infinite and for any infinite dimensional r.e. subspace $W$ of $V_{\infty}$, $W \cap S \neq \emptyset$. An r.e. subspace $M$ is maximal if the dimension of $V_{\infty}/M$ is infinite and for any r.e. subspace $W \supseteq M$, either the dimension of $V_{\infty}/W$ or the dimension of $W/M$ is finite. An r.e. subspace $M$ is supermaximal if the dimension of $V_{\infty}/M$ is infinite and for any r.e. subspace $W \supseteq M$, either the dimension of $V_{\infty}/W$ or the dimension of $W/M$ is finite. The $NP$ analogues of these notions in $st(V_{\infty})$ and $tal(V_{\infty})$ are the following. Let $A$ be an oracle, then an $NP^A$ subspace $S$ of $st(V_{\infty})$ (tal$(V_{\infty})$) is $NP^A$-simple if the dimension of $st(V_{\infty})/S$ (tal$(V_{\infty})/S$) is infinite and for any infinite $NP^A$ subspace $W$ of $st(V_{\infty})$ (tal$(V_{\infty})$), $W \cap S \neq \{st(0)\}$ ($W \cap S \neq \{tal(0)\}$). An $NP^A$ subspace $M$ is $NP^A$-maximal if the dimension of $st(V_{\infty})/M$ (tal$(V_{\infty})/M$) is infinite and for any $NP^A$ subspace $W$ of $st(V_{\infty})$ (tal$(V_{\infty})$), either the dimension of $st(V_{\infty})/W$ (tal$(V_{\infty})/W$) or the dimension of $W/M$ is finite. An $NP^A$ subspace $M$ is $NP^A$-supermaximal if the dimension of $st(V_{\infty})/M$ (tal$(V_{\infty})/M$) is infinite and for any $NP^A$ subspace $W$ of $st(V_{\infty})$ (tal$(V_{\infty})$), either $st(V_{\infty}) = W$ (tal$(V_{\infty}) = W$) or the dimension of $W/M$ is finite.

There is a slightly weaker notion than the $NP^A$-simple space which we will explore in this paper. Note that in the case of simple sets or simple subspaces, we can replace the infinite r.e. set $W$ or the infinite dimensional r.e. subspace $W$ by an infinite recursive set $W$ or an infinite dimensional recursive subspace. That is, every infinite r.e. set $W$ contains an infinite recursive set and every infinite dimensional r.e. subspace $V$ of $V_{\infty}$ contains an infinite dimensional recursive subspace. Thus an r.e. set $S$ of $V_{\infty}$ is simple iff $N - S$ is infinite and for any infinite recursive set $W$, $W \cap S \neq \emptyset$. Similarly an r.e. subspace $S$ of $V_{\infty}$ is simple iff the dimension of $V_{\infty}/S$ is infinite and for any infinite dimensional recursive subspace $W$ of $V_{\infty}$, $W \cap S \neq \emptyset$. Thus we make the following definition. Let $A$ be an oracle, then a $NP^A$ subspace $S$ of $st(V_{\infty})$ (tal$(V_{\infty})$) is $P^A$-simple if the dimension of $st(V_{\infty})/S$ (tal$(V_{\infty})/S$) is infinite and for any infinite dimensional $P^A$ subspace $W$ of $st(V_{\infty})$ (tal$(V_{\infty})$), $W \cap S \neq \{st(0)\}$ ($W \cap S \neq \{tal(0)\}$). It follows from [35] that there exists oracles $A$ such that there exists an infinite dimensional $NP^A$ subspace $V$ of tal$(V_{\infty})$ such that $V$ has no infinite dimensional subspace $W \in P^A$. Thus while a subspace $W$ which is $NP^A$-simple is certainly $P^A$-simple, it is not clear that every $P^A$-simple subspace of tal$(V_{\infty})$ is $NP^A$-simple.

In this paper we shall study the semilattice of $NP^A$ subspaces of the standard representation of $V_{\infty}$, $\mathcal{L}_{NP^A}(st(V_{\infty}))$, and the semilattice of $NP^A$ subspaces of the tally representation of $V_{\infty}$, $\mathcal{L}_{NP^A}(tal(V_{\infty}))$. The main result of this paper is to show that the existence of $P$-simple subspaces and $NP$-maximal subspaces is oracle dependent for either
st(V_\infty) or for tal(V_\infty). Thus we have an interesting contrast with the set case, namely, that there exists oracles A and B such that NP^A-maximal subspaces exist in st(V_\infty) and NP^B-maximal subspaces exist in tal(V_\infty), but there are no NP^A-maximal or NP^B-maximal sets. We also show that there are oracles A such that there are no NP^A-simple subspaces but we do not know if there is an oracle B for which NP^B-simple exists.

This paper is organized as follows. In Section 2, we shall deal with basic definitions and notations. Then in Section 3, we shall develop a series of results on the connections between subspaces V \in L^\text{NP^A}(st(V_\infty)) or subspaces V \in L^\text{NP^A}(tal(V_\infty)) and their bases which lie in P^A or NP^A. These results are necssary for the proofs of our main results concerning the oracle dependence of the existence of P^A-simple and NP^A-maximal subspaces and are of interest in their own right. Many of the results on the connections between subspaces NP^A and their bases which lie in P^A or NP^A were proved in [35] but only for tal(V_\infty). Thus is Section 3, we develop an analogous machinery for st(V_\infty). Finally, in Section 4, we shall prove our main results. We note that a number of results of this paper were announced in [30] for tal(V_\infty). In [30], we claimed that there exists an oracle A for which there is a V \in L^\text{NP^A}(tal(V_\infty)) which is both NP^A-simple and NP^A-maximal. Our proof of that result contained an error and we do not know whether there exists such an oracle A. Instead, in this paper, we prove that there exist oracles A and B for which there is a V \in L^\text{NP^A}(tal(V_\infty)) which is both P^A-simple and NP^A-maximal and there is a W \in L^\text{NP^P}(st(V_\infty)) which is both P^B-simple and NP^B-maximal. Similarly, there exists oracles C and D such that NPC \neq PC and no NPC-simple, PC-simple, or NPC-maximal subspaces of tal(V_\infty) exist and NP^D \neq P^D and no NP^D-simple, P^D-simple, or NP^D-maximal of st(V_\infty) exist.

2. Preliminaries

All sets are assumed to be over a fixed alphabet \Sigma = \{0, 1, \ldots, k - 1\} where k \geq 1 is the size of the underlying finite field F of V_\infty. For x \in \Sigma^*, the length of x is denoted by |x|. If A \subset \Sigma^*, then the cardinality of A is denoted by |A| and the complement of A is denoted by \overline{A}. A tally set T is any subset of \{0\}^*. We let N denote the natural numbers. We let < denote the usual order on \Sigma^*, i.e. x < y iff |x| < |y| or |x| = |y| and x lexicographically precedes y.

Our basic computation model is the standard multi-tape Turing machine (TM) acceptors and transducers of Hopcroft and Ullman [22]. An oracle machine is a multitape Turing machine M with a distinguished work tape, a query tape, and three distinguished states: QUERY, YES, and NO. At some step of a computation on an input string \omega, M may transfer into the state QUERY. In state QUERY, M transfers into the state YES if the string currently appearing on the query tape is in an oracle set A; otherwise, M transfers into the state NO. In either case, the query tape is instantly erased. The set of strings accepted by M relative to the oracle set A is L(M, A) = \{\omega\} there is an accepting computation of M on input \omega when the oracle set is A}. If A = \phi, we write L(M) instead of L(M, \phi).
A Turing machine $M$ is $t(n)$ time bounded for some function $t$ on the natural numbers if each computation of $M$ on inputs of size $n$ has length at most $t(n)$. If $t$ can be chosen to be a polynomial, then $M$ is polynomial time bounded. We let

$$\text{DTIME}(t(n)) = \{L(M) \mid M \text{ is deterministic and } t(n) \text{ time bounded}\}$$

$$\text{NTIME}(t(n)) = \{L(M) \mid M \text{ is nondeterministic and } t(n) \text{ time bounded}\}$$

$$P = \bigcup \{\text{DTIME}(p(n)) \mid p \text{ is a polynomial}\}$$

$$NP = \bigcup \{\text{NTIME}(p(n)) \mid p \text{ is a polynomial}\}$$

$$\text{DEXT} = \bigcup \{\text{DTIME}(2^c n) \mid c > 0\}$$

$$\text{NEXT} = \bigcup \{\text{NTIME}(2^c n) \mid c > 0\}$$

We fix enumerations $\{P_i\}_{i \in \mathbb{N}}$ and $\{N_i\}_{i \in \mathbb{N}}$ of the polynomial-time bounded deterministic oracle Turing machines and the polynomial-time bounded nondeterministic oracle Turing machines, respectively. We may assume that $p_i(n) = \max(2, n)^i$ is a strict upper bound on the length of any computation by $P_i$ or $N_i$ with any oracle $X$ on inputs of length $n$. $P_i^X$ and $N_i^X$ denote the oracle Turing machines using oracle $X$ and in an abuse of notation we shall denote $L(P_i, X)$ by simply $P_i^X$ and $L(N_i, X)$ by $N_i^X$. This given, $P^X = \{P_i^X : i \in \mathbb{N}\}$ and $NP^X = \{N_i^X : i \in \mathbb{N}\}$.

For $A, B \subseteq \Sigma^*$, we shall write $A \preceq_B^P B$ if there is a polynomial-time function $f$ such that for all $x \in \Sigma^*, x \in A$ iff $f(x) \in B$. We shall write $A \preceq^P B$ if $A$ is polynomial time Turing reducible to $B$. For $r$ equal to $m$ or $T$, we write $A \equiv^P_B$ if $A \preceq^P B$ and $B \preceq^P A$ and we write $A \not\equiv^P_B$ if not $A \equiv^P_B$ and not $B \equiv^P_A$.

We end this section with some basic definitions and notations for vector spaces. Let $V$ be either $V_\infty$, $st(V_\infty)$ or $tal(V_\infty)$, and $tal(V_\infty)$ even though technically the zero vectors of the three vector spaces are distinct objects. Then given a subset $A$ of $V$, we let $\text{space}(A)$ denote the subspace of $V$ generated by $A$. Given two subspaces $U$ and $W$ of $V$, we let $U + W$ denote the subspace generated by $U \cup W$. We shall write $W = U_1 + U_2$ if $W, U_1$ and $U_2$ are subspaces of $V$ such that $W = U_1 + U_2$ and $U_1 \cap U_2 = \{0\}$. We say $U$ is a complementary subspace of $W$ if $U \oplus W = V$. Given $x \in V$, we let $h(x)$ denote the height of $x$. We note that if $x \in st(V_\infty)$, then in polynomial time in $|x|$, we can produce the binary representations of the integers $a_1, \ldots, a_n$ such that $x = st((a_1, \ldots, a_n))$ with $a_n \neq 0$ so that we can find the height of $x$ in polynomial time in $|x|$. Similarly if $x \in tal(V_\infty)$, then in polynomial time in $|x|$, we can produce the tally representations of the integers $a_1, \ldots, a_n$ such that $x = tal((a_1, \ldots, a_n))$ with $a_n \neq 0$ so that we can find the height of $x$ in polynomial time in $|x|$.

3. Subspaces and bases

In this section, we shall explore the relation between the complexity of a subspace $V$ of either $st(V_\infty)$ or $tal(V_\infty)$ and the complexity of a basis of that subspace. Note that
since the universe of $st(V_{\infty})$ is $Bin(N)$, there is a natural order $<$ on the elements of $st(V_{\infty})$ inherited from the standard ordering of the natural numbers. Similarly, since the universe of $tal(V_{\infty})$ is $tal(N)$, there is a natural order $<$ on the elements of $st(V_{\infty})$ inherited from the standard ordering of the natural numbers. This given, we can now state some very useful definitions for our purposes. Recall that $e_1, e_2, \ldots$ is the standard basis for $V_{\infty}$. Thus $R(e_n) = k^{n-1}$.

**Definition 3.1.** Let $V$ be a subspace of $st(V_{\infty})$ or $tal(V_{\infty})$.

1. Call $B$ a **height increasing** basis of $V$ if $B$ is a basis for $V$ and for all $n \geq 1$, $B$ has at most one element of height $n$.

2. The **standard height increasing basis** of $V$, $B_V$, is defined by declaring that $x \in B_V$ iff $x \in V$ and there is no $y \in V$ such that $y < x$ and $h(y) = h(x)$.

3. The **standard height increasing complementary basis** of $V \subseteq tal(V_{\infty})$, $B_{\overline{V}}$, is defined in $tal(V_{\infty})$ by declaring that $tal(e_n) \in B_{\overline{V}}$ iff $tal(e_n) \notin V$ and there is no $y \in V$ such that $h(y) = n$. Similarly the standard height increasing complementary basis of $V \subseteq st(V_{\infty})$, $B_{\overline{V}}$, is defined in $st(V_{\infty})$ by declaring that $st(e_n) \in B_{\overline{V}}$ iff $st(e_n) \notin V$ and there is no $y \in V$ such that $h(y) = n$.

4. We call space $(B_{\overline{V}})$, the **standard complement** of $V$.

We note that there is a crucial difference between $st(V_{\infty})$ and $tal(V_{\infty})$ with respect to searches. That is, the vector of height $n$ with the smallest $R$ value is $e_n$ and $R(e_n) = k^{n-1}$. The vector of height $n$ with the largest $R$ value is $(k-1)e_1 + \cdots + (k-1)e_n$ and

$$R((k-1)e_1 + \cdots + (k-1)e_n) = \sum_{i=1}^{N} (k-1)k^{i-1} = k^n - 1.$$ 

Thus in $tal(V_{\infty})$, given a vector $v$ of height $n$, we can produce in polynomial time in $|v|$, a list of all vectors of height $n$ in $tal(V_{\infty})$. However, in $st(V_{\infty})$, given a vector $v$ of height $n$, it takes exponential time in $|v|$ to produce a list of all vectors of height $n$ in $st(V_{\infty})$. For this reason, the relation between the complexity of $V$, $B_V$, $B_{\overline{V}}$, and space($B_{\overline{V}}$) is very different in $tal(V_{\infty})$ than in $st(V_{\infty})$. For this reason, we shall divide this section into two subsections, one for $tal(V_{\infty})$ and one for $st(V_{\infty})$, and discuss the relation between the complexity of bases and subspaces for each case separately.

### 3.1. Bases and subspaces for $tal(V_{\infty})$.

Nerode and Remmel in [35] studied bases of NP-subspaces of $tal(V_{\infty})$ so we start by listing a number of results from that paper.

**Theorem 3.2** (Nerode and Remmel [35]). Let $V$ be a subspace of $tal(V_{\infty})$.

(a) If $B$ is a height increasing basis of $V$, then $V \leq_{\overline{V}} B$.

(b) $B_{\overline{V}} \leq_{\overline{V}} V$ and $B_{\overline{V}} \leq_{\overline{V}} V$. 
We note that in [35], we constructed a recursive subspace $V$ of $\text{tul}(V_\infty)$ such that neither $B_V \preceq^P_m V$ nor $V \preceq^P_m B_V$ hold so that we cannot replace $\preceq^P_T$ by $\preceq^P_m$ in the statement of Theorem 3.2.

**Theorem 3.3** (Nerode and Remmel [35]). (i) A subspace $V$ of $\text{tul}(V_\infty)$ is in $\mathbb{P}$ iff $V$ has a height increasing basis $B$ in $\mathbb{P}$.

(ii) If $V$ is a subspace of $\text{tul}(V_\infty)$ and $V \in \mathbb{P}$, then $V$ has a complementary subspace $W$ in $\mathbb{P}$.

**Theorem 3.4.** Suppose that $A$ is a height increasing independent set in $\mathbb{NP}$. Then $\text{space}(A) \in \mathbb{NP}$.

**Proof.** Note that if $A$ is a height increasing independent set, then $x \in \text{space}(A)$ iff $x \in \text{space}(\{y \in A : h(y) \leq h(x)\})$. Thus $x \in \text{space}(A)$ iff there are elements $b_1, \ldots, b_n$ of height $\leq h(x)$ and $\lambda_1, \ldots, \lambda_n \in F$ such that $x = \sum_{i=1}^{n} \lambda_i b_i$. Moreover, if $h(x) = m$, then $k^{m-1} \leq |x| \leq k^m - 1$ so that each $b_i$ must have length $\leq |x|$. Thus in nondeterministic polynomial time, we can construct computations which show that $b_i \in A$ and then verify that $x = \sum_{i=1}^{n} \lambda_i b_i$. Thus $\text{space}(A)$ is in $\mathbb{NP}$ if $A \in \mathbb{NP}$. □

**Theorem 3.5** (Nerode and Remmel [35]). Suppose $\mathbb{NP}^X = \mathbb{Co-NP}^X$ and $V$ is a subspace of $\text{tul}(V_\infty)$. Then

(i) $V \in \mathbb{NP}^X$ iff $V$ has a height increasing basis in $\mathbb{NP}^X$.

(ii) $V \in \mathbb{NP}^X$ implies $V$ has a complementary subspace $W$ in $\mathbb{NP}^X$.

**Proof.** By the relativized version of Theorem 3.4, if $V$ has a height increasing basis in $\mathbb{NP}^X$, then $V \in \mathbb{NP}^X$.

Next suppose that $V \in \mathbb{NP}^X$. Then given $x$, we can in polynomial time in $x$ produce a list of all vectors $w_0, \ldots, w_{k^{h(x)}-1}$ in $\text{tul}(V_\infty)$ of height at most $h(x)$. Since $\mathbb{NP}^X = \mathbb{Co-NP}^X$, there are $r$ and $s$ such that $V = \mathbb{NP}^X_r$ and $\text{tul}(V_\infty) - V = \mathbb{NP}^X_s$. It follows that in polynomial time in $|x|$, we can guess computations $c_0, \ldots, c_{k^{h(x)}-1}$ such that $c_i$ shows that $w_i \in \mathbb{NP}^X_r$ if $v_i \in V$ and $v_i \in \mathbb{NP}^X_s$ if $x \notin V$. Thus, given an $X$ oracle, in polynomial time in $|x|$, we can nondeterministically determine the membership of all vectors in $\text{tul}(V_\infty)$ of height at most $h(x)$ relative to $V$. Then $x$ is in $B_V$ iff $x$ is the least vector of height $h(x)$ in $V$ and $x$ is in $B_{\overline{V}}$ iff $x = \text{tul}(e_{h(x)})$ and no vector of height $h(x)$ is in $V$. Thus both $B_V$ and $B_{\overline{V}}$ are in $\mathbb{NP}^X$. Finally, by our argument above, $\text{space}(B_V) \in \mathbb{NP}^X$ since $B_{\overline{V}} \in \mathbb{NP}^X$. □

Next we will show, the assumption that $\mathbb{NP}^X = \mathbb{Co-NP}^X$ also eliminates the possibility of the existence of $\mathbb{NP}^X$-simple and $\mathbb{NP}^X$-maximal sets.

**Theorem 3.6.** Suppose that $\mathbb{NP}^X = \mathbb{Co-NP}^X$ and $V$ is an $\mathbb{NP}^X$ subspace of $\text{tul}(V_\infty)$ such that $\text{tul}(V_\infty)/V$ is infinite dimensional. Then $V$ is not $\mathbb{NP}^X$-simple and $V$ is not $\mathbb{NP}^X$-maximal.
Proof. By Theorem 3.5, it follows that $space(B_T) \in NP^X$ so that $V$ is not $NP^X$-simple. To see that $V$ is not $NP^X$ maximal, note that by our argument in Theorem 3.5, it follows that for any given $x \in NP^X$, we can nondeterministically from an $X$ oracle find a list of all elements $u_1 < \cdots < u_s$ of height $\leq h(x)$ which are in $B_T$ and a list of all elements $v_1 < \cdots < v_t$ of height $\leq h(x)$ which are in $B_F$. Thus we can form a new $NP^X$ height increasing independent set where $x \in C$ iff $x = u_i$ for some $i \leq s$ or $x = v_{2k}$ for some $2k \leq t$. It is then easy to see that both $tal(V_\infty / space(C))$ and $space(C)/V$ are infinite dimensional. It also follows from Theorem 3.5 that $space(C) \in NP^X$ so that $C$ witnesses that $V$ is not $NP^X$-maximal. 

Since Baker, Gill and Solovay [2] produced recursive oracles $X$ such that $NP^X \neq P^X$ but $NP^X = Co-NP^X$, we have the following.

Theorem 3.7. There exists a recursive oracle $A$ such that $NP^A \neq P^A$ and there are no $NP^A$-simple or $NP^A$-maximal subspaces of $tal(V_\infty)$.

We note that the construction of Theorem 3.6 does not construct a $P^X$-subspace $W$ such that $W \cap V = \{0\}$ since it is a priori possible that $space(B_T)$ does not contain an infinite dimensional subspace in $P^X$. Thus we do not automatically rule out the possibility of the existence of $P^X$-simple subspaces of $tal(V_\infty)$ with the assumption that $NP^X = Co-NP^X$. We shall see a bit later that there exist oracles $A$ such that no $NP^A$-simple, $P^A$-simple, or $NP^A$-maximal subspaces exists in $tal(V_\infty)$. However we first need to state a few more results from [35].

Theorem 3.8 (Nerode and Remmel [35]). Let $V$ be a recursively enumerable infinite dimensional subspace of $tal(V_\infty)$. Then the following are equivalent:

1. $V$ has a basis $C$ in $P$,
2. $V$ contains an infinite dimensional subspace $W$ in $P$,
3. $V$ contains an infinite height increasing independent subset $S$ in $P$.

Our next result will show that if $V$ has an infinite height increasing independent subset in $P$, then $V$ is not $P$-simple or $NP$-simple.

Theorem 3.9. Let $V$ be a recursive subspace of $tal(V_\infty)$ such that $V$ contains an infinite height increasing independent set $C$ in $P$. Then if the dimension of $tal(V_\infty)/V$ is infinite, there is an infinite height increasing independent set $D$ in $P$ such that $V \cap space(D) = \{0\}$.

Proof. Note that $B_T$ is recursive. Let $b_0, b_1, \ldots$ be a list of the elements of $B_T$ such that $h(b_0) < h(b_1) < \cdots$. Let $f$ be a recursive function such that $f(0^n) = b_n$. Similarly, let $c_0, c_1, \ldots$ be a list of elements of $C$ such that $h(c_0) < h(c_1) < \cdots$. Then let $d_s = b_s + tal c_{r(s)}$ where

$$r(s) = 1 + \sum_{i=0}^s h(b_i) + \text{the number of steps to compute } f(0), \ldots, f(s).$$
Then we claim that $D = \{d_0, d_1, \ldots \}$ is our required height increasing independent set. First observe that by our definition of $r(s)$, $r(s) > h(b_s)$ so that $h(d_s) = h(c_{r(s)})$. Also it is clear that $r(0) < r(1) < \cdots$ so that $h(d_0) < h(d_1) < \cdots$. Thus $D$ is a height increasing basis. Moreover, it is easy to see that $D$ is independent over $V$. Thus we need only show that $D$ is p-time. To decide whether a given $x \in \text{tal}(V_\infty)$ is in $D$, we first compute which elements $y$ with $h(y) \leq h(x)$ are in $C$. Now $C$ is a p-time set so that for all $z$ we can determine whether $z \in C$ in $\max(2, |z|)^m$ steps for some fixed $m$. Moreover, if $h(x) = n$, then $x = 0|\langle x \rangle$ where $k^{n-1} \leq |x| \leq k^n - 1$ so that it requires at most $2^m + 2^m + \sum_{j=2}^{k^{n-1}} j^m \leq \sum_{j=0}^{k^n-1} j^m < (k|\langle x \rangle| + 1)^m = \left((k|x| + 1)^2\right)^m$ steps to find the elements of $C$ of height less than or equal to $h(x)$. If no element of height $h(x)$ is in $C$, then clearly $x \notin D$. If there is an element of height $h(x)$ in $C$, then in polynomial time in $|x|$, we can find $r$ such that $h(c_r) = h(x)$. At this point, we start to compute the sequence of elements $f(0), f(1), \ldots$ in order for $r$ steps. Suppose that at the end of $r$ steps, we have successfully computed $f(0), \ldots, f(t)$. Note that if we are not successful in computing $f(0)$ by the end of $r$ steps, then $x \notin D$. Otherwise, see if there is some $s \leq t$ such that

$$r = 1 + \sum_{i=0}^{s} h(b_i) + \text{the number of steps to compute } f(0), \ldots, f(s).$$

If there is no such $s$, then $x \notin D$ and if there is such an $s$, then $x \in D$ iff $x = f(s) + \text{tal}_{c_r}$. It follows that we can decide if $x \in D$ in polynomial time in $|x|$ so that $D$ is a p-time height increasing independent set which is independent over $V$. □

**Corollary 3.10.** Let $V \in \text{NP}$ be a subspace of $\text{tal}(V_\infty)$ such that $V$ contains an infinite height increasing independent set $C$ in $\text{P}$. Then $V$ is not NP-simple or P-simple.

**Proof.** We may assume that $V$ is co-infinite dimensional since otherwise $V$ cannot be $\text{NP}^A$-simple or $\text{P}^A$-simple. We can thus use the proof of Theorem 3.9 to construct a p-time infinite height increasing independent set $D$ such that $D$ is independent over $V$. It follows by Theorem 3.3, that $\text{space}(D)$ is a p-time subspace of $\text{tal}(V_\infty)$. Since $D$ is independent over $V$, $\text{space}(D) \cap V = \{0\}$ so that $V$ is not NP-simple or P-simple. □

To prove that there exists a recursive oracle $B$ such that $\text{NP}^B \neq \text{P}^B$ and yet no $\text{NP}^B$-maximal, $\text{NP}^B$-simple, or $\text{P}^B$-simple subspaces exist, we need a result from [35].

**Theorem 3.11** (Nerode and Remmel [35]). There is a recursive oracle $B$ such that $\text{P}^B \neq \text{NP}^B$ and such that every infinite set $X$ which is p-time Turing reducible to a set $Y$ in $\text{NP}^B$ contains an infinite subset in $\text{P}^B$.

**Theorem 3.12.** There is a recursive oracle $B$ such that $\text{P}^B \neq \text{NP}^B$ and no $\text{NP}^B$-maximal, $\text{NP}^B$-simple, or $\text{P}^B$-simple subspaces of $\text{tal}(V_\infty)$ exist.
Proof. Let $B$ be the recursive oracle of Theorem 3.11. Let $V$ be an NP$^B$ subspace of $tal(V_{\infty})$ such that the dimension of $tal(V_{\infty})/V$ is infinite. By Theorem 3.2, $B_V$ is p-time Turing reducible to $V$ so that $B_V$ contains an infinite subset $E$ in P$^B$. Thus $E$ is an infinite height increasing independent set in P$^B$ so that by Theorem 3.2, $space(E)$ is an infinite dimensional subspace in P$^B$. Clearly, $space(E) \cap V = \{0\}$ so that $space(E)$ witnesses that $V$ is not P$^B$-simple or NP$^B$-simple. Moreover, since we can test whether $tal(e_1), \ldots, tal(e_n)$ are in $E$ in polynomial time in $|tal(e_n)|$, the set $E_2 = \{tal(e_n) \in E : card(E \cap \{tal(e_1), \ldots, tal(e_n)\})$ is even} is also a p-time height increasing independent set. We claim that $W = space(V \cup E_2)$ is a subspace of $tal(V_{\infty})$ which witnesses that $V$ is not NP$^B$-maximal. Note that $B_V \cup E_2$ is a height increasing basis for $W$ and that $E - E_2 \subseteq B_V$. Thus $W \supseteq V$ and the dimensions of both $tal(V_{\infty})/W$ and $W/V$ are infinite. Because $B_V \cup E_2$ is a height increasing basis for $W$, it follows that $x \in W$ iff there exists a $b \in V$ and an $e \in space(E_2)$ such that $x = b + tal e$ and $h(b), h(e) < h(x)$. Thus given a $B$-oracle, we can nondeterministically guess $b$ and $e$ of length $\leq k|x|$ and the computation which shows that $b \in V$, and then verify in polynomial time that $x = b + tal e$ and $e \in space(E_2)$. Thus $W \in$ NP$^B$ and hence $V$ is not NP$^B$-maximal. □

We note that in light of Theorem 3.8, it also follows that for the oracle $B$ of Theorem 3.11, every NP$^B$ subspace $V$ of $tal(V_{\infty})$ has a basis in P$^B$. We note that in [35], we proved that there exists a recursive oracle $A$ such that there exists an NP$^A$ subspace of $tal(V_{\infty})$ which had no basis in P$^A$. Also in [35], we constructed exponential time subspaces of $tal(V_{\infty})$ which have no basis in P.

3.2. Bases and subspaces of $st(V_{\infty})$

It will be convenient to use an alternative representation of $st(V_{\infty})$. That is, if the underlying field $F$ has $k$ elements, then given an element of $x \in Bin(N) - \{0\}$, we can compute its $k$-ary expansion $x = a_0 + a_1 \cdot k + \cdots + a_n k^n$ where $0 \leq a_i \leq k - 1$ for all $i$ and $a_n \neq 0$ in polynomial time $|x|$. Thus we can define a polynomial time isomorphism from $\Psi$ which maps $Bin(N)$ onto $ST_k = \{0\} \cup \{0, \ldots, k - 1\}^* \{1, \ldots, k - 1\}$ by $\Psi(0) = 0$ and for $x \in Bin(N) - \{0\}$, $\Psi(x) = a_0 \cdots a_k$ where $x = a_0 + a_1 \cdot k + \cdots + a_n k^n$. We can then use $\Psi$ to induce operations of sum $+_st_k$ and scalar multiplication for each $\lambda \in F$, $\lambda_{st_k}$, from the operations $+_st_k$ and $\lambda_{st_k}$ on $st(V_{\infty})$ to turn $ST_k$ into a vector space isomorphic to $V_{\infty}$. Thus we shall implicitly identify $st(V_{\infty})$ with the polynomial time structure $(ST_k, +_{st_k}, \lambda_{st_k}, \ldots, (k - 1)_{st_k})$. The main advantage of this identification is that we will now have the property that for a nonzero $x$ in $st(V_{\infty})$, the length of $x$ will equal the height $h$.

As pointed out in the introduction to this section, there is a significant difference between $st(V_{\infty})$ and $tal(V_{\infty})$ with regard to searches. Indeed many of the proofs of the propositions and theorems in the previous subsection relied on the fact that given an $x \in tal(V_{\infty})$, we could produce a list of all elements $tal(V_{\infty})$ of height $\leq h(x)$ in polynomial time in $|x|$. This is no longer the case in $st(V_{\infty})$. That is, if $x \in tal(V_{\infty})$ and $h(x) = n$, then $k^{n-1} \leq |x| \leq k^n - 1$ while if $x \in st(V_{\infty})$, then $h(x) = |x|$ so that there
are \( k^{|x|} - 1 \) elements of height less than or equal to \( h(x) \) in \( st(V_{\infty}) \). Thus in \( st(V_{\infty}) \), we cannot find all the elements of height less than or equal to \( h(x) \) in a \( p \)-time height increasing set \( S \) in polynomial time \( |x| \). However, there is a special class of \( p \)-time independent sets of \( st(V_{\infty}) \), which we call strongly \( p \)-time independent sets, which do have most of the useful properties possessed by \( p \)-time height increasing bases of \( tal(V_{\infty}) \).

**Definition 3.13.** An independent set \( B \subseteq st(V_{\infty}) \) is called **strongly \( p \)-time** if

(i) \( B \) is a \( p \)-time set,

(ii) \( B \) is height increasing, and

(iii) If \( B = \{ b_0, b_1, \ldots \} \) where \( h(b_0) < h(b_1) < \cdots \), then there is a polynomial time function \( f \) such that for all \( n > 0 \)

(iiiia) \( f(1^n) = b_k \) if \( h(b_k) = n \) and \( B \) has an element of height \( n \), and

(iiiib) \( f(1^n) = \emptyset \) if \( B \) has no element of height \( n \).

We note that condition (iii) allows us to find, for any \( x \in st(V_{\infty}) \), all elements of \( b \) of height \( \leq h(x) \) in polynomial time in \( |x| \). That is, given \( x \in st(V_{\infty}) \), \( h(x) = |x| \) and we can compute \( f(1), f(1^2), \ldots, f(1^{h(x)}) \) in polynomial time \( |x| \). Then \( \{ b : b \in B \wedge h(b) \leq h(x) \} = \{ f(1^n) : n \leq |x| \wedge f(1^n) \neq \emptyset \} \). As noted above, any \( p \)-time height increasing independent set \( B \) in \( tal(V_{\infty}) \) also has the property that, for any \( x \), we can find all elements of \( B \) of height \( \leq h(x) \) in polynomial time in \( |x| \). Thus condition (iii) is specifically designed to give us this property which holds for all \( p \)-time height increasing bases in \( tal(V_{\infty}) \) automatically. It is easy to see that our standard basis \( \{ e_1, e_2, e_3, \ldots \} \) of \( st(V_{\infty}) \) is strongly \( p \)-time.

Our next proposition lists several basic properties of subspaces generated by subsets of a strongly \( p \)-time basis.

**Theorem 3.14.** Let \( B \) be a strongly \( p \)-time basis of \( st(V_{\infty}) \) and suppose that \( S \subseteq B \). Then

(i) \( S \in P \) iff \( space(S) \in P \).

(ii) \( S \in NP \) iff \( space(S) \in NP \).

(iii) \( S \in Co-NP \) iff \( space(S) \in Co-NP \).

(iv) \( S \equiv_P space(S) \).

**Proof.** Since \( S = space(S) \cap B \), it follows that \( S \leq_P space(S) \) and \( S \) is in \( P(NP, Co-NP) \) if \( space(S) \) is in \( P(NP, Co-NP) \).

Let \( f \) be the \( p \)-time function such that \( f(1^n) = b_n \), where \( b_n \) is the element of height \( n \) in \( B \). Then, given an \( x \in st(V_{\infty}) \) of height \( n \), we can compute \( f(1) = b_1, \ldots, f(1^n) = b_n \) and test \( b_1, \ldots, b_n \) for membership in \( S \), all in time polynomial in \( |x| \). Thus in polynomial time in \( |x| \), we can find \( \{ s_1, \ldots, s_k \} \), where \( \{ b_{s_1}, \ldots, b_{s_k} \} = \{ y \in S : h(y) \leq h(x) \} \). Moreover, the fact that \( B \) is a height increasing basis means that \( x = \sum_{i=1}^{|x|} \lambda_i b_i \) for some \( \lambda_1, \ldots, \lambda_{|x|} \) in \( F \). Now, suppose that \( |x| = n \), then we can write \( x = x_1 \cdots x_n \) where all \( x_i \in F \) and and each \( b_i = b_{i_1} \cdots b_{i_n} \) where \( b_{i,j} \in F \). Then we
can solve the matrix equation over $F$

$$BY = X,$$

where $B = (b_{i,j})$, $Y$ is a column vector of unknowns, and $X$ is a column vector $(x_1, \ldots, x_n)$ in polynomial time in $n = |n|$. Thus in polynomial time in $|x|$, we can find $\lambda_1, \ldots, \lambda_k$ such that $x = \sum_{i=1}^{[n]} \lambda_ib_i$. This given,

$$x \in \text{space}(S) \iff \{i : \lambda_i \neq 0\} \subseteq \{s_1, \ldots, s_k\}.$$

It then easily follows that $\text{space}(S) \leq_T S$ and $\text{space}(S)$ is in P (NP, Co-NP) if $S$ is in P (NP, Co-NP). □

Our next result is a weak analogue for $st(V_\infty)$ of Theorem 3.3.

**Theorem 3.15.** Let $V$ be a subspace of $st(V_\infty)$ with strongly p-time basis $R$. The $R \cup B_\varphi$ is a strongly p-time basis for $st(V_\infty)$ and both $V$ and space($B_\varphi$) are in P.

**Proof.** Let $f$ be the p-time function such that
1. $f(1^n) = 0$ if $R$ has no element of height $n$, and
2. $f(1^n) = r_n$, where $r_n$ is the unique element of height $n$ in $R$ otherwise.

Recall that $(B_\varphi) = \{e_n : V \text{ has no element of height } n\}$. Thus $(B_\varphi) = \{e_n, f(1^n) = 0\}$, from which it follows easily that $(B_\varphi) \in P$. Hence $R \cup (B_\varphi)$ is a height increasing basis of $st(V_\infty)$ in P. Next, define $g : \{0,1\}^* \to st(V_\infty)$ by
1. $g(x) = f(x)$ if $x \notin \{1\}^*$,
2. $g(x) = f(x)$ if $x \in \{1\}^*$ and $f(x) \neq \emptyset$,
3. $g(x) = st(e|x|)$ otherwise.

Then, clearly, $g$ is a p-time function such that for all $n$, $g(1^n)$ is an element of height $n$ in $R \cup (B_\varphi)$. Thus $R \cup (B_\varphi)$ is a strongly p-time basis for $st(V_\infty)$. It now follows from Proposition 3.14 that $V$ and space($B_\varphi$) are in P since they are generated by p-time subsets of a strongly p-time basis of $st(V_\infty)$. □

Our next theorem shows that no extra condition on height increasing basis, such as condition (iii), is required to generate subspaces of $st(V_\infty)$ in NP.

**Theorem 3.16.** Let $B$ be a height increasing independent set of $st(V_\infty)$ which is in NP. Then space($B$) is in NP.

**Proof.** The key property that a height increasing basis has is that if $x \in \text{space}(B)$, then $x \in \text{space} (\{b \in B : h(b) \leq h(x)\})$. That is, $x$ must be generated by the elements of height $\leq h(x)$ in $B$ if $x \in \text{space}(B)$. Thus to see that space($B$) $\in$ NP, we simply guess the elements of $B$ of height $\leq h(x)$, say $\{b_1, \ldots, b_k\} = \{b \in B : h(b) \leq h(x)\}$, where $h(b_1) < \cdots < h(b_k)$. Now, for all nonzero $y \in st(V_\infty)$, $h(y) = |y|$ so $|b_i| \leq |x|$ for all $i$ and $k \leq |x|$. Then we perform a nondeterministic polynomial time computation to check if $b_1, \ldots, b_k$ are all in $B$. Finally, we use our polynomial time dependence algorithm to check whether $x \in \text{space}(\{b_1, \ldots, b_k\})$. Thus space($B$) is in NP. □
Theorem 3.17. Suppose \( \mathsf{NP}^X = \mathsf{Co-NP}^X \) and \( V \) is a subspace of \( \text{st}(V_\infty) \). Then

(i) \( V \in \mathsf{NP}^X \) iff \( V \) has a height increasing basis in \( \mathsf{NP}^X \).

(ii) \( V \in \mathsf{NP}^X \) implies \( V \) has a complementary subspace \( W \) in \( \mathsf{NP}^X \).

Proof. By the relativized version of Theorem 3.16, if \( V \) has a height increasing basis in \( \mathsf{NP}^X \), then \( V \in \mathsf{NP}^X \).

Next suppose that \( V \in \mathsf{NP}^X \). Thus \( \text{st}(V_\infty) - V \) is in \( \mathsf{Co-NP}^X \) and hence is in \( \mathsf{NP}^X \).

It is easy to see \( H(V) = \{ \text{st}(e_n) : \exists x \in V \text{ and } h(x) = n \} \) is also in \( \mathsf{NP}^X \). Note that \( B = \{ \text{st}(e_n) : n \geq 1 \} \) is a strongly p-time basis for \( \text{st}(V_\infty) \). Moreover, is easy to see that \( B_V = B - H(V) \) is in \( \mathsf{Co-NP}^X \) and hence is in \( \mathsf{NP}^X \). Thus by the relativized version of Theorem 3.16, \( \text{space}(B_V) \) is in \( \mathsf{NP}^X \). Note also that \( \text{st}(V_\infty) - V \) is the union of the \( \mathsf{NP}^X \) sets \( \{ x \in \text{st}(V_\infty) : x \in V \text{ and } \exists y(|y| = |x|, y \in V, y < x) \} \) and \( \text{st}(V_\infty) - V \) so that \( B_V \) is in \( \mathsf{Co-NP}^X \) and hence is in \( \mathsf{NP}^X \). Thus \( B_V \in \mathsf{NP}^X \) so that \( V \) has a height increasing basis in \( \mathsf{NP}^X \). \( \Box \)

Next we will show that the assumption \( \mathsf{NP}^X = \mathsf{Co-NP}^X \) also eliminates the possibility of the existence of \( \mathsf{NP}^X \)-simple and \( \mathsf{NP}^X \)-maximal sets in \( \text{st}(V_\infty)/V \).

Theorem 3.18. Suppose that \( \mathsf{NP}^X = \mathsf{Co-NP}^X \) and \( V \) is an \( \mathsf{NP}^X \) subspace of \( \text{st}(V_\infty) \) such that \( \text{st}(V_\infty)/V \) is infinite dimensional. Then \( V \) is not \( \mathsf{NP}^X \)-simple and \( V \) is not \( \mathsf{NP}^X \)-maximal.

Proof. By Theorem 3.17, it follows that \( \text{space}(B_V) \in \mathsf{NP}^X \) so that \( V \) is not \( \mathsf{NP}^X \)-simple. To see that \( V \) is not \( \mathsf{NP}^X \)-maximal, note that by our argument in Theorem 3.17, it follows that for any given \( x \in \text{st}(V_\infty) \), we can nondeterministically find, from an \( X \) oracle, a list of all elements \( u_1 < \cdots < u_s \) of height \( \leq h(x) \) which are in \( B_V \) and a list of all elements \( v_1 < \cdots < v_t \) of height less than or equal to \( h(x) \) which are in \( B_{\overline{V}} \) since both \( B_V \) and \( B_{\overline{V}} \) are in \( \mathsf{NP}^X \). Thus we can form a new \( \mathsf{NP}^X \) height increasing independent set where \( x \in C \) iff \( x = u_i \) for some \( i \leq s \) or \( x = v_{2k} \) for some \( 2k \leq t \). It is then easy to see that both \( \text{st}(V_\infty/\text{space}(C)) \) and \( \text{space}(C)/V \) are infinite dimensional. It also follows from Theorem 3.4 that \( \text{space}(C) \in \mathsf{NP}^X \) so that \( C \) witnesses that \( V \) is not \( \mathsf{NP}^X \)-maximal. \( \Box \)

We note that Baker et al. [2] produced recursive oracles \( X \) such that \( \mathsf{NP}^X \neq \mathsf{P}^X \) but \( \mathsf{NP}^X = \mathsf{Co-NP}^X \). Thus we have the following.

Theorem 3.19. There exists a recursive oracle \( A \) such that \( \mathsf{NP}^A \neq \mathsf{P}^A \) and there are no \( \mathsf{NP}^A \)-simple or \( \mathsf{NP}^A \)-maximal subspaces of \( \text{st}(V_\infty) \).

The result of our next theorem is analogue of Theorem 3.9 for \( \text{st}(V_\infty) \).

Theorem 3.20. Let \( V \) be a recursive co-infinite dimensional subspace of \( \text{st}(V_\infty) \) such that \( V \) contains an infinite strongly p-time height increasing independent set \( C \). Then
there is an infinite strongly p-time height increasing independent set $D$ such that $V \cap \text{space}(D) = \{0\}$.

**Proof.** Note that $B_\mathbb{P}$ is recursive. Let $b_0, b_1, \ldots$ be a list of elements of $B_\mathbb{P}$ such that $h(b_0) < h(b_1) < \cdots$. Let $f$ be a recursive function such that $f(0^n) = b_n$. Let $g$ be a p-time function such that $g(1^n) = 0$ if $C$ has no element of height $n$ and $g(1^n) \in C$ be the unique element of $C$ of height $n$ if $C$ has an element of height $n$. Let $C = \{c_0, c_1, \ldots\}$ where $h(c_0) < h(c_1) < \cdots$. Then let $d_s = b_s +_{st} c_{r(s)}$ where

$$r(s) = 1 + \sum_{i=0}^{s} h(b_i) + \text{the number of steps to compute } f(0), \ldots, f(s).$$

Then we claim that $D = \{d_0, d_1, \ldots\}$ is our required height increasing independent set. First observe that by our definition of $r(s)$, $r(s) > h(b_s)$ so that $h(d_s) = h(c_{r(s)})$. Also it is clear that $r(0) < r(1) < \cdots$ so that $h(d_0) < h(d_1) < \cdots$. Thus $D$ is a height increasing basis. Moreover, it is easy to see that $D$ is independent over $V$. Thus we need only show that $D$ is strongly p-time. To this end, we construct a p-time function $t$ as follows. If $x \neq 1^n$ for some $n$, then define $t(x) = 0$. If $x = 1^n$, we first compute which elements $y$ with $h(y) \leq n$ are in $C$. This requires us to compute $g(1), \ldots, g(1^n)$ which we can compute in polynomial time in $|x|$. If no element of height $n$ is in $C$, then clearly no element of height $n$ is in $D$ so we define $t(x) = \emptyset$. If there is an element of height $n$ in $C$, then in polynomial time in $|x|$, we can find $r$ such that $c_r = g(1^n)$. At this point, we start to compute the sequence of elements $f(0), f(1), \ldots$ in order for $r$ steps. Suppose that at the end of $r$ steps, we have successfully computed $f(0), \ldots, f(t)$. Note that if we are not successful in computing $f(0)$ by the end of $r$ steps, then no element of height $n$ is in $D$ so we let $t(x) = \emptyset$. Otherwise, see if there is some $s \leq t$ such that

$$r = 1 + \sum_{i=0}^{s} h(b_i) + \text{the number of steps to compute } f(0), \ldots, f(s).$$

If there is no such $s$, then again no element of height $n$ is in $D$ so we let $t(x) = \emptyset$. If there is such an $s$, then we set $t(x) = f(s) +_{st} c_r \in D$. It follows that $t$ is a p-time function which witnesses that $D$ is a strongly p-time height increasing independent set which is independent over $V$. \[\square\]

**Corollary 3.21.** Let $V$ be an NP co-infinite dimensional subspace of $st(V_\infty)$ such that $V$ contains an infinite strongly p-time height increasing independent set $C$. Then $V$ is not NP-simple or P-simple.

**Proof.** Use the proof of Theorem 3.20 to construct a strongly p-time infinite height increasing independent set $D$ such that $D$ is independent over $V$. It follows by Theorem 3.14, that $\text{space}(D)$ is a p-time subspace of $(V_\infty)$. Since $D$ is independent over $V$, $\text{space}(D) \cap V = \{0\}$ so that $V$ is not NP-simple or P-simple. \[\square\]
Theorem 3.22. There is a recursive oracle B such that \( P^B \neq NP^B \) and no \( NP^B \)-maximal, \( NP^B \)-simple, or \( P^B \)-simple subspaces of \( st(V_\infty) \) exist.

Proof. Let B be the recursive oracle of Theorem 3.11. Let V be an \( NP^B \) subspace of \( st(V_\infty) \) such that the dimension of \( st(V_\infty)/V \) is infinite. Note that the set \( S = \{ \text{st}(e_n) : \exists y(|y| = n \land y \in V) \} \) is \( NP^B \) and the set \( B = \{ \text{st}(e_n) : n \geq 1 \} \) is in P so that \( B' = \{ \text{st}(e_n) : \exists y(|y| = n \land y \in V) \} = B - S \) is p-time Turing reducible to \( S \in NP^B \). Hence \( B' \) contains infinite \( P^B \) subset E. Clearly, \( E \) is an infinite height increasing independent set and we can determine if \( st(e_n) \in E \) in p-time in \(|st(e_n)| = n\). Thus we can define a polynomial time function \( f \) such that \( f(x) = 0 \) if \( x \neq 1^n \), \( f(1^n) = st(e_n) \) if \( st(e_n) \in E \), and \( f(1^n) = \emptyset \) otherwise. Then \( f \) witnesses the fact that \( E \) is a strongly p-time independent set. It follows from Theorem 3.2, \( space(E) \) is a subspace in \( P^B \). Clearly, \( space(E) \cap V = \{ 0 \} \) so that \( space(E) \) witnesses that \( V \) is not \( P^B \)-simple or \( NP^B \)-simple. Moreover, since we can test whether \( st(e_1), \ldots, st(e_n) \) are in \( E \) in polynomial time in \(|e_n|\), the set \( E_2 = \{ st(e_n) \in E : \text{card}(E \cap \{ st(e_1), \ldots, st(e_n) \}) \text{ is even} \} \) is also a p-time height increasing independent set. We claim that \( W = space(V \cup E_2) \) is a subspace of \( st(V_\infty) \) which witnesses that \( V \) is not \( NP^B \)-maximal. Note that \( B_V \cup E_2 \) is a height increasing basis for \( W \) and that \( E_2 \subseteq B \). Thus \( W \subseteq V \) and the dimensions of both \( tal(V_\infty)/W \) and \( W/V \) are infinite. Because \( B_V \cup E_2 \) is a height increasing basis for \( W \), it follows that \( x \in W \) iff there exist \( b \in V \) and \( e \in space(E_2) \) such that \( x = b + st e \) and \( h(b), h(e) \leq h(x) \). Thus given a B-oracle, we can nondeterministically guess \( b \) and \( e \) of length \( \leq |x| \) and a computation which shows that \( b \in V \), and then verify in polynomial time that \( x = b + st e \) and \( e \in space(E_2) \). Thus \( W \in NP^B \) and hence \( V \) is not \( NP^B \)-maximal.

Our next result is a weak analogue of Theorem 4.4 of [35] for \( st(V_\infty) \).

Theorem 3.23. Let \( V \) be an r.e. infinite dimensional subspace of \( st(V_\infty) \). Suppose that there exists an infinite strongly p-time independent subset \( I \subseteq V \). Then \( V \) has a basis in P.

Proof. Let \( B = \{ b_0 < b_1 < \cdots \} \) where \( h(b_0) < h(b_1) < \cdots \). Let \( f \) be a p-time function such that

1. \( f(1^n) = \emptyset \) if \( B \) has no element of height \( n \) and
2. \( f(1^n) = b_n \) where \( b_n \) is the unique element of height \( n \) in \( B \) otherwise.

Note that for any \( x \in st(V_\infty) \), \(|x| = h(x) \) so that we can find all elements \( b_{i_0}, \ldots, b_{i_k} \) of \( B \) of height \( \leq h(x) \) in polynomial time in \(|x| \). That is, we can compute \( f(1), \ldots, f(1^{|x|}) \) in \( q(|x|) \) steps for some fixed polynomial \( q \) and \( \{ b_{i_0}, \ldots, b_{i_k} \} = \{ f(1^j) : j \leq |x| \land f(1^j) \neq \emptyset \} \). Every r.e. subspace of \( V_\infty \) has a recursive basis [16]. So let \( X = \{ x_0 < x_1 < x_2, \ldots \} \) be a recursive basis for \( V \). Let \( st(V_\infty) = \{ v_0 < v_2 < v_2 < \cdots \} \). Intuitively we would like to define a basis \( I \) for \( V \) inductively in stages. At stage \( 0 \), we would like to add \( b_{i_0} \) and \( b_{i_0} + x_0 \) to \( I \) for an \( i_0 \) so large that
(i) \( h(b_0) > h(x_0) \).

(ii) We have enough time so that if we run \(|b_i| - n\) steps of the Turing machine \( M \)
which computes the characteristic function of \( X \) on \( v_n \) for \( n = 0, 1, \ldots, |b_i| \), then \( M \)
has converged on all \( y \in V_\infty \) with \( y < x_0 \). Then at stage 1, we would like to pick \( i_1 \)
large enough so that if we run \( M \) for \(|b_i| - n\) steps on \( v_n \) for \( n = 0, \ldots, |b_i| \), then \( M \)
has converged on all \( y \in V_\infty \) with \( y < x_1 \) and \( h(b_i) > h(b_0) + h(x_0) + h(x_1) \).
Then if \( x_1 \not\in (\{b_0, b_0 + x_0\})^* \), we can add \( b_i \) and \( b_i + x_1 \) to \( I \). Continuing in this way
we can use the strongly \( p \)-time independent set \( B \) to modify the basis \( X \) to produce a
polynomial time basis for \( V \).

More formally, to decide if \( x \in I \), proceed as follows. Let \( h(x) = n \) and run \(|x| - i \)
steps of \( M \) on \( v_i \) for \( i = 0, \ldots, |x| \). Let \( z_0 \) be the largest \( z \leq |x| \) such that \( M \)
converges on \( v_y \) for all \( y \leq z \) and let \( x_0, \ldots, x_m \) be the elements of \( X \) in \( \{v_y : y \leq z_0\} \). This takes
\((|x|/2) \) steps and hence is polynomial in \(|x| \). Also let \( b_0, \ldots, b_p \) be the elements of \( B \) of
height less than or equal to \( |x| \). Again we can find these in polynomial time in \(|x| \). Now, if \( h(b_p) \neq n \), so that \( B \) has no elements of height \( n \), then \( x \not\in I \). Otherwise,
proceed inductively to define a sequence \( i_0, i_1, \ldots, i_\ell \) and an increasing sequence of sets
\( I_0 \subseteq I_1 \subseteq \cdots \subseteq I_\ell \) as follows.

Let \( i_0 \) be the least \( i \leq p \) such that if \( x_0 = v_{r_0} \), then for all \( q < r_0 \), \( M \) converges on
\( v_q \) in less than \(|b_1| - q \) steps and \( h(b_i) > h(x_0) \). Set \( I_0 = \{b_0, b_0 + x_0\} \).

Suppose that \( i_0, i_1, \ldots, i_s \) and \( I_s \) have been defined. Let \( i_{s+1} \) be the least \( i \leq m \) such
that if \( a_{s+1} = \mu a (a \leq p \land x_a \not\in (I_s)^*) \) and \( x_{a_{s+1}} = v_r \), then for all \( q < r \), \( M \) converges on
\( v_q \) in less than \(|b_1| - q \) steps, \( h(b_i) > h(z) \) for all \( z \in I_s \), and \( h(b_i) > h(x_{a_{s+1}}) \). Set
\[
I_{s+1} = I_s \cup \{b_{i_{s+1}}, b_{i_{s+1}} + x_{a_{s+1}}\}.
\]

Finally, let \( \ell \) be the least integer \( \leq m \) such that \( i_{\ell+1} \) is not defined. It is easy to
see that we can compute this sequence in polynomial time in \(|x| \). Then we put \( x \)
into \( B \) iff \( x \in I_\ell \). It is easy to see that for each \( s \), our choice of \( I_{s+1} \) ensures that
\( \{b_{i_{s+1}}, b_{i_{s+1}} + x_{a_{s+1}}\} \) is independent over \( I_s \). Thus it is easy to prove by induction that
\( I_s \) is an independent set and that \( \{x_0, \ldots, x_s\} \subseteq space(I_s) \). Thus the above procedure
defines a polynomial time set \( I = \bigcup_s I_s \) such that \( I \subseteq V \). But as \( X \subseteq space(I) \) and \( I \)
is independent, it follows that \( I \) is a polynomial time basis for \( V \). \( \square \)

Our next result shows that if a subspace \( V \) of \( st(V_\infty) \) has \( p \)-time basis, then the
Turing degree of \( V \) is only limited by the fact that it must be the Turing degree of an
r.e. set.

**Theorem 3.24.** Given any r.e. Turing degrees \( \delta \), there exists an r.e. subspace \( V \) of
\( st(V_\infty) \) such that \( V \) has degree \( \delta \) and \( V \) has a basis in \( P \).

**Proof.** Let \( B_1 = \{st(e_{2n}) : n \geq 1\} \). Clearly, \( B_1 \) is a strongly \( p \)-time independent set.
Then for any given r.e. degree \( \delta \), let \( B_\delta \) be an infinite r.e. subset of \( \{st(e_{2n+1}) : n \geq 0\} \).
of degree $\delta$. It is easy to see that the Turing degree of $V_\delta = \text{space}(B_1 \cup B_\delta)$ is $\delta$. By Theorem 3.23, $V_\delta$ has a basis in $P$, since $V_\delta$ contains $B_1$. \qed

Next we show that it is not true that every r.e. subspace of $\text{st}(V_\infty)$ has a basis in $P$.

**Theorem 3.25.** There is a subspace $V$ of $\text{st}(V_\infty)$ such that $V \in \text{DEXT}$ and $V$ has no basis in $P$.

**Proof.** Let $P_0, P_1, \ldots$ be our effective list of all polynomial time subsets of $\{0, \ldots, k - 1\}^*$ and recall that $p_i(n) = \max(2, n)^i$ is a strict upper bound on the length of any computation of $P_i$ started on inputs of length $n$. Note that for $n \geq 2^n$, we have $2^n \geq n^i + 1$ for all $n$.

We construct a set $E \subseteq \{\text{st}(e_n) : n \geq 1\}$ in stages so that $V = \text{space}(E)$ is our desired subspace. Let $E_n$ denote those elements of $E$ put into $E$ by the end of stage $n$. At any given stage $n$ of our construction, we will ensure that $E_n - E_{n-1} \subseteq \{\text{st}(e_n)\}$ so that at stage $n$ our basic decision is whether or not to add $\text{st}(e_n)$ to $E$. By a proof which is similar to Theorem 3.14, it is easy to show $V \in \text{DEXT}$ if $E \in \text{DEXT}$. We shall construct an infinite $E$ to meet the following set of requirements:

$R_e$: Either $P_e$ is not an infinite independent set $\subseteq \text{st}(V_\infty)$ or there is an $x \in P_e$ such that $h(x) = n$ and $\text{st}(e_n) \notin E$.

Note that if $\text{st}(e_n) \notin E$, then $\text{space}(E)$ has no elements of height $n$, so that meeting requirement $R_e$ will ensure that $P_e$ is not a basis for $\text{space}(E)$.

Our strategy for meeting a single requirement $R_e$ is as follows. At stage $n$, we will search all the elements of $\{0, \ldots, k - 1\}^*$ of length $\leq n$ and find all elements of $P_e$ of length $\leq n$. Say $\{p_0, \ldots, p_k\} = \{x \in P_e : |x| \leq n\}$. There are four cases.

Case 1: For some $i$, $p_i \not\in \text{st}(V_\infty)$.

Case 2: $\{p_0, \ldots, p_k\}$ are dependent.

Case 3: $\exists i \exists m \leq n(h(p_i) = m)$ and $\text{st}(e_m) \notin E_{n-1}$.

Case 4: Not case 1, 2 or 3.

If cases 1, 2 occur or case 3 occurs for some $m < n$, then we will have automatically ensured that requirement $R_e$ is met so we do not have to take any action at stage $n$ to meet requirement $R_e$. Otherwise if case 3 holds so that there is a $p_i \in P_{e,n}$ with $h(p_i) = n$, then we simply want to ensure that $\text{st}(e_n) \notin E$, for this we will ensure that requirement $R_e$ is met. If Case 4 occurs, then we want to put $\text{st}(e_n) \in E$. Our idea in this case is that if $P_e$ were a basis of $\text{space}(E)$, then since $\text{st}(e_n) \notin \text{space}(\{x \in P_e : h(x) \leq n\})$, it must be the case the linear combination of elements of $P_e$ which equals $\text{st}(e_n)$ must involve some element $x \in P_e$ such that $h(x) > n$. Thus, there will be some later stage at which we will be in case 3 for $R_e$. Now, if there is an $m < n$ such that either case 1 or case 2 holds at stage $m$, then $P_e$ will not be an independent subset of $\text{st}(V_\infty)$ so that we will say requirement $R_e$ is cancelled at stage $n - 1$. If there is an $m < n$ such that case 3 holds and $\text{st}(e_m) \notin E_{n-1}$ and for some $p_i \in P_{e,n-1}$, $h(p_i) = m$, then we will say $R_e$ is satisfied at stage $n - 1$. Otherwise, if there is an $m < n$ such
that case 4 holds at stage $m$ and $st(e_m) \in E$, and $st(e_m) \notin \text{space}\{x \in P_e : h(x) \leq n - 1\}$, then we require $R_e$ to be active at stage $n - 1$. If $R_e$ is neither cancelled, satisfied, nor active at stage $n - 1$, then we say $R_e$ is inactive at stage $n - 1$.

**Construction of $E$.**

**Stage 0:** Let $E_0 = \emptyset$.

**Stage $n > 0$:** Suppose $E_{n-1} = \{st(e_{i_1}), \ldots, st(e_{i_k})\}$, where $1 \leq i_1 < \cdots < i_k < n$. For each $e$ such that $2^e \leq n$, find $P_e \cap \{x \in \{0, \ldots, k - 1\}^* : |x| \leq n\} = P_{e,n}$.

Since there are $\leq k^{n+1} - 1$ elements of $\{0, \ldots, k - 1\}^*$ of length $\leq n$ and $\max(2(n)^t) \leq 2^n$ for all such $e$, it takes at most $k^{2n+1}$ steps to find all such $P_{e,n}$. Note that the dimension of $\{x \in st(V_\infty) : h(x) \leq n\}$ is $n$, so that there are at most $n$ independent elements in $P_{e,n}$ for any such $e$. Thus if $P_{e,n} \not\subseteq st(V_\infty)$ or card $(P_{e,n}) > n$, then $P_e$ is not an independent subset of $st(V_\infty)$ so that requirement $R_e$ is automatically satisfied at stage $n$. Next use the polynomial time dependence algorithm for $st(V_\infty)$ to compute whether the remaining $P_{e,n}$ are independent and the dependence of $st(e_j)$ on $P_{e,n}$ for each $j \leq n$. Note that for all remaining $e$, card $(P_{e,n}) \leq n$ and $x \in P_{e,n}$ implies $|x| \leq n$ so that $|st(e_j)| + \sum_{x \in P_{e,n}} |x| \leq n^2 + n$. Since computing the dependence algorithm is polynomial time in the sum of the lengths of the elements of the set, it is easy to see that for some fixed polynomial $q$, we can compute all such dependencies in $q(n)$ steps.

Thus for some polynomial $p$, we can compute for each $e$ with $2^e \leq n$, whether $R_e$ is cancelled, satisfied, active, or inactive at stage $n - 1$ and whether $st(e_n) \in \text{space}(P_{e,n})$ in $k^{2n+1} + p(n)$ steps given $E_{n-1}$.

Now look for the least requirement $R_e$ such that $2^e \leq n$ and $R_e$ is not cancelled or satisfied or active at stage $n - 1$ and $P_{e,n}$ has an element of height $n$. If there is no such $e$, then set $E_n = E_{n-1} \cup \{st(e_n)\}$. In this case, all requirements $R_e$ with $2^e \leq n$ will either be cancelled, satisfied, or active at stage $n$. If there is such an $e$, let $e(n)$ be the least such $e$. Then if for some $i < e(n)$, $R_i$ is inactive at stage $n - 1$, let $E_n = E_{n-1} \cup \{st(e_n)\}$. If there is no such $i$, then set $E_n = E_{n-1}$.

This completes the construction of $E$. We now prove three lemmas which will show that $E$ has the desired properties.

**Lemma 3.26.** $E \in DEXT$.

**Proof.** Note that once we have computed for each $e$ with $2^e \leq n$, whether $R_e$ is satisfied, active, or inactive at stage $n - 1$ and whether $st(e_n) \in \text{space}(P_{e,n})$, the rest of our construction at stage $n$ takes at most $r(n)$ steps for some fixed polynomial $r$. Thus stage $n$ of our construction takes at most $nk^{2n+1} + p(n) + r(n)$ steps for any $n$ given $E_{n-1}$. Of course to decide if $st(e_n) \in E$, we must recompute stages 0, 1, 2, \ldots, $n - 1$ to find $E_{n-1}$ so that to decide if $st(e_n) \in E$ requires $\sum_{k=0}^n (jk^{2k+1} + p(j) + r(j)) \leq 2^{cn}$ steps for some constant $c$. Thus $E \in DEXT$ and $V = \text{space}(E) \in DEXT$. □

**Lemma 3.27.** For each $e \geq 0$, either $P_e$ is finite or there is a stage $n$ at which $R_e$ is satisfied or cancelled at stage $n$. 
Proof. We proceed by induction on $e$. Suppose for all $i < e$ either $P_i$ is finite or $R_i$ is satisfied or cancelled at some stage $s$. Note that once a requirement $R_i$ is satisfied or cancelled at stage $s$, it will be satisfied or cancelled at all stages $t \geq s$. Thus, there is a stage $n$ so large that for all $i < e$, either $R_i$ is satisfied or cancelled at stage $n$ or $x \in P_i$ implies $h(x) = |x| < n$. We may assume that $P_e$ is an infinite independent set of $st(V_{\infty})$ since otherwise $P_e$ is finite or $P_e$ not an independent subset of $st(V_{\infty})$ and hence $R_e$ will be eventually be cancelled at some stage $s$. So let $s > n$ be a stage such that there is an $x \in P_e$ with $|x| = s$. Then at stage $s$, either $R_e$ is satisfied or cancelled at stage $s - 1$ or $R_e$ is the least requirement which is not satisfied or cancelled at stage $s - 1$ such that $\exists x \in P_e(|x| = n)$ and hence $e(s) = e$. Now if there is a $j < e$ such that $R_j$ is inactive at stage $s - 1$, then by our choice of $s > n$, it must be the case that $st(e_j) \not\in \text{space}(P_j)$. Then we will put $st(e_j)$ into $E$ and $st(e_j)$ will witness that $R_j$ is active at stage $t$ for all $t \geq s$. Then consider a stage $s_1 > s$ such that $\exists x \in P_e(|x| = s_1)$. Then either $R_e$ is satisfied or cancelled at stage $s_1 - 1$ or once again we will have $e(s_1) = e$. But in the latter case, we will be guaranteed that for all $j < e$, $R_j$ is satisfied, cancelled, or active at stage $s_1 - 1$ so that our construction ensures $st(e_n) \not\in E$ and $R_e$ is satisfied at stage $s_1$. Thus in any case, either $P_e$ is finite or there is a stage at which $R_e$ becomes satisfied or cancelled. \[\square\]

Lemma 3.28. $E$ is infinite.

Proof. Suppose, on the contrary, that $E = \{st(e_{i_0}), \ldots, st(e_{i_k})\}$ for some $i_0 < \cdots < i_k$. Then for some $k$, $P_k = E$. Then let $n$ be a stage large enough so that for all $j \leq k$ either $R_j$ is satisfied or cancelled at stage $n$ or $P_j$ is finite and $x \in P_j$ implies $h(x) = |x| < n$. Then consider stage $n + 1$. If $e(n + 1)$ is defined at stage $n + 1$, then it must be that $e(n + 1) > k$. If $R_k$ is not active at stage $n$, then our construction would ensure $st(e_{n+1}) \in E$ contradicting the fact that $E = P_k$. If $R_k$ is active at stage $n$, then there must be some $st(e_j) \in E_n$ such that $st(e_j) \not\in \text{space}(P_{k,n}) = \text{space}(P_k)$. Thus in either case, our construction would ensure $P_k \neq F$. Thus $E$ is infinite. So the lemma is proven. \[\square\]

Thus $\text{space}(E)$ is an infinite dimensional space in DEXT which has no p-time basis. \[\square\]

4. Properties of $\text{NP}^X$-subspaces

In this section we shall study various properties of the lower semilattice of $\text{NP}^X$-subspaces of $tal(V_{\infty})$ and $st(V_{\infty})$ for various oracles $X$. Our first result shows that in contrast to the collection of r.e. subspaces which is closed under both intersection ($\cap$) and sum ($+$) and hence forms a lattice, the collection of $\text{NP}^X$-subspaces of either $tal(V_{\infty})$ and $st(V_{\infty})$ is only closed under intersection and hence only forms a lower semilattice.
Theorem 4.1. There exist two polynomial time subspaces \( W \) and \( V \) of \( \text{tal}(V_\infty) \) (\( \text{st}(V_\infty) \)) such that \( W \cap V = \{0\} \) and \( W + V \) is not recursive.

Proof. The proof that we present below works equally well for both \( \text{tal}(V_\infty) \) and \( \text{st}(V_\infty) \). Thus we shall write a generic proof where \( V_\infty \) may be interpreted as either \( \text{tal}(V_\infty) \) or \( \text{st}(V_\infty) \) and the standard basis \( e_1, e_2, \ldots \) may be interpreted as either the standard basis \( \text{tal}(e_1), \text{tal}(e_2), \ldots \) of \( \text{tal}(V_\infty) \) or the standard basis \( \text{st}(e_1), \text{st}(e_2), \ldots \) of \( \text{st}(V_\infty) \) as appropriate.

By a result of Metakides and Nerode [26], a subspace \( V \) of \( V_\infty \) is recursive iff \( V \) is r.e. and \( V \) has an r.e. complementary space. It is easy to see that we can form an effective list \( (A_0, B_0), (A_1, B_1), \ldots \) of all pairs of r.e. subspaces \( W_i \) and \( W_j \) of \( V_\infty \) such that \( W_i \cap W_j = \{0\} \). That is, if \( W_0, W_1, \ldots \) is an effective list of all r.e. subspaces of \( V_\infty \) and \( W_i^n \) denotes the set of elements enumerated into \( W_i \) after \( n \) steps, then \( (A_i, B_i) \) is the pair of r.e. subspaces given by letting \( (A_i, B_i) \) be \( (W_k, W_1) \) iff \( i = \langle k, l \rangle \) and \( W_k \cap W_l = \{0\} \) or letting \( (A_i, B_i) \) be \( (\text{space}(W_k^n), \text{space}(W_1^n)) \) where \( n \) is the least \( m \) such that \( \text{space}(W_k^{n+1}) \cap \text{space}(W_1^{n+1}) \neq \{0\} \) if \( W_k \cap W_l \neq \{0\} \).

Given the list \( (A_0, B_0), (A_1, B_1), \ldots \), we shall construct \( W \) and \( V \) so that \( W + V \not\equiv A_i \) for any \( i \) such that \( A_i + B_i = V_\infty \). Thus \( W + V \) will not be recursive. In the construction that follows we will in fact construct two p-time height increasing disjoint independent sets \( K \) and \( L \) so that \( W = \text{space}(K) \) and \( V = \text{space}(L) \) will be our desired polynomial time subspaces. Let \( r_0, r_1, \ldots \) be a list of all prime numbers in increasing order. Our idea is to use the vectors \( e_i + e_i, e_i, e_i, e_i, e_i \) for some \( i \geq 0 \) and \( n \geq 1 \), and the only vectors which will be placed into \( L \) will be of the form \( e_i, e_i \) for some \( i \geq 0 \) and \( n \geq 1 \). In fact, for any fixed \( i \) either

\[ K \cap \{e_i + e_i, 2^n | n \geq 1\} = \emptyset \]

and

\[ L \cap \{e_i, 2^n | n \geq 1\} = \emptyset \]

or there will be an \( m \) such that

\[ K \cap \{e_i + e_i, 2^n | n \geq 1\} = \{e_i + e_i, 2^n\} \]

and

\[ L \cap \{e_i, 2^n | n \geq 1\} = \{e_i, 2^n\}. \]

Note that in the standard representation of \( V_\infty \), \( L \) will be a polynomial time subset in the strongly p-time height increasing basis \( \{\text{st}(e_n) : n > 0\} \) and \( K \) will be a polynomial time subset of the strongly p-time height increasing independent set \( \{e_k + e_k, 2^n : k \text{ is odd and } n \geq 1\} \) so that \( L \) and \( K \) themselves will be strongly p-time independent sets. Thus by Theorem 3.14, \( W \) and \( V \) will be polynomial time subspaces of \( \text{st}(V_\infty) \).

In the tally representation of \( V_\infty \), \( K \) and \( L \) will be polynomial time height increasing
independent sets so that by Theorem 3.2, \( W \) and \( V \) will be polynomial time subspaces of \( \text{tal}(V_\infty) \).

Now to decide if \( e_r + e_r,2^n \in K \) and \( e_r,2^n \in L \), we run the enumerations of \( A_i \) and \( B_i \) for \( m \) steps. Let \( A_i^m \) and \( B_i^m \) denote those elements enumerated into \( A_i \) and \( B_i \), respectively, after \( m \) steps. If \( m > |e_r| \) and \( \text{space}(A_i^m) + \text{space}(B_i^m) \neq \text{space}(A_i^{m-1}) + \text{space}(B_i^{m-1}) \), then we place \( e_r + e_r,2^n \) into \( K \) and \( e_r,2^n \) into \( L \) if \( e_r \in \text{space}(A_i^m) + \text{space}(B_i^m) - \text{space}(A_i^m) \). Otherwise we place neither \( e_r + e_r,2^n \) into \( K \) nor \( e_r,2^n \) into \( L \).

Using the fact that in \( m \) steps, we can at most enumerate \( m \) vectors which are of length at most \( m \) and the fact that Gauss elimination is polynomial time in the dimensions of the matrix, it is easy to see that both \( K \) and \( L \) are p-time height increasing independent sets.

Now suppose \( e_r + e_r,2^n \in K \) and \( e_r,2^n \in L \). Since \( A_i \cap B_i = \{0\} \), we know that each element \( v \in \text{space}(A_i) \setminus \text{space}(B_i) \) has a unique expression in the form \( v = a + b \) with \( a \in \text{space}(A_i) \) and \( b \in \text{space}(B_i) \). By our construction, it follows that \( e_r \in \text{space}(A_i^m) + \text{space}(B_i^m) - \text{space}(A_i^m) \) so that \( e_r \notin \text{space}(A_i) \). But clearly \( e_r \in \text{space}(K) + \text{space}(L) \), so that \( A_i \neq \text{space}(K) + \text{space}(L) \).

Suppose there is no \( m \) such that \( e_r + e_r,2^n \in K \) and \( e_r,2^n \in L \). Then either there is no \( m \) such that \( e_r \in \text{space}(A_i^m) + \text{space}(B_i^m) \) in which case \( \text{space}(A_i) + \text{space}(B_i) \neq V_\infty \) so that we do not have to worry about \( A_i \) and \( B_i \), or \( e_r \in \text{space}(A_i^m) \) for some \( m \) (in which case \( e_r \notin \text{space}(K) + \text{space}(L) \) so again \( \text{space}(K) + \text{space}(L) \neq A_i \)).

We note that Breitbart [5] proved that if \( R \) is any infinite recursive set in \( \{0,1\}^* \), then there exists a set \( S \) in \( \mathbb{P} \) such that both \( S \cap R \) and \( R - S \) are infinite. This result shows that there can be no NP-maximal sets since if \( M \in \text{NP} \) and \( R = \{0,1\}^* - M \) is infinite, then certainly \( R \) is an infinite recursive set. Thus there is a set \( S \in \mathbb{P} \) such that both \( S \cap R \) and \( R - S \) are infinite. But then \( W = S \cup M \) is a set in \( \mathbb{NP} \) such that both \( W - M \) and \( \{0,1\}^* - M \) are infinite so that \( M \) is not NP-maximal. Our next result shows that the analogue of Breitbart's splitting theorem holds for recursive subspaces of \( \text{tal}(V_\infty) \) and \( \text{st}(V_\infty) \).

**Theorem 4.2.** Let \( V \) be an infinite dimensional recursive subspace of \( \text{tal}(V_\infty) \) (or \( \text{st}(V_\infty) \)). Then there exist subspaces \( B_0 \) and \( B_1 \) in \( \mathbb{P} \) such that \( B_0 \cap B_1 = \{0\} \), \( B_0 + B_1 = \text{tal}(V_\infty) \) (or \( \text{st}(V_\infty) \)), and both \( B_0 \cap V \) and \( B_1 \cap V \) are infinite dimensional.

**Proof.** Once again the proof that we present below works equally well for both \( \text{tal}(V_\infty) \) and \( \text{st}(V_\infty) \). Thus we shall write a generic proof where \( V_\infty \) may be interpreted as either \( \text{tal}(V_\infty) \) or \( \text{st}(V_\infty) \) and the standard basis \( e_1, e_2, \ldots \) may be interpreted as either the standard basis \( \text{tal}(e_1), \text{tal}(e_2), \ldots \) of \( \text{tal}(V_\infty) \) or the standard basis \( \text{st}(e_1), \text{st}(e_2), \ldots \) of \( \text{st}(V_\infty) \) as appropriate. Also we shall denote \( a_0, a_1, \ldots \) by \( \text{tal}(0), \text{tal}(1), \ldots \) if we are interpreting \( V_\infty \) as \( \text{tal}(V_\infty) \) and denote by \( \text{st}(0), \text{st}(1), \ldots \) if we are interpreting \( V_\infty \) as \( \text{st}(V_\infty) \).

Given a nonzero \( x \in V_\infty \), we can uniquely express \( x \) in the form \( x = \lambda_1 e_1 + \cdots + \lambda_n e_n \) where \( \lambda_1, \ldots, \lambda_n \) are nonzero elements of \( F \). In such a situation, we write that the
support of $x$, $\text{supp}(x)$, equals $\{e_i, \ldots, e_n\}$. We shall need the following lemma which is proved in [39].

**Lemma 4.3.** Let $V$ be an infinite dimensional vector space in $V_\infty$. For all $n$, there exists a nonzero $v_n \in V$ such that $\text{supp}(v_n) \cap \{e_1, \ldots, e_n\} = \emptyset$.

Let $V$ be a recursive subspace of $V_\infty$. Thus characteristic function $\chi_V$ is computed by some $\phi_e$ where $\phi_e$ is the $e$th partial recursive function. Let $A$ be the algorithm which first computes $\phi_e(a_1)$ and $\text{supp}(a_1)$, then computes $\phi_e(a_2)$ and $\text{supp}(a_2)$, then computes $\phi_e(a_3)$ and $\text{supp}(a_3)$, etc. This given, we define an increasing sequence $s_0 < s_1 < \cdots$ by induction as follows.

Let $s_0$ be the least $s$ such that if we run algorithm $A$ for $s$ steps, we find an $a_i$ and $\text{supp}(a_i)$ where $\phi_e(a_i) = 1$. (Note that in $st(V_\infty)$, $|st(e)| = n$ and in $tal(V_\infty)$, $|tal(e)| = k^n-1$ so that certainly $\text{supp}(a_i) \subseteq \{e_1, \ldots, e_{s_0}\}$ since we cannot compute $\text{supp}(a_i)$ in $s$ steps if $e_i \notin \text{supp}(a_i)$ for $t > s$.)

Having defined $s_0 < \cdots < s_n$, let $s_{n+1}$ be the least $s > s_n$ such that we run the algorithm $A$ for $s$ steps, we find an $a_i$ and $\text{supp}(a_i)$ where $\phi_e(a_i) = 1$ and $\text{supp}(a_i) \cap \{e_1, \ldots, e_{s_n}\} = \emptyset$. (Note that in $st(V_\infty)$, $|st(e)| = n$ and in $tal(V_\infty)$, $|tal(e)| = k^n-1$ so that certainly $\text{supp}(a_i) \subseteq \{e_1, \ldots, e_{s_n+1}\}$ since we cannot compute $\text{supp}(a_i)$ in $s$ steps if $e_i \notin \text{supp}(a_i)$ for $t > s_{n+1}$.)

It easily follows from Lemma 4.3 that $s_n$ is defined for all $n \geq 0$. We let $A = \{e_i : i \leq s_0 \text{ or } \exists n(s_{2n-1} < i \leq s_{2n})\}$. It is easy to see that the set $\{1^{s_n} : n \geq 0\}$ is a p-time set so that $A$ and $A'$ are p-time height increasing independent sets if we interprete $V_\infty$ as $tal(V_\infty)$ and $A$ and $A'$ are strongly p-time independent sets if we interprete $V_\infty$ as $st(V_\infty)$. Thus in either case $\text{space}(A)$ and $\text{space}(A')$ are in $\text{P}$. Moreover, it is easy to see from our definition of the sequence $s_0 < s_1 < \cdots$ that both $\text{space}(A)$ and $\text{space}(A')$ contain infinite independent subsets of $V$. Thus both $\text{space}(A) \cap V$ and $\text{space}(A') \cap V$ are infinite dimensional. Thus if we let $B_0 = \text{space}(A)$ and $B_1 = \text{space}(A')$, then $B_0$ and $B_1$ satisfy the requirements of the theorem. \hfill $\Box$

We note that unlike the set case, Theorem 4.2 does not exclude the possibility of the existence of NP-maximal sets. That is, suppose $V$ is an infinite and co-infinite dimensional subspace of $tal(V_\infty)$. Then the complementary subspace of $V$, $\text{space}(B_\bot)$, is certainly recursive so that there exists a pair of polynomial time complementary subspaces, $U$ and $W$, so that $U \cap \text{space}(B_\bot)$ and $W \cap \text{space}(B_\bot)$ are infinite dimensional. However, in this case, we can not make the conclusion that $V + U$ is an NP subspace which witnesses that $V$ is not NP-maximal for two reasons. First there is no guarantee that $V + U$ is co-infinite dimensional and second, in the light of Theorem 4.1, there is no guarantee that $U + V$ is in NP. Indeed our next results will show that there are oracles $A$ for which NP$^A$-maximal sets exist. Similar remarks hold for $st(V_\infty)$.

**Theorem 4.4.** There exists an r.e. oracle $Y$ and a subspace $V$ of $tal(V_\infty)$ which is both $\text{P}^Y$-simple and NP$^Y$-maximal.
Proof. We shall construct $Y$ so that $M = \{0^n|\exists x \in \{0,1\}^* (|x| = n \text{ and } x \in Y)\}$ is our desired subspace. Clearly $M \in \text{NP}^Y$.

To ensure that $M$ is co-infinite dimensional we must meet the following set of requirements.

$$T_j : \text{card}(\{|n| \in Y \text{ contains no strings } x \text{ with } k^n \leq |x| < k^{n+1} - 1\}) \geq j$$

Thus $T_j$ says there are at least $j$ heights $n$ so that $M$ contains no strings of height $n$. So meeting requirement $T_j$ ensures $\dim(V_{\infty}/M) \geq j$.

To ensure that $M$ is $\text{P}^Y$-simple, we shall meet the following two sets of requirements. Given any subset $V \subseteq \text{tal}(V_{\infty})$, let $h(V) = \{n : \exists x \in V (h(x) = n)\}$.

$$S_j : \text{If } N^Y_i \text{ is an infinite dimensional subspace of } \text{tal}(V_{\infty}) \text{ such that } h(N^Y_i) - h(M) \text{ is infinite, then } M \cap N^Y_i \neq \{0\}.$$  

Now suppose that $P^Y_i$ is an infinite dimensional subspace $\text{tal}(V_{\infty})$. Note that meeting all the requirements $S_j$ will ensure that either $P^Y_i \cap M \neq \{0\}$ or $h(P^Y_i) \subseteq^* h(M)$ where for any two sets $A$ and $B$ where we write $A \subseteq^* B$ iff there is a finite set $F$ such that $A \subseteq (B \cup F)$. Now suppose that $h(P^Y_i) \subseteq^* h(M)$ and let $B_i$ be the standard height increasing basis for $P^Y_i$. By Lemma 3.2, $B_i$ is in $\text{P}^Y$. Then clearly we can modify $B_i$ by possibly deleting a finite set of elements to form a new height increasing basis $C_i$ such that $h(M) \supseteq \{n : \exists x \in C_i (h(x) = n)\}$. Thus $C_i$ will also be in $\text{P}^Y$ and by Lemma 3.2, $\text{space}(C_i)$ will also be in $\text{P}^Y$. Hence if $h(P^Y_i) \subseteq^* h(M)$, then there exists some $j$ such that $P^Y_j$ is an infinite dimensional subspace of $\text{tal}(V_{\infty})$ and $h(P^Y_j) \subseteq h(M)$. Thus to ensure that $M$ is $\text{P}^Y$ simple, it will be enough to ensure that we meet the following set of requirements.

$$R_i : \text{If } P^Y_i \text{ is an infinite dimensional subspace of } \text{tal}(V_{\infty}), \text{ then } h(P^Y_i) \nsubseteq h(M).$$

Finally, to ensure $M$ is NP maximal, we shall meet the following set of requirements. Let $\langle \cdot, \cdot \rangle$ be some standard recursive pairing function which maps $N \times N$ onto $N$.

$$Q_{\langle i, n \rangle} : \text{If } N^Y_i/M \text{ is infinite dimensional and } N^Y_i \supseteq M, \text{ then there is an } x \in N^Y_i \text{ such that } x + \text{tal}(e_n) \in M.$$ 

Note that if $N^Y_i \supseteq M$ and $\dim(N^Y_i/M)$ is infinite, then meeting all the requirements $Q_{\langle i, n \rangle}$ will ensure that $\text{tal}(e_n) \in N^Y_i$ for all $n$ so that $N^Y_i = \text{tal}(V_{\infty})$. Thus in fact, $M$ will be $\text{NP}^Y$-supermaximal.

We shall rank our requirements with those of highest priority coming first as $T_0, S_0, R_0, Q_0, T_1, S_1, R_1, Q_1, \ldots$.

In the construction that follows, we shall let $Y_s$ denote the set of elements enumerated into $Y$ by the end of stage $s$ and

$$M_s = \{0^n|\exists x \in \{0,1\}^* (|x| = l \text{ and } x \in Y_s)\}. $$

We shall ensure that for each $s$, $M_s$ is a finite dimensional subspace of $\text{tal}(V_{\infty})$ and that $h(M_s)$ is contained in $\{1, \ldots, s\}$. For any stage $s$, we let $CH_s = \{n^Y_1 < n^Y_2 < \cdots \}$
be the set of complementary heights for $M_s$, i.e. the set of all heights $n$ so that there are no elements of $\text{tal}(V_{\infty})$ of heights $n$ in $M_s$.

At any given stage $s$, we shall pick out at most one requirement $A_j$ where $A_j$ will be one of the requirements $S_j$, $R_j$, or $Q_j$ and take an action to meet that requirement. We shall then say that $A_j$ received attention at stage $s$. The action that we take to meet the requirement $A_j$ of the form $S_j$ or $Q_j$ will always be of the same form. That is, we shall put some elements into $Y$ at stage $s$ and possibly restrain some elements from entering $Y$ for the sake of the requirement. We shall let $\text{res}(A_j, s)$ denote the set of elements that are restrained from entering $Y$ at stage $s$ for the sake of requirement $A_j$. We say that requirement $A_j$ of the form $S_j$ or $Q_j$ is satisfied at stage $s$, if there is a stage $s' < s$ such that $A_j$ has received attention at stage $s'$ and $\text{res}(A_j, s') \cap Y_s = \emptyset$.

The actions that we take to meet the requirements $R_j$ will be slightly different. First, we shall declare that all $R_j$ are in a passive state at the start of our construction. We would like to find an element $x \in P_j$ of height $n$ such that $n \notin h(M_s)$. If we can find such an $x$, then we will restrain all $y$ such that $k^{n-1} \leq |y| \leq k^n - 1$ plus all elements not in $Y_s$ which are queried of the oracle $Y_s$ during the computation of $P_j(x)$ from entering $Y$ for the sake of requirement $R_j$. Thus if we ensure that $\text{res}(R_j, s) \cap Y_s = \emptyset$, then $M$ will have no element of height $n$ and $x \in P_j$ so that $h(P_j) \subseteq h(M)$. If we take such an action for $R_j$ at stage $s$, then we will say that $R_j$ has received attention at stage $s$ and declare the state of $R_j$ to be active. Then for all $t > s$, we will say that an active $R_j$ is satisfied at stage $t$, if $\text{res}(R_j, s) \cap Y_t = \emptyset$. However if $R_j$ is injured at some stage $t > s$ in the sense that $\text{res}(R_j, s) \cap Y_t \neq \emptyset$, then $R_j$ will return to a passive state. If we cannot find such an $x$, we will attempt to force $h(P_j)$ to be finite. That is, since we will ensure that $h(M_{s-1}) \subseteq \{0, \ldots, s-1\}$ for all $s$, $M_{s-1}$ will have no element of height $s$. Recall that we are assuming that for $n \geq 0$, the run time of computations of $P_j(x)$ for any oracle $X$ is bounded $\max(2, n)^j$ for any string of length $n$. Then for $n \geq 2$, we let $b_n$ be the largest $i$ such that for all $k^{n-1} \leq r \leq k^n - 1$,

$$(k^n)^{i+2} < 2(k^{n-2}).$$

Note that it is easy to see that $\lim_{s \to \infty} b_s = \infty$. Our idea is that elements of height $n$ in $\text{tal}(V_{\infty})$ are of the form $0'$ where $k^{n-1} \leq r \leq k^n - 1$. Our strategy at the end of stage $s - 1$ for $s \geq 2$ will be to ensure that for all $R_j$ with $j \leq b_s$ which are in a passive state and have the property that $P_j^{s_i-1}(0') = 0$ for all $k^{a-1} \leq r \leq k^a - 1$, we restrain all elements which are not in $Y_{s-1}$ and which are queried of the oracle $Y_{s-1}$ in such computations from entering $Y$ for the sake of $R_j$. This action will force $h(P_j)$ to be finite if $R_j$ is in a passive state at stage $s$ for all but finitely many $s$. For any fixed $j \leq b_s$, the maximum restraint imposed for $R_j$ is if we restrained all elements not in $Y_{s-1}$ which are queried of the oracle $Y_{s-1}$ in some computation $P_j^{s_i-1}(0') = 0$ with $1 \leq r \leq k^n - 1$. Since the total number of steps used in all these computations is at most

$$2^j + \sum_{i=2}^{k^j} j^i \leq k^s \cdot (k^s)^j = (k^s)^{j+1}.$$
then clearly we could have restrained at most \((k^s)^{(j+1)}\) elements from entering \(Y\) for the sake of \(R_j\). Thus at stage \(s\), we will have restrained at most
\[
\sum_{i=0}^{b_s} (k^n)^{i+1} < (k^n)^{b_s+2} < 2^{k^{s-1}}
\]
elements for entering \(Y\) for the sake of some passive requirement \(R_j\) with \(j \leq b_s\) at stage \(s-1\). Hence for any given \(r\) with \(k^{n-1} \leq r < k^n - 1\), we will have restrained at most \(2^{r-1}\) elements of length \(r\) from entering \(Y\) for such \(R_j\)'s.

**Construction.**

Stages 0,1: Let \(Y_0 = Y_1 = \emptyset\) so that \(M_0 = M_1 = \emptyset\). Let \(res(A_j, 0) = res(A_j, 1) = \emptyset\) for all requirements \(A_j\) of the form \(S_j\), \(R_j\), or \(Q_j\).

Stage \(s\) with \(s \geq 2\): Let \(A_j\) be the highest priority requirement among \(S_0, R_0, Q_0, \ldots, S_s, R_s, Q_s\) such that

Case 1: \(A_j = S_j\) and \(S_j\) is not satisfied at stage \(s-1\) and there exists an \(l\) with
\[0 < h(0^l) \leq s\] such that
(a) \(0^l \in N_{Y_s-1}^j\),
(b) \(h(0^l) \in C_{Y_s-1}^j\) and \(h(0^l) > n_{Y_s-1}^j\), and
(c) for each \(0^n \in space(\{0^l\} \cup M_{Y_s-1}^j) - M_{Y_s-1}^j\), there is a string \(x_n \in \{0,1\}^*\) such that
\[|x_n| = |0^n| = n\] and \(x_n\) is not restrained from \(Y\) by any requirement of higher priority than \(S_j\) at stage \(s-1\) nor is \(x_n\) queried of the oracle in some fixed computation of \(N_{Y_s-1}^j\) which accepts \(0^l\).

Case 2: \(A_j = R_j\) and \(R_j\) is not satisfied at stage \(s-1\) and there exists an \(l\) with
\[0 < h(0^l) \leq s\] such that
(i) \(0^l \in P_{Y_s-1}^j\) and
(ii) \(h(0^l) \in C_{H_s-1}^j\) and \(h(0^l) > n_{Y_s-1}^j\).

Case 3: \(A_j = Q_j\) and \(Q_j\) is not satisfied at stage \(s-1\), and if \(j = (e, n)\), there exists an \(l\) with \(0 \leq h(0^l) \leq s\) such that
(I) \(0^l \in N_{Y_s-1}^j\),
(II) \(h(0^l) \in C_{H_s-1}^j\) and \(h(0^l) > \text{max}(n, n_{Y_s-1}^j)\), and
(III) For each \(0^m \notin space(\{0^l + \text{tail}(e_n)\} \cup M_{Y_s-1}^j) - M_{Y_s-1}^j\), \(h(0^m) > n_{Y_s-1}^j\) and there is a string \(x_m\) of length \(m\) in \(\{0,1\}^*\) which is not restrained from \(Y\) by any requirement of higher priority than \(Q_j\) at stage \(s-1\) nor is \(x_m\) queried in some fixed computation of \(N_{Y_s-1}^j\) which accepts \(0^l\).

If there is no such requirement \(A_j\), let \(Y_s = Y_{s-1}\). Also for all requirements \(A_j\) of the form \(S_j\) or \(Q_j\) and for all requirements \(A_j\) of the form \(R_j\) where either \(R_j\) is satisfied at stage \(s-1\) or \(j > b_{s+1}\), let \(res(A_j, s) = res(A_j, s-1)\). Declare that a requirement \(R_j\) is active at stage \(s\) if \(R_j\) is active at stage \(s-1\). For any \(R_j\) with \(j \leq b_{s+1}\) which is currently passive and has the property that \(P_{Y_s}^j(0^r) = 0\) for all \(k^s \leq r < k^{s+1} - 1\), let \(res(R_j, s) = res(R_j, s-1)\) union the set of all \(y \notin Y_s\) such that \(y\) is queried of the oracle in one of the computations \(P_{Y_s}^j(0^r)\) where \(k^s \leq r \leq k^{s+1} - 1\).
If there is such a requirement $A_j$, we have three cases.

**Case 1:** $A_j = S_j$. 

Let $l_s$ denote the least $l$ corresponding to $S_j$. Then for each $0^n \in \text{space}\left(\{0^i\} \cup M_{s-1}\right) - M_{s-1}$, pick the least string $\alpha_n$ such that $|\alpha_n| = n$, $\alpha_n$ is not restrained from $Y$ by any requirement of higher priority than $S_j$, at stage $s - 1$ nor is $\alpha_n$ queried of the oracle $Y_{s-1}$ in the computation of $N_j^{Y_{s-1}}$ which accepts $0^i$ and put $\alpha_n$ into $Y$. This will ensure that if $M_{s-1}$ is a finite dimensional subspace of $V_\infty$, then $M_s$ will also be a finite dimensional subspace of $V_\infty$. Note that the assumption that $h(0^i) \in CH_{s-1}$ ensures that all $0^n \in \text{space}\left(\{0^i\} \cup M_{s-1}\right) - M_{s-1}$ have the property that $h(0^n) \geq h(0^i)$. That is, such a $0^n$ must be of the form $0^n = 0^i + \tau m$ where $m \in M_{s-1}$ and since $h(m) \neq h(0^i)$, it must be the case that $h(0^n) \geq h(0^i)$. Thus $h(M_s) \cap \{n_s^{l_s-1}, \ldots, n_s\} = \emptyset$ and hence for all $i \leq j_s$, $n_i^{l_s-1} = n_i$. Let $\text{res}(S_j, s)$ equal the set of all strings not in $Y_{s-1}$ which are queried of the oracle $Y_{s-1}$ in the computation of $N_j^{Y_{s-1}}$ which accepts $0^i$ and say $S_j$ receives attention at stage $s$. Also for all requirements $A_j$ of the form $S_j$ or $Q_j$ and for all requirements $A_j$ of the form $R_j$ where either $R_j$ is satisfied at stage $s - 1$ or $j > b_{s+1}$, let $\text{res}(A_j, s) = \text{res}(A_j, s - 1)$ if $Y_s \cap \text{res}(A_j, s - 1) = \emptyset$ and let $\text{res}(A_j, s) = \emptyset$ if $Y_s \cap \text{res}(A_j, s - 1) \neq \emptyset$. Declare that a requirement $R_j$ is active at stage $s$ iff $R_j$ is active at stage $s - 1$ and $Y_s \cap \text{res}(R_j, s - 1) = \emptyset$. For any $R_j$ with $j \leq b_{s+1}$ which is currently passive and has the property that $P_j^{Y}(0^r) = 0$ for all $k^s \leq r < k^{s+1} - 1$, let $\text{res}(R_j, s)$ equal $\text{res}(R_j, s - 1)$ union the set of all $y \in Y_s$ such that $y$ is queried of the oracle $Y_s$ in one of the computations $P_j^{Y}(0^r)$ where $k^s \leq r < k^{s+1} - 1$.

**Case 2:** $A_j = R_j$. 

Let $l_s$ denote the least $l$ corresponding to $j_s$ and $n_s = h(0^i)$. We then say that $R_j$, is active and receives attention at stage $s$. We let $Y^s = Y^{s-1}$ and $\text{res}(R_j, s)$ consist of all elements $y$ with $k^{n_s-1} \leq |y| < k^{n_s} - 1$ and all elements which are not in $Y_{s-1}$ and which are queried of the oracle $Y_{s-1}$ in the computation $P_j^{Y_{s-1}}(0^i)$. Note that if $\text{res}(R_j, s) \cap Y = \emptyset$, then $M$ will have no elements of height $n_s = h(0^i)$ but $0^i \in P_j^Y$. Also for all requirements $A_j$ of the form $S_j$ or $Q_j$ and for all requirements $A_j$ of the form $R_j$ where $j \neq j_s$ and where either $R_j$ is satisfied at stage $s - 1$ or $j > b_{s+1}$, let $\text{res}(A_j, s) = \text{res}(A_j, s - 1)$ if $Y_s \cap \text{res}(A_j, s - 1) = \emptyset$ and let $\text{res}(A_j, s) = \emptyset$ if $Y_s \cap \text{res}(A_j, s - 1) \neq \emptyset$. For $j \neq j_s$, declare that a requirement $R_j$ is active at stage $s$ iff $R_j$ is active at stage $s - 1$ and $Y_s \cap \text{res}(R_j, s - 1) = \emptyset$. For any $R_j$ with $j \leq b_{s+1}$ which is currently passive and has the property that $P_j^{Y}(0^r) = 0$ for all $k^s \leq r < k^{s+1} - 1$, let $\text{res}(R_j, s)$ equal $\text{res}(R_j, s - 1)$ union the set of all $y \in Y_s$ such that $y$ is queried of the oracle $Y_s$ in one of the computations $P_j^{Y}(0^r)$ where $k^s \leq r < k^{s+1} - 1$.

**Case 3:** $A_j = Q_j$. 

Let $j_s = \langle e_s, n_s \rangle$ and $l_s$ denote the least $l$ corresponding to $j_s$. Then for each $0^n \in \text{space}\left(\{0^i + \tau \text{tail}(e_n)\} \cup M_{s-1}\right) - M_{s-1}$, pick the least string $\alpha_m$ such that $|\alpha_m| = m$, and $\alpha_m$ is not restrained from $Y$ by any requirement of higher priority than $Q_{j_s}$, at stage $s - 1$ nor is $\alpha_m$ queried of the oracle $N_{e_s}^{Y_{s-1}}$ which accepts $0^i$ and put $\alpha_m$ into $Y$. Once again this will ensure that $M_s$ is a finite dimensional subspace of $V_\infty$. Note that since $h(0^i) > n_s = h(\text{tail}(e_n))$, it follows that $h(0^i + \tau \text{tail}(e_n)) = h(0^i)$. Thus as in case 1, the assumption that $h(0^i) \in CH_{s-1}$ ensures that all $0^n \in \text{space}\left(\{0^i\}$
+essail(e_n) \cup M_s - 1) - M_s have the property that \( h(0^i) > h(0^j) \). Let \( res(Q_j, s) \) equal the set of all strings which are not in \( Y_{s - 1} \) which are queried of the oracle in the computation of \( N_{s - 1} \) which accepts \( 0^j \) and say \( Q_j \) receives attention at stage \( s \). Also for all requirements \( A_j \) of the form \( S_j \) or \( Q_j \) and for all requirements \( A_j \) of the form \( R_j \) where either \( R_j \) is satisfied at stage \( s - 1 \) or \( j > b_s + 1 \), let \( res(A_j, s) = res(A_j, s - 1) \) if \( Y_s \cap res(A_j, s - 1) = \emptyset \) and let \( res(A_j, s) = \emptyset \) if \( Y_s \cap res(A_j, s - 1) \neq \emptyset \). Declare that a requirement \( R_j \) is active at stage \( s \) iff \( R_j \) is active at stage \( s - 1 \) and \( Y_s \cap res(R_j, s - 1) = \emptyset \). For any \( R_j \) with \( j \leq b_s + 1 \) which is currently passive and has the property that \( P_j^{Y_s}(0^r) = 0 \) for all \( k^s \leq r \leq k^{s + 1} - 1 \), let \( res(R_j, s) \) equal \( res(R_j, s - 1) \) union the set of all \( y \in Y_s \) such that \( y \) is queried of the oracle \( Y_s \) in one of the computations \( P_j^{Y_s}(0^r) \) where \( k^s \leq r \leq k^{s + 1} - 1 \).

This completes the construction of \( Y \).

**Lemma 4.5.** Each requirement of the form \( S_j, R_j, \) or \( Q_j \) receives attention at most finitely often.

**Proof.** We proceed by induction on \( j \). Suppose that \( s_0 \) is such that there is no stage \( s \geq s_0 \) such that one of \( S_0, R_0, Q_0, \ldots, S_j, R_j, Q_j \) receives attention at stage \( s \). Then if there is a \( t > s_0 \) such that \( S_{j+1} \) receives attention at stage \( t \), then by construction \( S_{j+1} \) is satisfied at stage \( t \) and \( res(S_{j+1}, t) \cap Y_t = \emptyset \). However, it is easy to see from our construction that for \( s > t \), \( res(S_{j+1}, s) = res(S_{j+1}, t) \) and \( res(S_{j+1}, s) \cap Y_s = \emptyset \) unless some requirement of higher priority than \( S_{j+1} \) receives attention at stage \( s \). Since this never happens by our choice of \( s_0 \), \( S_{j+1} \) can receive attention at most once after stage \( s_0 \). Thus there must be a stage \( s_1 \) such that there is no stage \( s \geq s_1 \) such that one of \( S_0, R_0, Q_0, \ldots, S_j, R_j, Q_j, S_{j+1} \) receives attention at stage \( s \). A similar argument will show that \( R_{j+1} \) can receive attention at most once after stage \( s_1 \). Thus there must be a stage \( s_2 \) such that there is no stage \( s \geq s_2 \) such that one of \( S_0, R_0, Q_0, \ldots, S_j, R_j, Q_j, S_{j+1}, R_{j+1} \) receives attention at stage \( s \). Finally, a similar argument will show that \( Q_{j+1} \) can receive attention at most once after stage \( s_2 \). Thus each of the requirements \( S_j, R_j, \) or \( Q_j \) can receive attention only finitely often. \( \square \)

**Lemma 4.6.** \( \dim(tal(V_\infty)/M) \) is infinite.

**Proof.** We prove by induction that \( \dim(tal(V_\infty)/M) \geq k \) for all \( k \). That is, let \( t_0 \) be a stage such that no requirement \( S_0, R_0, Q_0, \ldots, S_k, R_k, Q_k \) receives attention at any stage \( s \geq t_0 \). Since \( M_{t_0} \) is finite dimensional, \( n_i^k \) is defined for all \( i \). Hence, \( M_i \) contains no strings of height \( n \) for \( n = n_1^{t_0}, \ldots, n_k^{t_0} \). But no requirement \( S_j, R_j, \) or \( Q_j \) with \( j > k \) can force elements of height \( n < n_k^k \) into \( M \) at any stage \( s \). Hence by our choice of \( t_0 \), there can be no strings of heights \( n \) for \( n = n_1^{t_0}, \ldots, n_k^{t_0} \) in \( M \). Thus \( \dim(tal(V_\infty)/M) \geq k \). \( \square \)

**Lemma 4.7.** \( M \) is \( V^{+} \)-simple.
Proof. First we show that if $N_f^Y$ is a subspace of $tal(V_\infty)$ such that $h(N_f^Y) - h(M)$ is infinite, then $N_f^Y \cap M \neq \{0\}$. On the contrary, assume $N_f^Y$ is such that $h(N_f^Y) - h(M)$ is infinite and $N_f^Y \cap M = \{0\}$. Note that since $M$ is co-infinite dimensional by Lemma 4.6, it follows that $n_i = \lim_{s\to\infty} n_i^s$ exists for all $i$. Let $s_0$ be a stage large enough so that $n_i^s = n_i$ for $i \leq j$ and none of the requirements $S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1}$ receives attention after stage $s_0$. Let $U_{s_0}$ denote the set of all $O^a$ such that there exists a requirement $A_i$ among $S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1}$ which is satisfied at stage $s_0$ such that there exists an $x \in res(A_i, s_0)$ with $|x| = n$. Our choice of $s_0$ ensures that if $n \not\in U_{s_0}$, then no string $x$ of length $n$ is ever restrained from $Y$ by a requirement of higher priority than $S_j$ which is satisfied at some stage $t > s_0$. Also our choice of $s_0$ ensures that $n_i = n_i^s$ for all $i \leq j$ and $t > s_0$. Next let $t_0$ be such that

1. $t_0 > \max(\{h(y) : y \in U_{s_0}\} \cup \{2, s_0, n_j\})$,
2. $b_{t_0} > j$, and
3. $2^{r-1} > r^j$ for all $r > t_0$.

Note that for any $t > t_0$, our construction ensures that the number of strings of length $r$ where $k^{r-1} \leq r \leq k^j - 1$ which are restrained by some requirement $R_i$ with $i < j$ which is passive at stage $t$ is less than $2^{r-1}$. Moreover, we are assuming that any successful computation of the oracle machine $N_f^Y$ for any oracle $X$ on a string of length $r > 2$ takes at most $r^j$ steps. Thus our choice of $t_0$ ensures that if $t > t_0$ and $O^a \in N_f^Y$ is a string of height $> t_0$, then there is at least one string $x_a \in \{0, 1\}^*$ of length $x$ which is not restrained from $Y$ by any requirement of higher priority than $S_j$ at stage $t$ nor is queried of the oracle $Y$ in some fixed computation which shows that $0^x \in N_f^Y$. Since $h(N_f^Y) - h(M)$ is infinite, there must exist a $n^a \in N_f^Y$ such that $h(0^n) > t_0$ and $h(0^n) \not\in h(M)$. Then there must be some stage $s > t_0$ such that $0^n \in N_f^{Y_{s-1}}$. Note that at stage $s$, each $0^n \in space(\{0^n\} \cup M_{s-1}) - M_{s-1}$ has the property that $h(0^n) \geq h(0^m) > t_0$ and thus there is at least one string $x_m$ of length $m$ which is not restrained from $Y$ by any requirement of higher priority than $S_j$ at stage $s - 1$ nor is queried of the oracle $Y_s$ in some fixed computation which shows that $0^n \in N_f^{Y_{s-1}}$. Thus $0^n$ witnesses that $S_j$ is a candidate to receive attention at stage $s$. Thus either $S_j$ is satisfied at stage $s - 1$ or $S_j$ is highest priority requirement among $S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1}$ which can receive attention at stage $s$. In either case, it follows that $S_j$ will be satisfied at stage $s$. Thus there will be some $0^n \in (N_f^Y \cap M_s) - \{0\}$ such that all elements which are queried of the oracle $Y_s$ in some computation which shows that $0^n \in N_f^{Y_{s-1}}$ and which are not $Y_t$ are in $res(S_j, s)$. However, our choice of $t_0$ ensures we can never put any element of $res(S_j, s)$ into $Y$ after stage $s$ so that $0^n$ will witness that $N_f^Y \cap M \neq \{0\}$.

Remark. We note that the assumption that $h(N_f^Y) - h(M)$ is infinite seems to be crucial in this argument. That is, if we merely assume that $dim(N_f^Y/M)$ is infinite, then it may be the case that whenever there exists a $0^n \in N_f^Y$ such that $h(0^n) > t_0$ and $0^n \not\in M$, then at a stage $s > t_0$ where $0^n \in N_f^{Y_{s-1}}$, there is some $0^n \in M_{s-1}$ such that $h(0^n) = h(0^n)$. In such a situation it is possible that $h(0^n +_{al} 0^n)$ is much less than $h(0^n)$. That is, it may be possible that some element in $0^n \in space(\{0^n\} \cup M_{s-1}) - M_{s-1}$ has height so small that all strings of length $x$ are queried of the oracle during any computation
which shows that $0^x \in N_k^{Y_{s-1}}$. Then it will be impossible to put a string of length $x$ into $Y_s$ so as to ensure that $0^x \in M$, while maintaining the computation to ensure that $0^x \in N_k^Y$.

To continue our proof of the lemma, we can now assume that if $P_j^Y$ is an infinite dimensional subspace of $tal(V_{\infty})$ such that $P_j^Y \cap M = \{0\}$, then $h(P_j^Y) \subseteq h(M)$ is finite. By our argument preceding the construction, it would then follow that there is some $j$ such that $P_j^Y$ is an infinite dimensional subspace of $tal(V_{\infty})$ and $h(P_j^Y) \subseteq h(M)$. Let $s_1$ be a stage large enough so that $n_i^{s_1} = n_i$ for $i \leq j$ and none of the requirements $S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1}, S_j$ receives attention after stage $s_1$. Let $U_{s_1}$ denote the set of all $0^x$ such that there exists a requirement $A_i$ among $S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1}, S_j$ which is satisfied at stage $s_1$ and there exists an $x \in \text{res}(A_i, s_1)$ with $|x| = n$. Our choice of $s_1$ ensures that if $n \notin U_{s_1}$, then no string $x$ of length $n$ is ever restrained from $Y$ by a requirement of higher priority than $R_j$ which is satisfied at some stage $t > s_0$. Also our choice of $s_1$ ensures that $n_i = n_i^{s_1}$ for all $i \leq j$ and $t > s_0$. Next let $t_1$ be such that

(1) $t_1 > \max\{h(y) : y \in U_{s_1}\} \cup \{2, s_1, n_{t-1}\}$.
(2) $b_{t_1} > j$, and
(3) $2^{r-1} > r^j$ for all $r > t_1$.

Now we claim that there can be no stage $t > t_1$ at which $R_j$ is satisfied at stage $t$. That is, if $R_j$ is satisfied at stage $t$, there must be some $s \leq t$ such that $R_j$ receives attention at stage $s$ and there is a $0^x \in P_j^{Y_{s-1}}$ such that $q = h(0^x) \in CH_{s-1}$ and $\text{res}(R_j, s) \cap Y = \emptyset$. But then our choice of $t > t_1$ ensures that $\text{res}(R_j, s) \cap Y = \emptyset$ which means that $M$ can have no string of height $q$ while $0^x \in P_j^{Y_{s-1}}$. But then $0^x$ witnesses that $h(P_j^Y) \subseteq h(M)$ which contradicts our assumption that $h(P_j^Y) \subseteq h(M)$. Thus it must be the case that for all stages $t > t_1$, $R_j$ is in a passive state. It follows that for all $t > t_1$, there can be no $r$ with $k^t \leq r \leq k^{t+1} - 1$ such that $P_j^Y(0^r) = 1$ since otherwise at stage $t + 1$, there is some $r$ with $k^t \leq r \leq k^{t+1} - 1$ such that $P_j^Y(0^r) = 1$. But then at stage $t + 1$, $0^r$ witnesses that $R_j$ is a candidate to receive attention at stage $t + 1$. By our choice of $t > t_1$, it would follow that $R_j$ is the highest priority requirement among $S_0, R_0, Q_0, \ldots, S_{t+1}, R_{t+1}, Q_{t+1}$ which could receive attention at stage $t + 1$ so that $R_j$ would receive attention at stage $t + 1$ which we have already ruled out. Thus it must be the case that for all $r$ with $k^t \leq r \leq k^{t+1} - 1$, $P_j^Y(0^r) = 0$. But then our choice of $t > t_1$ ensures that $j \leq b_{t+1}$ and hence all elements which are not in $Y_j$ which are queried of the oracle $Y_j$ during one of the computations $P_j^Y(0^r) = 0$ where $k^t \leq r \leq k^{t+1} - 1$ are put into $\text{res}(R_j, t)$. Again the fact that $t > t_1$ ensures that $\text{res}(R_j, t) \cap Y = \emptyset$ so that for all $r$ with $k^t \leq r \leq k^{t+1} - 1$, $P_j^Y(0^r) = 0$. That is, $P_j^Y$ has no string of length $t + 1$ for any $t > t_1$ and hence $h(P_j^Y)$ is finite. Thus there can be no such $P_j^Y$ such that $P_j^Y$ is an infinite dimensional subspace of $tal(V_{\infty})$ and $h(P_j^Y) \subseteq h(M)$. But this means that there can be no $r$ such that $P_j^Y$ is an
Lemma 4.8. $M$ is NP$^Y$-maximal.

**Proof.** By our remarks preceding the construction, we need only show that we meet all the requirements $Q_{(e,n)}$. So assume $N^Y_e$ is a subspace of $tal(V_{\infty})$ such that $(N^Y_e/M)$ is infinite dimensional and $N^Y_e \supseteq M$. Let $j = (e,n)$ and let $s_2$ be a stage such that $n_i = n_i^t$ for $i \leq j$ and none of $S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1}, S_j, R_j$ receive attention after stage $s_2$. Let $U_{s_2}$ denote the set of all $0^n$ such that there exists a requirement $A_i$ among $S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1}, S_j, R_j$ which is satisfied at stage $s_2$ and there exists an $x \in res(A_i, s_2)$ with $|x| = n$. Our choice of $s_2$ ensures that if $n \notin U_{s_2}$, then no string $x$ of length $n$ is ever restrained from $Y$ by a requirement of higher priority than $Q_j$ which is satisfied at some stage $t > s_2$. Also our choice of $s_2$ ensures that $n_i = n_i^t$ for all $i \leq j$ and $t > s_2$. Next let $t_2$ be such that

1. $t_2 > \max\{|h(y) : y \in U_{s_2}\} \cup \{2, s_2, n_{i-1}\}$,
2. $b_{t_2} > j$, and
3. $2^{r-1} > r'$ for all $r > t_2$.

Note that for any $t > t_2$, our construction ensures that the number strings of length $r$ where $k^{t-1} \leq r < k^{t-1} - 1$ which are restrained by some requirement $R_i$ with $i < j$ which is passive at stage $t$ is less than $2^{-r-1}$. Moreover, we are assuming that any successful computation of the oracle machine $N^Y_j$ for any oracle $X$ on a string of length $r \geq 2$ takes at most $r'$ steps. Thus our choice of $t_2$ ensures that if $t > t_2$ and $0^t \in N^Y_j$ is a string of height $> t_2$, then there is at least one string $x \in \{0,1\}^*$ of length $x$ which is not restrained from $Y$ by any requirement of higher priority than $Q_j$ at stage $t$ nor is queried of the oracle $Y_t$ in some fixed computation which shows that $0^t \in N^Y_j$.

Next observe that since $dim(N^Y_e/M)$ is infinite and $N^Y_e \supseteq M$, it must be the case that $h(N^Y_e) - h(M)$ is infinite. That is, let $A = \{a_0, a_1, \ldots\}$ be an infinite set of elements of $N^Y_e$ which is independent over $M$. Then consider some fixed $a_i \in A$ and suppose $a_i = \sum_{i=1}^{q} \lambda_i tie, \ldots, q, \lambda_q \neq 0$, and $j_1 < \cdots < j_q$ Thus $h(a_i) \geq j_q$. Now if there exists an $m_1 \in M$ such that $h(a_i) = h(M)$, then $m_1 = \sum_{i \leq j} \beta_i tie, \ldots, \beta_{j_q}$ where $\beta_l \in F$ for all $l$ and $\beta_{j_q} \neq 0$. But then $a_i^1 = a_i - \sum_{l=0}^{q} \beta_{j_q} m_1$ is an element of $N^Y_e - M$ such that $h(a_i^1) < h(a_i)$. Now, if there exists an $m_2 \in M$ such that $h(a_i^1) = h(m_2)$, then once again there is some $\gamma \in F$ such that $a_i^2 = a_i^1 - \sum \gamma m_2$ is an element of $N^Y_e - M$ with $h(a_i^2) < h(a_i^1) < h(a_i)$. If we continue in this fashion, we must eventually find some $a_i^k = a_i + \sum \gamma v_k$ where $v_k \in M$ such that $h(a_i^k) \notin h(M)$. That is, we can replace our original independent set $A$ over $M$ by a set $A' = \{a_0, a_1, \ldots\}$ where for all $i$, $a_i - \sum v_k \in M$ and $h(a_i^k) \notin h(M)$. But then $A'$ is an infinite subset of $N^Y_e$ which is independent over $M$. Thus there is no finite set $F$ such that $space(M \cup F) \supseteq A'$. This implies that $h(A') = \{h(a_i) : i \geq 0\}$ must be infinite since otherwise there clearly would be a finite set $F$ such that $space(M \cup F) \supseteq A'$. But by construction $h(A') \subseteq h(N^Y_e) - h(M)$ so that $h(N^Y_e) - h(M)$ must be infinite.
Since \( h(N^Y_e) - h(M) \) is infinite, there must exist a \( 0^q \in N^Y_e \) such that \( h(0^q) > t_2 \), \( h(0^q) > n \), and \( h(0^q) \notin h(M) \). Then there must be some stage \( s > t_2 \) such that \( 0^q \in N^Y_{e-1} \).

Note that at stage \( s \), each \( 0^m \in \text{space}(\{0^q + \text{tal}(e_n)\} \cup M_{s-1}) \) has the property that \( h(0^m) \geq h(0^q + \text{tal}(e_n)) = h(0^q) > t_2 \) and thus there is at least one string \( x_m \) of length \( m \) which is not restrained from \( Y \) by any requirement of higher priority than \( Q_j \) at stage \( s-1 \) nor is queried of the oracle \( Y_{s-1} \) in some fixed computation which shows that \( 0^q \in N^Y_{f-1} \). Thus \( 0^q \) witnesses that \( Q_j \) is a candidate to receive attention at stage \( s \). Hence either \( Q_j \) is satisfied at stage \( s-1 \) or \( Q_j \) is the highest priority requirement among \( S_0, R_0, Q_0, \ldots, S_n, R_n, Q_s \), which can receive attention at stage \( s \). In either case, it follows that \( Q_j \) will be satisfied at stage \( s \). Thus there will be some \( 0^q \in N^Y_j \) such that \( 0^q + \text{tal}(e_n) \in M_s \) and all elements which are queried of the oracle in some computation which shows that \( 0^q \in N^Y_j \) and which are not \( Y \), are in \( \text{res}(Q_j, s) \).

However, our choice of \( t_2 \) ensures we can never put any element of \( \text{res}(Q_j, s) \) into \( Y \) after stage \( s \) so that \( 0^q \in N^Y_j \) and hence requirement \( Q_j \) is met. Thus \( M \) will be \( \text{NP}^Y \)-supermaximal and hence will be \( \text{NP}^Y \)-maximal. 

We note that \( M \) constructed in Theorem 4.4 has a number of interesting properties besides being \( \text{NP}^Y \)-maximal and \( \text{P}^Y \)-simple. First of all, it is easy to check that in meeting the requirements \( S_j \) we made no use of the fact that \( N^Y_j \) was a subspace of \( \text{tal}(V_\infty) \) but only that \( N^Y_j \) was a subset of \( \text{tal}(V_\infty) \). Similarly, it is easy to check that in meeting the requirements \( R_j \) we made no use of the fact that \( P^Y_j \) was a subspace of \( \text{tal}(V_\infty) \) but only that \( P^Y_j \) was a subset of \( V_\infty \). Thus meeting all the requirements \( R_j \) ensures that there is no infinite subset \( W \) of \( \text{tal}(V_\infty) \) in \( \text{P}^Y \) such that \( h(W) \leq h(M) \). Thus \( M \) does not contain any infinite \( \text{P}^Y \) set and hence \( M \) does not have a basis in \( \text{P}^Y \). We also claim that \( \text{tal}(V_\infty) - M \) does not have any infinite subsets in \( \text{P}^Y \). That is, suppose that \( P^Y_j \subseteq \text{tal}(V_\infty) - M \). Now it cannot be that \( h(P^Y_j) - h(M) \) is infinite since otherwise there is an \( i \) such that \( P^Y_j \cap N^Y_i \) and the fact that we met requirement \( S_i \) would mean that \( P^Y_j \cap M \neq \{0\} \). Thus \( h(P^Y_j) \leq h(M) \). Let \( Q = h(\text{space}(A)) - h(P^Y_j) \).

Then clearly

\[
S = \{x \in P^Y_j : h(x) \notin Q\}
\]

is an infinite set in \( \text{P}^Y \) such that \( h(S) \subseteq h(\text{space}(A)) \). Since meeting all the requirements \( R_j \) rule out the existence of such an \( S \), \( \text{tal}(V_\infty) - M \) does not contain an infinite set in \( \text{P}^Y \). Recall that a set of strings \( S \) is called \( \text{P}^Y \)-immune if \( S \) has no infinite subset in \( \text{P}^Y \). Thus both \( M \) and \( \text{tal}(V_\infty) - M \) are \( \text{P}^Y \)-immune.

Note also that by Theorem 3.7, the fact that \( M \) is \( \text{NP}^Y \)-maximal implies that \( \text{NP}^Y \neq \text{Co-NP}^Y \) and hence that \( \text{P}^Y \neq \text{NP}^Y \). Thus we have proved the following.

**Corollary 4.9.** There exists an r.e. oracle \( Y \) and a subspace \( M \) of \( \text{tal}(V_\infty) \) such that

1. \( \text{P}^Y \neq \text{NP}^Y \) and \( \text{NP}^Y \neq \text{Co-NP}^Y \),
2. \( M \in \text{NP}^Y \),
3. \( M \) is \( \text{P}^Y \)-immune and hence has no basis in \( \text{P}^Y \),
4. $\text{tail}(V_\infty) - M$ is $P^Y$-immune, and
5. $M$ is both $P^Y$-simple and $NP^Y$-supermaximal.

Our next task is to prove an analogue of Theorem 4.4 for $st(V_\infty)$. Recall that we are assuming that the field $F = \{0, \ldots, k - 1\}$ where $0$ is the zero element of $F$ and that we are using our alternative representation of $st(V_\infty)$. That is, we defined a polynomial time isomorphism from $\Psi$ which maps $\text{Bin}(N)$ onto $ST_k = \{0\} \cup \{0, \ldots, k - 1\}^*$ $\{1, \ldots, k - 1\}$ by $\Psi(0) = 0$ and for $x \in \text{Bin}(N) - \{0\}$, $\Psi(x) = a_0 \cdots a_k$ where $x = a_0 + a_1 \cdot k + \cdots + a_n k^n$. We then used $\Psi$ to induce operations of sum + and scalar multiplication for each $\lambda \in F$, $\lambda_{ST_k}$, from the operations $+$ on $st(V_\infty)$ to turn $ST_k$ into a vector space isomorphic to $V_\infty$. Thus we shall implicitly identify $st(V_\infty)$ with the polynomial time structure $(ST_k, +, 0_{ST_k}, \ldots, (k - 1)_{ST_k})$. The main advantage of this identification is that for a nonzero $x$ in $st(V_\infty)$, the length of $x$ will equal the height $x$.

Theorem 4.10. There exists an r.e. oracle $D$ such that there exists an $NP^D$-supermaximal $P^D$-simple subspace in $st(V_\infty)$.

Proof. Our construction will proceed in stages. We let $D_s$ be the set of elements enumerated into $D$ by the end of stage $s$. For any given $x \in \{0, \ldots, k - 1\}^*$ with $|x| \geq 1$, we let $C_x$ denote the set of all strings of length $8|x| + 2$ of $\{0, \ldots, k - 1\}^*$ of the form $x 10^{|x|} 1\sigma$ where $\sigma$ is any string of length $3|x|$ in $\{0, \ldots, k - 1\}^*$. Note that there are $k^{|x|}$ strings in $C_x$ for any $x \in st(V_\infty)$. Let $C_\emptyset = \{\emptyset\}$. It is then easy to see that if $x \neq y$, then $C_x \cap C_y = \emptyset$.

We then define $A = \{x : C_x \cap D \neq \emptyset\}$. Thus $A$ will be in $NP^D$. Our idea is to define $D$ so that $A$ is a height increasing independent subset of $st(V_\infty)$. Then by the relativized version of Theorem 3.16, $\text{space}(A) \in NP^D$. Our construction of $D$ will ensure that $\text{space}(A)$ is our desired $P^D$-simple $NP^D$-supermaximal space. Let $A_s = \{x : C_x \cap D_s \neq \emptyset\}$. At each stage $s$, we shall let $B_s = \{\epsilon_n : A_s$ has no element of height $n\}$. Our construction will ensure that at each stage $s$, $A_s \cup B_s$ is a height increasing basis of $st(V_\infty)$. We define $b_i^s$ for all $i$ and $s$ so that $B_s = \{b_0^s, b_1^s, \ldots\}$ where $ht(b_0^s) < ht(b_1^s) < \cdots$.

To ensure that $\text{space}(A)$ is co-infinite dimensional we must meet the following set of requirements.

$T_j : \text{card}\{n : D \text{ contains no strings } x \text{ with } |x| = 8n + 2\} \geq j$

Thus $T_j$ says there are at least $j$ heights $n$ so that $A$ contains no strings of height $n$. So meeting requirement $T_j$ ensures $\dim(st(V_\infty)/\text{space}(A)) \geq j$.

To ensure that $\text{space}(A)$ is $P^D$-simple, we shall meet the following two sets of requirements. Given any subset $V \subseteq st(V_\infty)$, let $h(V) = \{n : \exists x \in V \ (h(x) = n)\}$

$S_j :$ If $N^D_i$ is an infinite dimensional subspace of $st(V_\infty)$ such that
$h(N^D_i) - h(\text{space}(A))$ is infinite, then $\text{space}(A) \cap N^D_i \neq \emptyset$. 
Now suppose that $P^D_i$ generates an infinite dimensional subspace of $st(V_\infty)$ which is in $NP^D$. Note that meeting all the requirements $S_j$ will ensure that either $\text{space}(P^D_i) \cap \text{space}(A) \neq \{0\}$ or $h(\text{space}(P^D_i)) \subseteq h(\text{space}(A))$. Now suppose that $h(\text{space}(P^D_i)) \subseteq h(\text{space}(A))$ and let $U = h(P^D_i) - h(\text{space}(A))$. If $U = \emptyset$, then $h(P^D_i) \subseteq h(\text{space}(A))$.

Otherwise, $U$ is a finite set so let $U = \{n_0, \ldots, n_q\}$ and let $x_0, \ldots, x_q$ be elements of $\text{space}(P^D_i)$ such that $h(x_i) = n_i$. Note that any $x \in st(V_\infty)$ is a string of the form $x = a_1 \cdots a_{|x|}$ where $a_j \in \{0, \ldots, k-1\}$. Then we define the full height of $x$, $fh(x) = \{n : 1 \leq n \leq |x| \text{ and } a_n \neq 0\}$. Then it is easy to see that given any $x \in \text{space}(P^D_i)$ there exists some $\lambda_1, \ldots, \lambda_q$ in $F$ such that $fh(x - \sum_{i=1}^q \lambda_i x_i) \cap U = \emptyset$. That is, if $x = a_1 \cdots a_{|x|}$ where $|x| \geq n_q$ and $a_{n_q} \neq 0$ and $x_q = a_q \cdots a_{n_q}$ where $a_{n_q} \neq 0$, then $x' = x - \sum_{i=1}^{n_q} x_i = b_0 \cdots b_{|x|}$ where $b_{n_q} = 0$ so that $n_q \notin fh(x')$. Now if $b_{n_{q-1}} \neq 0$ and $x_{q-1} = a_{1,q-1} \cdots a_{n_{q-1},q-1}$ where $a_{n_{q-1},q-1} \neq 0$, then $x'' = x' - \sum_{i=1}^{n_{q-1}} c_i x_{q-1} = c_0 \cdots c_{|x|}$ where $c_{n_q} = b_{n_q} = 0$ and $c_{n_{q-1}} = 0$ so that neither $n_q$ nor $n_{q-1}$ is in $fh(x'')$.

Continuing in this way we can construct our desired linear combination $\sum_{j=1}^q \lambda_j x_i$ such that $fh(x - \sum_{j=1}^q \lambda_j x_i) \cap U = \emptyset$. Now let $Q = \{x \in \text{space}(P^D_i) : fh(x) \cap U = \emptyset\}$. It is easy to see that $Q$ is a subspace of $P^D_i$ and our argument above shows that $\text{space}(P^D_i) = \text{space}(\{x_1, \ldots, x_q\}) \oplus Q$. Thus $Q$ is an infinite dimensional subspace of $st(V_\infty)$ such that $h(Q) \subseteq h(\text{space}(A))$. Let $T$ be the set of all $y$ such that $fh(y) \cap U = \emptyset$, $|y| > k|\lambda_q|$, and there exists an $x \in P^D_i$ and $z \in \text{space}(\{x_1, \ldots, x_q\})$ such that $x + st z = y$. Note that $\text{space}(\{x_1, \ldots, x_q\})$ has exactly $k^q$ elements since $\{x_1, \ldots, x_q\}$ is a height increasing basis for $\text{space}(\{x_1, \ldots, x_q\})$. Thus given any $y$ with $|y| > k|\lambda_q|$ in polynomial time in $|y|$ we can find all $y + st w$ such that $w \in \text{space}(\{x_1, \ldots, x_q\})$. Now for any $w \in \text{space}(\{x_1, \ldots, x_q\})$, $h(w) \leq h(x_q) = |x_q| < k|\lambda_q|$ so that $h(y + st w) = h(y)$. Thus it takes at most $k^q(|y|)$ steps to test all such $y + st w$ for membership in $P^D_i$ given an oracle $D$. But then $y \in T$ iff $\{y + st w : w \in \text{space}(\{x_1, \ldots, x_q\})\} \cap P^D_i \neq \emptyset$.

Thus it follows that $T$ is in $P^D$ and clearly $T$ generates an infinite dimensional subspace of $Q$. Thus there must be some $j$ such that $P^D_j$ generates an infinite dimensional subspace of $st(V_\infty)$ and $h(\text{space}(P^D_j)) \subseteq h(\text{space}(A))$. Thus to ensure that $\text{space}(A)$ is in $P^D$ simple, it will be enough to ensure that we meet the following set of requirements.

$R_i$ : If $P^D_i$ generates an infinite dimensional subspace of $st(V_\infty)$, then $h(P^D_i) \not\subseteq h(\text{space}(A))$.

Finally, to ensure $\text{space}(A)$ is $NP$-supermaximal, we shall meet the following set of requirements. Let $(,)$ be some standard recursive pairing function which maps $N \times N$ onto $N$. $Q_{(i,n)} :$ If $N^D_i/\text{space}(A)$ is an infinite dimensional space and $N^D_i \supseteq \text{space}(A)$, then there is an $x \in N^D_i$ such that $x + st(e_n) \in \text{space}(A)$. 
Note that if $N^D_i \supseteq \text{space}(A)$ and $\dim(N^D_i/\text{space}(A))$ is infinite, then meeting all the requirements $Q_{(i,n)}$ will ensure that $\text{st}(e_n) \in N^D_i$ for all $n$ so that $N^D_i = \text{st}(V_\infty)$.

We shall rank our requirements with those of highest priority coming first as $T_0, S_0, R_0, Q_0, T_1, S_1, R_1, Q_1, \ldots$.

As in the construction of Theorem 4.4, at any given stage $s$, we shall pick out at most one requirement $E_j$ where $E_j$ will be one of the requirements $S_j$, $R_j$, or $Q_j$ and take an action to meet that requirement. We shall then say that $E_j$ received attention at stage $s$. The action that we take to meet the requirement $E_j$ of the form $S_j$ or $Q_j$ will always be of the same form. That is, we shall put some elements into $D$ at stage $s$ and possibly restrain some elements from entering $D$ for the sake of the requirement. We shall let $\text{res}(E_j, s)$ denote the set of elements that are restrained from entering $D$ at stage $s$ for the sake of requirement $E_j$.

We say that requirement $E_j$ of the form $S_j$ or $Q_j$ is satisfied at stage $s$, if there is a stage $s' < s$ such that $E_j$ has received attention at stage $s'$ and $\text{res}(E_j, s') \cap D_s = \emptyset$.

The actions that we take to meet the requirements $R_j$ will essentially be the same as in the construction of Theorem 4.4. First, we shall declare that all $R_j$ are in a passive state at the start of our construction. We would like to find an element $x \in P^D_j$ of height $n$ such that $n \not\in h(\text{space}(A_s))$. If we can find such an $x$, then we will restrain all $y$ such that $|y| = 8n + 2$ and $y \in C_x$ for some $x \in \text{st}(V_\infty)$ of height $n$ plus all elements not in $D_s$ which are queried of the oracle during the computation of $P^D_j(x)$ from entering $D$ for the sake of requirement $R_j$. Then if we ensure that $\text{res}(R_j, s) \cap D = \emptyset$, then $A$ will have no elements of height $n$ and $x \in P^D_j$ so that $h(P^D_j) \not\subseteq h(\text{space}(A))$. If we take such an action for $R_j$ at stage $s$, then we will say that $R_j$ has received attention at stage $s$ and declare the state of $R_j$ to be active. Then for all $t > s$, we will say that an active $R_j$ is satisfied at stage $t$, if $\text{res}(R_j, s) \cap D_t = \emptyset$. However, if $R_j$ is injured at some stage $t > s$ in the sense that $\text{res}(R_j, s) \cap D_t \neq \emptyset$, then $R_j$ will return to a passive state. If we cannot find such an $x$, we will attempt to force $h(P^D_j)$ to be finite. That is, since we will ensure that $h(\text{space}(A_{s-1})) \subseteq \{0, \ldots, s - 1\}$ for all $s$, $A_{s-1}$ will have no elements of height $s$. Recall that we are assuming that for $n \geq 2$, the run time of computations of $P^X_j(y)$ for any oracle $X$ is bounded $\max(2, n)^j$ for any string of length $n$. Then for $n \geq 2$, we let $d_n$ be the largest $i$ such that for all $r$, $n^{i+2} < k^n$.

Note that it is easy to see that $\lim_{s \rightarrow \infty} b_s = \infty$. Our idea is that elements of height $n$ in $\text{st}(V_\infty)$ are just the elements of length $n$. Our strategy at the end of stage $s - 1$ for $s \geq 2$ is that for all $R_j$ with $j \leq d_s$ which are in a passive state and have the property that $P^D_j(x) = 0$ for all $x \in \text{st}(V_\infty)$ of length $n$, we will restrain all elements which are not in $D_{s-1}$ and which are queried in such computations from entering $D$ for the sake of $R_j$. This action will force $h(P^D_j)$ to be finite if $R_j$ is in a passive state at stage $s$ for all but finitely many $s$. For any fixed $j \leq b_s$, the maximum restraint imposed for $R_j$ is if we restrained all elements not in $D_{s-1}$ which are queried of the oracle $D_{s-1}$ in some computation $P^D_{j-1}(x) = 0$ with $1 \leq |x| \leq n$.
and $x \in st(V_{\infty})$. Since the total number of steps used in all these computations is at most

\[ 2^j + \sum_{i=2}^{s} k^i j \leq \sum_{i=2}^{s} k^i j = k^s s^{j+1}, \]

then clearly we could have restrained at most $k^s s^{j+1}$ elements from entering $D$ for the sake of $R_j$. Thus at stage $s$, we will have restrained at most

\[ \sum_{i=0}^{d_s} k^s s^{i+1} < k^s s^{d_s+2} < k^s k^s = k^{2s} \]

elements from entering $D$ for the sake of some passive requirement $R_j$ with $j \leq b_j$ at stage $s - 1$. Hence for any given $x$ with $|x| = n$, we will have restrained less than $k^{2s}$ elements of $C_x$ from entering $D$ for such $R_j$'s.

**Construction.**

**Stages 0, 1:** Let $D_0 = D_1 = \emptyset$ so that $A_0 = A_1 = \emptyset$. Let $res(E_j, 0) = res(E_j, 1) = \emptyset$ for all requirements $E_j$ of the form $S_j$, $R_j$, or $Q_j$.

**Stage $s$ with $s \geq 2$:**

Let $E_j$ be the highest priority requirement among $S_0, R_0, Q_0, \ldots, S_s, R_s, Q_s$ such that

**Case 1:** $E_j = S_j$ and $S_j$ is not satisfied at stage $s - 1$ and there exists an $x \in st(V_{\infty})$ with $0 < |x| \leq s$ such that

(a) $x \in N_j^{d_s-1}$,

(b) $|x| \notin h\left(\text{space}(A_{s-1})\right)$ and $|x| > |b_j^{s-1}|$, and

(c) there exists a $y \in C_x$ such that $y$ is not restrained from $D$ by any requirement of higher priority than $S_j$ at stage $s - 1$ and $y$ is not queried of the oracle $D_{s-1}$ in some fixed computation which shows that $x \in N_j^{D_{s-1}}$.

**Case 2.** $E_j = R_j$ and $R_j$ is not satisfied at stage $s - 1$ and there exists an $x \in st(V_{\infty})$ with $0 \leq |x| \leq s$ such that

(i) $|x| \notin h\left(\text{space}(A_{s-1})\right)$ and

(ii) $x \in P_j^{D_{s-1}}$.

**Case 3.** $E_j = Q_j$ and $Q_j$ is not satisfied at stage $s - 1$, and if $j = (e, n)$, there exists an $x$ with $0 \leq |x| \leq s$ such that

(I) $x \in P_j^{D_{s-1}}$,

(II) $|x| \notin h\left(\text{space}(A_{s-1})\right)$, $|x| > |b_j^{s-1}|$, and $|x| > n$, and

(III) there exists a $y \in C_{x+st(e_n)}$ such that $y$ is not restrained from $D$ by any requirement of higher priority than $S_j$ at stage $s - 1$ and $y$ is not queried of the oracle $D_{s-1}$ in some fixed computation which shows that $P_j^{D_{s-1}}(x)$.

If there is no such requirement $E_j$, let $D_s = D_{s-1}$. Also for all requirements $E_j$ of the form $S_j$ or $Q_j$ and for all requirements $E_j$ of the form $R_j$ where either $R_j$ is satisfied at stage $s - 1$ or $j > d_{s+1}$, let $res(E_j, s) = res(E_j, s - 1)$. Declare that a requirement $R_j$ is active at stage $s$ iff $R_j$ is active at stage $s - 1$. For any $R_j$ with $j \leq d_{s+1}$ which
is currently passive and has the property that $P^D_f(x) = 0$ for all $x \in st(V_\infty)$ of length $s + 1$, let $\text{res}(R_j, s)$ equal $\text{res}(R_j, s - 1)$ union the set of all $y \notin D_s$ such that $y$ is queried of the oracle $D_s$ in one of the computations $P^D_f(x)$ where $x \in st(V_\infty)$ of length $s + 1$.

If there is such a requirement $E_j$, we have three cases.

Case 1. $E_j = S_{j_s}$.

Let $x_s$ denote the least $x$ corresponding to $S_{j_s}$. Then pick the least string $\alpha_{x_s} \in C_{x_s}$ such that $\alpha_{x_s}$ is not restrained from $D$ by any requirement of higher priority than $S_{j_s}$ at stage $s - 1$ nor is $\alpha_{x_s}$ queried of the oracle $D_{s-1}$ in the computation of $N^{D_{s-1}}_j$ which accepts $x_s$ and put $\alpha_{x_s}$ into $D$. Let $\text{res}(S_{j_s}, s) = \text{res}(S_{j_s}, s - 1) \cup \{y \in D_s : y \text{ is queried of the oracle } D_{s-1} \text{ in one of the computations } q(x) \text{ where } x \in st(V_\infty) \text{ of length } s + 1\}$. Let $S_{j_s}$ receive attention at stage $s$. Also for all requirements $E_j$ of the form $S_j$ or $Q_j$ and for all requirements $E_j$ of the form $R_j$ where $j \neq j_s$ and where either $R_j$ is satisfied at stage $s - 1$ or $j > d_{s+1}$, let $\text{res}(E_j, s) = \text{res}(E_j, s - 1)$ if $D_s \cap \text{res}(E_j, s - 1) = \emptyset$ and let $\text{res}(E_j, s) = \emptyset$ if $D_s \cap \text{res}(E_j, s - 1) \neq \emptyset$. Declare that a requirement $R_j$ is active at stage $s$ iff $R_j$ is active at stage $s - 1$ and $D_s \cap \text{res}(R_j, s) = \emptyset$. For any $R_j$ with $j \leq d_{s+1}$ which is currently passive and has the property that $P^D_f(z) = 0$ for all $z \in st(V_\infty)$ of length $s + 1$, let $\text{res}(R_j, s)$ equal $\text{res}(R_j, s - 1)$ union the set of all $y \in D_s$ such that $y$ is queried of the oracle $D_s$ in one of the computations $P^D_f(z)$ where $z \in st(V_\infty)$ and $|z| = s + 1$.

Case 2. $E_j = R_j$.

Let $x_s$ denote the least $x$ corresponding to $j_s$. We then say that $R_{j_s}$ is active and receives attention at stage $s$. We let $D_s = D_{s-1}$ and $\text{res}(R_{j_s}, s)$ consist of all elements $y$ of length $8|x_s| + 2$ which are in some $C_z$ such that $z \in st(V_\infty)$ and $|z| = |x_s|$ and all elements which are not in $D_{s-1}$ and which are queried of the oracle $D_{s-1}$ in the computation $P^{D_{s-1}}_{j_s}(x) = 1$. Note that if $\text{res}(R_{j_s}, s) \cap D = \emptyset$, then $A$ will have no elements of height $|x_s|$ but $x_s \in P^D_{j_s}$. Also for all requirements $E_j$ of the form $S_j$ or $Q_j$ and for all requirements $E_j$ of the form $R_j$ where $j \neq j_s$ and where either $R_j$ is satisfied at stage $s - 1$ or $j > b_{s+1}$, let $\text{res}(E_j, s) = \text{res}(E_j, s - 1)$ if $D_s \cap \text{res}(E_j, s - 1) = \emptyset$ and let $\text{res}(E_j, s) = \emptyset$ if $D_s \cap \text{res}(E_j, s - 1) \neq \emptyset$. Declare that a requirement $R_j$ is active at stage $s$ iff $R_j$ is active at stage $s - 1$.

Case 3. $E_j = Q_{j_s}$.

Let $j_s = (e_{j_s}, n_{j_s})$ and $x_s$ denote the least $x$ corresponding to $j_s$. Then pick the least string $\alpha_{x_s} \in C_{x_s+n_{e_{j_s}}}$, and $\alpha_{x_s}$ is not restrained from $D$ by any requirement of higher priority than $Q_{j_s}$ at stage $s - 1$ nor is $\alpha_{x_s}$ queried of the oracle $D_{s-1}$ in the computation of $P^{D_{s-1}}_{e_{j_s}}(x_s)$ and put $\alpha_{x_s}$ into $D$. Let $\text{res}(Q_{j_s}, s)$ consists of all strings which are not in $D_{s-1}$ which are queried of the oracle $D_{s-1}$ in the computation of $P^{D_{s-1}}_{e_{j_s}}(x_s)$ and say $Q_{j_s}$ receives attention at stage $s$. Also for all requirements $E_j$ of the form $S_j$ or $Q_j$ and for all requirements $E_j$ of the form $R_j$ where either $R_j$ is satisfied at stage $s - 1$ or $j > d_{s+1}$, let $\text{res}(E_j, s) = \text{res}(E_j, s - 1)$ if $D_s \cap \text{res}(E_j, s - 1) = \emptyset$ and let $\text{res}(E_j, s) = \emptyset$ if $D_s \cap \text{res}(E_j, s - 1) \neq \emptyset$. Declare that a requirement $R_j$ is active at stage $s$ iff $R_j$ is
active at stage \( s - 1 \) and \( D_s \upharpoonright res(R_j, s - 1) = \emptyset \). For any \( R_j \) with \( j \leq d_{s+1} \) which is currently passive and has the property that \( P_j^{D_s}(x) = 0 \) for all \( x \in st(V_{\infty}) \) of length \( s + 1 \), let \( res(R_j, s) \) equal \( res(R_j, s - 1) \) union the set of all \( y \notin D_s \) such that \( y \) is queried of the oracle \( D_s \) in one of the computations \( P_j^{D_s}(x) \) where \( x \in st(V_{\infty}) \) and \( |x| = s + 1 \).

This completes the construction of \( D \). We note that \( A \) is a height increasing independent set in \( N^{P^D} \) since our construction ensures that we can never put two elements of the same height in \( A \). Thus by Theorem 3.16, \( space(A) \in N^{P^D} \).

**Lemma 4.11.** Each requirement of the form \( S_j, R_j, \) or \( Q_j \) receives attention at most finitely often.

**Proof.** We proceed by induction on \( j \). Suppose that \( s_0 \) is such that there is no stage \( s \geq s_0 \) such that one of \( S_0, R_0, Q_0, \ldots, S_j, R_j, Q_j \) receives attention at stage \( s \). Then if there is a \( t > s_0 \) such that \( S_{j+1} \) receives attention at stage \( t \), then by construction \( S_{j+1} \) is satisfied at stage \( t \) and \( res(S_{j+1}, t) \cap D_t = \emptyset \). However, it is easy to see from our construction that for \( s > t \), \( res(S_{j+1}, s) = res(S_{j+1}, t) \) and \( res(S_{j+1}, s) \cap D_s = \emptyset \) unless some requirement of higher priority than \( S_{j+1} \), receives attention at stage \( s \). Since this never happens by our choice of \( s_0 \), \( S_{j+1} \) will be satisfied at all stages \( s > t \). Thus \( S_{j+1} \) will be satisfied at all stages \( s > t \). Hence, \( S_{j+1} \) can receive attention at most once after stage \( s_0 \). Thus, there must be a stage \( s_1 \) such that there is no stage \( s \geq s_1 \) such that one of \( S_0, R_0, Q_0, \ldots, S_j, R_j, Q_j, S_{j+1} \) receives attention at stage \( s \). A similar argument will show that \( R_{j+1} \) can receive attention at most once after stage \( s_1 \). Thus, there must be a stage \( s_2 \) such that there is no stage \( s \geq s_2 \) such that one of \( S_0, R_0, Q_0, \ldots, S_j, R_j, Q_j, S_{j+1}, R_{j+1} \) receives attention at stage \( s \). Finally, a similar argument will show that \( Q_{j+1} \) can receive attention at most once after stage \( s_2 \). Thus each of the requirements \( S_j, R_j, \) or \( Q_j \) can receive attention only finitely often. \( \square \)

**Lemma 4.12.** \( \dim(st(V_{\infty})/space(A)) \) is infinite.

**Proof.** We prove by induction that \( \dim(st(V_{\infty})/V) \geq k \) for all \( k \). That is, let \( t_0 \) be a stage such that no requirement \( S_0, R_0, Q_0, \ldots, S_k, R_k, Q_k \) receives attention at any stage \( s \geq t_0 \). Since \( space(A) \mid_0 \) is finite dimensional, \( b_i^k \) is defined for all \( i \). Hence \( space(A) \mid_0 \) contains no strings of height \( n \) for \( n = |b_i^1|, |b_i^2| \). But no requirement \( S_j, R_j, \) or \( Q_j \) with \( j > k \) can force elements of height \( n < |b_i^k| \) into \( A \) at any stage \( s \). Hence, by our choice of \( t_0 \), there can be no strings of heights \( n \) for \( n = |b_i^1|, |b_i^2| \) in \( A \). Thus \( \dim(st(V_{\infty})/space(A)) \geq k \). \( \square \)

**Lemma 4.13.** \( space(A) \) is \( P^D \)-simple.

**Proof.** First we show that if \( N_j^D \) is a subspace of \( st(V_{\infty}) \) such that \( h(N_j^D) - h(space(A)) \) is infinite, then \( N_j^D \cap space(A) \neq \{0\} \). For a contradiction assume \( N_j^D \) is such that \( h(N_j^D) - h(space(A)) \) is infinite and \( N_j^D \cap space(A) = \{0\} \). Note that since \( space(A) \) is
co-infinite dimensional by Lemma 4.12, it follows that \( b_i = \lim_{x \to \infty} b_i^x \) exists for all \( i \).

Let \( s_0 \) be a stage large enough so that \( b_i^x = b_i \) for \( i \leq j \) and none of the requirements \( S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1} \) receives attention after stage \( s_0 \). Let \( U_{s_0} \) denote the set of all \( n \) such that there exists a requirement \( E_i \) among \( S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1} \) which is satisfied at stage \( s_0 \) such that there exists an \( \alpha \in \text{res}(E_i, s_0) \) with \( |\alpha| = 8n + 2 \). Our choice of \( s_0 \) ensures that if \( n \notin U_{s_0} \), then no string \( \alpha \) of length \( 8n + 2 \) is ever restrained from \( D \) by a requirement of higher priority than \( S_j \) which is satisfied at some stage \( t > s_0 \). Also our choice of \( s_0 \) ensures that \( b_i = b_i^x \) for all \( i \leq j \) and \( t > s_0 \). Next let \( t_0 \) be such that

1. \( t_0 > \max(U_{s_0} \cup \{ 2, s_0, |b_j| \}) \times x \),
2. \( d_{t_0} > j \), and
3. \( k^2 > r^j \) for all \( r > t_0 \).

Note that for any \( t > t_0 \), our construction ensures that the number strings of length \( 8t + 2 \) which are restrained by some requirement \( R_i \) with \( i < j \) which is passive at stage \( t \) is less than \( k^2 \). Moreover, we are assuming that any successful computation of the oracle machine \( N_{j}^{X} \) for any oracle \( X \) on a string of length \( r \geq 2 \) takes at most \( r^j \) steps. Thus our choice of \( t_0 \) ensures that if \( t > t_0 \) and \( x \in N_{j}^{D_t} \) is a string of height \( > t_0 \), then there is at least one string \( x_k \in C_k \) which is not restrained from \( D \) by any requirement of higher priority than \( S_j \) at stage \( t \) nor is queried of the oracle \( D_t \) in some computation which shows that \( x \in N_{j}^{D_t} \). Since \( h(N_{j}^{D_t}) - h(\text{space}(A)) \) is infinite, there must exist an \( x \in N_{j}^{D_t} \) such that \( |x| > t_0 \) and \( |x| \notin h(\text{space}(A)) \). Then there must be some stage \( s > t_0 \) such that \( x \in N_{j}^{D_{s-1}} \). Thus \( x \) witnesses that \( S_j \) is a candidate to receive attention at stage \( s \). Thus either \( S_j \) is satisfied at stage \( s - 1 \) or \( S_j \) is the highest priority requirement among \( S_0, R_0, Q_0, \ldots, S_s, R_s, Q_s \) which can receive attention at stage \( s \). In either case, it follows that \( S_j \) will be satisfied at stage \( s \). Thus there will be some \( x \in N_{j}^{D_{s}} \cap A_s \) such that all elements which are queried of the oracle \( D_s \) in some computation which shows that \( x \in N_{j}^{D_{s}} \) and which are not \( D_s \) are in \( \text{res}(S_j, s) \). However, our choice of \( t_0 \) ensures that we can never put any element of \( \text{res}(S_j, s) \) after stage \( s \) so that \( x \) will witness that \( N_{j}^{D_{s}} \cap \text{space}(A) \neq \{ \emptyset \} \).

**Remark.** We note that again the assumption that \( h(N_{j}^{D}) - h(\text{space}(A)) \) is infinite seems to be crucial in this argument. That is, if we merely assume that \( \text{dim}(N_{j}^{D}/\text{space}(A)) \) is infinite, then it may be the case that whenever there exists an \( x \in N_{j}^{D} \) such that \( |x| > t_0 \) and \( x \notin \text{space}(A) \), then at a stage \( s > t_0 \) where \( x \in N_{j}^{D_{s-1}} \), there may be some \( y \in A_{s-1} \) such that \( |x| = |y| \). But then we cannot add \( x \) to \( A \), because then \( A \) will not be a height increasing basis. If \( A \) is not height increasing, then we cannot be certain that \( \text{space}(A) \in \text{NP}^{D} \).

To continue our proof of the lemma, we can now assume that if \( P_{j}^{D} \) generates an infinite dimensional subspace of \( st(V_{\infty}) \) which is in \( \text{NP}^{D} \) such that \( P_{j}^{B} \cap \text{space}(A) = \{ \emptyset \} \), then \( h(P_{j}^{D}) - h(\text{space}(A)) \) is finite. By our argument preceding the construction, it would then follow that there is some \( j \) such that \( P_{j}^{D} \) generates an infinite dimensional subspace of \( st(V_{\infty}) \) and \( h(P_{j}^{D}) \subseteq h(\text{space}(A)) \). We shall now show that there can be no
such \( j \). On the contrary, assume that \( P^D_j \) generates an infinite dimensional subspace of \( \text{st}(V_{\infty}) \) and \( h(P^D_j) \subseteq h(\text{space}(A)) \). Let \( s_1 \) be a stage large enough so that \( b^n_i = b_i \) for \( i \leq j \) and none of the requirements \( S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1}, S_j \) receives attention after stage \( s_1 \). Let \( U_{s_1} \) denote the set of all \( n \) such that there exists a requirement \( E_i \) among \( S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_{j-1}, S_j \) which is satisfied at stage \( s_1 \) such that there exists an \( \alpha \in \text{res}(E_i, s_1) \) with \( |x| = 8n + 2 \). Our choice of \( s_1 \) ensures that if \( n \notin U_{s_1} \), then no string \( \alpha \) of length \( 8n + 2 \) is ever restrained from \( D \) by a requirement of higher priority than \( R_j \) which is satisfied at some stage \( t > s_1 \). Also our choice of \( s_1 \) ensures that \( b_i = b'_i \) for all \( i \leq j \) and \( t > s_1 \). Next let \( t_1 \) be such that

1. \( t_1 > \max(U_{s_1} \cup \{2, s_1, n_{i-1}\}) \),
2. \( d_{t_1} > j \), and
3. \( k^r > r^j \) for all \( r > t_1 \).

Now we claim that there can be no stage \( t > t_1 \) at which \( R_j \) is satisfied at stage \( t \). That is, if \( R_j \) is satisfied at stage \( t \), there must be some \( s \leq t \) such that \( R_j \) receives attention at stage \( s \) and there is a \( x \in P^D_{t-1} \) such that \( |x| \notin h(\text{space}(A_{t-1})) \) and \( \text{res}(R_j, s) = \text{res}(R_j, t) \) contains all strings of length \( 8|x| + 2 \) which are in some \( C_z \) where \( z \in \text{st}(V_{\infty}) \) and \( |z| = |x| \) plus all strings which are not in \( D_{s-1} \) which are queried of the oracle \( D_{s-1} \) in the computation \( P^D_{t-1}(x) = 1 \) and \( \text{res}(R_j, s) \cap D_{t-1} = \emptyset \). But then our choice of \( t > t_1 \) ensures that \( \text{res}(R_j, s) \cap D = \emptyset \) which means that \( \text{space}(A) \) can have no strings of height \( |x| \) while \( x \in P^D_j \). But then \( x \) witnesses that \( h(P^D_j) \subseteq h(\text{space}(A)) \) which contradicts our assumption that \( h(P^D_j) \subseteq h(\text{space}(A)) \). Thus it must be the case that for all stages \( t > t_1 \), \( R_j \) is in a passive state. But then it must also be the case that for all \( t > t_1 \), there can be no \( x \in \text{st}(V_{\infty}) \) of length \( t + 1 \) such that \( P^D_j(x) = 1 \) since otherwise \( x \) would witness that \( R_j \) is a candidate to receive attention at stage \( t + 1 \). By our choice of \( t > t_1 \), it would follow that \( R_j \) is the highest priority requirement among \( S_0, R_0, Q_0, \ldots, S_{t+1}, R_{t+1}, Q_{t+1} \) which could receive attention at stage \( t + 1 \) so that \( R_j \) would receive attention at stage \( t + 1 \) which we have already ruled out. Thus it must be the case that for all \( x \in \text{st}(V_{\infty}) \) of length \( t + 1 \), \( P^D_j(x) = 0 \). But then our choice of \( t > t_1 \) ensures that \( j \leq d_{t+1} \) and hence all elements which are not in \( D_t \) which are queried of the oracle \( D_t \) during one of the computations \( P^D_j(x) = 0 \) where \( x \in \text{st}(V_{\infty}) \) and \( |x| = t + 1 \) are put into \( \text{res}(R_j, t) \). Again the fact that \( t > t_1 \) ensures that \( \text{res}(R_j, t) \cap D = \emptyset \) so that for all \( x \in \text{st}(V_{\infty}) \) with \( |x| = t + 1 \), \( P^D_j(x) = 0 \). That is, \( P^D_j \) has no strings in \( \text{st}(V_{\infty}) \) of length \( t + 1 \) for any \( t > t_1 \) and hence \( h(\text{space}(P^D_j)) \) is finite. Thus there can be no such \( P^D_j \) such that \( P^D_j \) generates an infinite dimensional subspace of \( \text{st}(V_{\infty}) \) and \( h(P^D_j) \subseteq h(\text{space}(A)) \). But this means that there can be no \( r \) such that \( P^D_r \) is an infinite dimensional subspace of \( \text{st}(V_{\infty}) \) and \( P^D_r \cap \text{space}(A) = \{0\} \). Thus \( \text{space}(A) \) is \( P^D \)-simple as claimed. 

**Lemma 4.14.** \( \text{space}(A) \) is \( N^D_{\text{max}} \)-maximal.

**Proof.** By our remarks preceding the construction, we need only show that we meet all the requirements \( Q_{(e,n)} \). So assume \( N^D_e \) is a subspace of \( \text{st}(V_{\infty}) \) such that \( (N^D_e / \text{space}(A)) \) is infinite dimensional and \( N^D_e \supseteq \text{space}(A) \). Let \( j = (e,n) \) and let \( s_2 \) be a
stage such that \( b_i = b_i^2 \) for \( i \leq j \) and none of the requirements \( S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_j \) receive attention after stage \( s_2 \). Let \( U_{s_2} \) denote the set of all \( n \) such that there exists a requirement \( E_i \) among \( S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_j, S_j, R_j \) which is satisfied at stage \( s_2 \) such that there exists an \( \alpha \in \text{res}(E_i, s_2) \) with \( |\alpha| = 8n + 2 \). Our choice of \( s_2 \) ensures that if \( n \notin U_{s_2} \), then no string \( \alpha \) of length \( 8n + 2 \) is ever restrained from \( D \) by a requirement of higher priority than \( Q_j \) which is satisfied at some stage \( t > s_2 \). Also our choice of \( s_2 \) ensures that \( b_i = b_i^2 \) for all \( i \leq j \) and \( t > s_2 \). Next let \( t_2 \) be such that

1. \( t_2 > \max(U_{s_2} \cup \{2, s_2, |b_j|\}) \),
2. \( d_{t_2} > j \), and
3. \( k^t > r^j \) for all \( r > t_2 \).

Note that for any \( t > t_0 \), our construction ensures that the number of strings of length \( 8t + 2 \) which are restrained by some requirement \( R_i \) with \( i < j \) which is passive at stage \( t \) is less than \( k^{2t} \). Moreover, we are assuming that any successful computation of the oracle machine \( N_j^X \) for any oracle \( X \) on a string of length \( r \geq 2 \) takes at most \( r^j \) steps. Thus our choice of \( t_2 \) ensures that if \( t > t_2 \) and \( x \in N_{j}^{D_i} \) is a string of height \( > t_0 \), then there is at least one string \( x \in C_x \) which is not restrained from \( D \) by any requirement of higher priority than \( Q_j \) at stage \( t \) nor is queried of the oracle \( D_t \) in some computation which shows that \( x \in N_{j}^{D_i} \).

Note that our argument in Lemma 4.8 shows that \( h(N_{j}^{D_i}) - h(\text{space}(A)) \) is infinite since \( \text{dim}(N_{e}^{D}/\text{space}(A)) \) is infinite and \( N_{e}^{D} \supset \text{space}(A) \). Since \( h(N_{j}^{D_i}) = h(\text{space}(A)) \) is infinite, there must exist an \( x \in N_{j}^{D_i} \) such that \( |x| > t_0, |x| > n \), and \( |x| \notin h(\text{space}(A)) \). Then there must be some stage \( s > t_0 \) such that \( x \in N_{j}^{D_i} \). Thus \( x \) witnesses that \( Q_j \) is a candidate to receive attention at stage \( s \). Thus either \( Q_j \) is satisfied at stage \( s - 1 \) or \( Q_j \) is the highest priority requirement among \( S_0, R_0, Q_0, \ldots, S_{j-1}, R_{j-1}, Q_j \) which can receive attention at stage \( s \). In either case, it follows that \( Q_j \) will be satisfied at stage \( s \). Thus there will be some \( x \in N_{j}^{D_i} \) such that \( x + st(e_n) \in A_x \) and all elements which are queried of the oracle \( D_s \) in some computation which shows that \( x \in N_{j}^{D_i} \) and which are not \( D_s \) are in \( \text{res}(Q_j, s) \). However, our choice of \( t_2 \) ensures we can never put any element of \( \text{res}(Q_j, s) \) into \( D \) after stage \( s \) so that \( x \in N_{j}^{D_i} \) and hence requirement \( Q_j \) is met. Thus \( \text{space}(A) \) will be \( \text{NP}^{D} \)-supermaximal and hence will be \( \text{NP}^{D} \)-maximal. □

We note that \( \text{space}(A) \) constructed in Theorem 4.10 has a number of interesting properties besides being \( \text{NP}^{D} \)-supermaximal and \( \text{P}^{D} \)-simple. First of all, meeting all the requirements \( R_j \) ensures that \( \text{space}(A) \) is \( \text{P}^{D} \)-immune. That is, if \( P_i^{D} \) is an infinite subset of \( \text{space}(A) \), then certainly \( P_i^{D} \) generates an infinite dimensional subspace of \( \text{st}(V_{\infty}) \) and \( h(P_i^{D}) \subseteq h(\text{space}(A)) \) which would violate requirement \( R_i \). Also as in the construction of Theorem 4.4, it is easy to check that in meeting the requirements \( S_j \), we made no use of the fact that \( N_i^{D} \) was a subspace of \( \text{st}(V_{\infty}) \) but only that \( N_j^{D} \) was a subset of \( \text{st}(V_{\infty}) \). We claim that \( \text{st}(V_{\infty}) - \text{space}(A) \) does not have any infinite subsets in \( \text{P}^{D} \). That is, suppose that \( P_i^{D} \subseteq \text{st}(V_{\infty}) - \text{space}(A) \). Now, it cannot be that \( h(P_j^{D}) \subseteq h(\text{space}(A)) \) is infinite since otherwise there is an \( i \) such that \( P_j^{D} = N_i^{D} \) and
the fact that we met requirement $S_i$ would mean that $P_i^D \cap \text{space}(A) \neq \{0\}$. Thus $h(P_i^D) \subseteq h(\text{space}(A))$. Let $Q = h(\text{space}(A)) - h(P_i^D)$. Then clearly

$$S = \{x \in P_i^D : h(x) \notin Q\}$$

is an infinite set in $P_i^D$ which generates an infinite dimensional subspace of $st(V_\infty)$ and $h(S) \subseteq h(\text{space}(A))$. Since meeting all the requirements $R_j$ rules out the existence of such an $S$, $st(V_\infty) - \text{space}(A)$ does not contain an infinite set in $P_i^D$. Thus $\text{space}(A)$ and $st(V_\infty) - \text{space}(A)$ are $P_i^D$-immune.

Note also that by Theorem 3.7, the fact that $\text{space}(A)$ is $NP_i^D$-maximal implies that $NP_i^D \neq \text{Co-NP}_i^D$ and hence that $P_i^D \neq NP_i^D$. Thus we have proved the following.

**Corollary 4.15.** There exists an r.e. oracle $D$ and a subspace $V$ of $st(V_\infty)$ such that

1. $P_i^D \neq NP_i^D$ and $NP_i^D \neq \text{Co-NP}_i^D$,
2. $V \in NP_i^D$,
3. $V$ is $P_i^D$-immune and hence has no basis in $P_i^D$.
4. $st(V_\infty) - V$ is $P_i^D$-immune, and
5. $V$ is both $P_i^D$-simple and $NP_i^D$-supermaximal.

5. Conclusions

In this paper we initiated the study of the lower semilattice of NP and P subspaces of both the standard polynomial time representation and the tally polynomial time representation of a countably infinite dimensional vector space $V_\infty$ over a finite field $F$. Our results show that there exists oracles $A$ and $B$ such that $NP^A \neq P^A$ and $NP^A$-maximal and $P^A$-simple subspaces exist in $tal(V_\infty)$ and $NP^B \neq P^B$ and $NP^B$-maximal and $P^B$-simple subspaces exist in $st(V_\infty)$ and there exists an oracle $C$ such that $NP^C \neq P^C$ and no $NP^C$-maximal, $P^C$-simple, or $NP^C$-simple subspaces exist in either $tal(V_\infty)$ or $st(V_\infty)$. Thus arguments which relativize cannot prove the existence or nonexistence of NP-maximal or P-simple subspaces of either $tal(V_\infty)$ or $st(V_\infty)$ even if we assume that NP $\neq$ P. We note that the situation is completely different if the underlying field $F$ is infinite for Bäuerle [3] proved that if $V_\infty$ is a recursive presentation of a countably infinite dimensional vector space $V_\infty$ over an infinite field $F$ which has a recursive dependence algorithm, then there exists a polynomial time supermaximal subspace $V$ of $V_\infty$.

In [35], we studied various properties concerning the connections between the complexity of subspaces $V$ of $tal(V_\infty)$ and the complexity of their bases. In that work, the notion of a polynomial time height increasing independent set played a crucial role. Moreover, we were able to show that there exists an oracle $A$ such that $NP^A \neq P^A$ and every $NP^A$ subspace of $tal(V_\infty)$ has a basis in $P^A$ and there is an oracle $B$ such that $NP^B \neq P^B$ and there exists a $NP^B$ subspace of $tal(V_\infty)$ which has no basis in $P^B$. In this paper, we strengthen the latter result by showing that there exists an oracle $B$ such that there is an $NP^B$ subspace $V$ of $tal(V_\infty)$ such that $V$ and $tal(V_\infty) - V$ are $P^B$-immune.
Also in this paper, we have shown that there is a natural analogue of p-time height increasing independent sets of \( \text{tal}(V_\infty) \) in the setting of \( \text{st}(V_\infty) \), namely strongly height increasing independent sets. We showed that many of the results about p-time height increasing independent sets in \( \text{tal}(V_\infty) \) have natural analogues for strongly height increasing independent sets in \( \text{st}(V_\infty) \). We also proved that there is an oracle \( C \) such that \( \text{NP}^C \neq \text{P}^C \) and there exists an \( \text{NP}^C \)-maximal and \( \text{P}^C \)-simple subspace \( V \) of \( \text{st}(V_\infty) \) such that both \( V \) and \( \text{st}(V_\infty) - V \) are \( \text{P}^C \)-immune so that \( V \) certainly has no basis in \( \text{P}^C \).

Again we note that the situation is completely different if the underlying field \( F \) is infinite. For example, in [35], we showed that with some mild extra assumptions about the polynomial time presentation of the infinite field \( F \), in both the standard polynomial time representation and the tally polynomial time representation of a countably infinite dimensional vector space \( V_\infty \) over an infinite polynomial time field \( F \), every r.e. subspace has a polynomial time basis.

Finally, we observe that results about NP and P subspaces of \( \text{tal}(V_\infty) \) naturally extend to results about \( \text{NEXT} \) and \( \text{DEXT} \) subspaces of \( \text{st}(V_\infty) \). That is, for \( S \) a subset of \( \{0, \ldots, k-1\}^* \), we write \( S \in \text{DEXT}^X \) (\( S \in \text{NEXT}^X \)) if there is an oracle machine \( M \in \text{DEXT} \) (\( M \in \text{NEXT} \)) such that \( S = L(M, X) \). Given a natural number \( n \), let \( \text{bin}(n) \) denote the binary representation of \( n \) and \( \text{tal}(n) \) the tally representation of \( n \). Then it is well known, see [1] for example, that if \( A \) is any subset of the natural numbers \( N \) and \( \text{Bin}(A) = \{ \text{bin}(n) : n \in A \} \) and \( \text{Tal}(A) = \{ \text{tal}(n) : n \in A \} \), then \( \text{Tal}(A) \in \text{P} \) iff \( \text{Bin}(A) \in \text{DEXT} \) and \( \text{Tal}(A) \in \text{NP} \) iff \( \text{Bin}(A) \in \text{NEXT} \). A similar result holds for \( \text{tal}(V_\infty) \) and \( \text{st}(V_\infty) \).

**Proposition 5.1** (Nerode and Remmel [35]). Let \( V \subseteq V_\infty \) and \( X \subseteq \{0, \ldots, k-1\}^* \). Let \( \text{st}(V) = \{ \text{st}(v) : v \in V \} \) and \( \text{tal}(V) = \{ \text{tal}(v) : v \in V \} \). Then

(i) \( \text{tal}(V) \in \text{P}^X \) iff \( \text{st}(V) \in \text{DEXT}^X \) and

(ii) \( V \in \text{NP}^X \) iff \( \text{st}(V) \in \text{NEXT}^X \).

Given Proposition 5.1, we can easily transfer results between the tally representation of \( V_\infty \) and the standard representation of \( V_\infty \). For a typical example, say that a subspace \( M \) of \( \text{st}(V_\infty) \) is \( \text{NEXT}^A \)-maximal if \( M \in \text{NEXT}^A \), \( \text{dim}(\text{st}(V_\infty)/M) \) is infinite, and for any subspace \( W \) of \( \text{st}(V_\infty) \) in \( \text{NEXT}^A \) containing \( M \), either \( \text{dim}(\text{st}(V_\infty)/W) \) is finite or \( \text{dim}(W/M) \) is finite. Then Theorem 3.22 and Theorem 4.10 show that the question of the existence of \( \text{NEXT} \)-maximal subspaces is oracle dependent.

**Theorem 5.2.** There is a recursive oracle \( A \) and an r.e. oracle \( B \) such that the following hold.

(i) \( \text{NEXT}^A \neq \text{DEXT}^A \) and \( \text{NEXT}^B \neq \text{DEXT}^B \).

(ii) There are no \( \text{NEXT}^A \)-maximal subspaces of \( \text{st}(V_\infty) \).

(iii) There is an \( \text{NEXT}^B \)-maximal subspace \( W \) of \( \text{st}(V_\infty) \).

In the same way, all the results in this paper about \( \text{P}^X \) and \( \text{NP}^X \) subspaces of \( \text{tal}(V_\infty) \) can be transfered to results \( \text{DEXT}^X \) and \( \text{NEXT}^X \) subspaces of \( \text{st}(V_\infty) \).
References

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