On Backward Product of Stochastic Matrices

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Abstract

We study the ergodicity of backward product of stochastic and doubly stochastic matrices by introducing the concept of absolute infinite flow property. By defining and exploring the properties of a rotational transformation of a stochastic chain, we show that absolute infinite flow property is necessary for ergodicity of any chain of stochastic matrices. Then, we establish that absolute infinite flow property is equivalent to ergodicity for doubly stochastic chains. Furthermore, we investigate the limiting behavior of a doubly stochastic chain and show that product of doubly stochastic matrices is convergent up to a permutation sequence.

Keywords: Product of stochastic matrices, doubly stochastic matrices, ergodicity.

1 Introduction

The study of forward product of an inhomogeneous chain of stochastic matrices is closely related to the limiting behavior, especially ergodicity, of inhomogeneous Markov chains. The earliest study on the forward product of inhomogeneous chains of stochastic matrices is the work of Hajnal in [1]. Motivated by a homogeneous Markov chain, Hajnal formulated the concepts of ergodicity in weak and strong senses for inhomogeneous Markov chains and developed some sufficient conditions for both weak and strong ergodicity of such chains. Using the properties of scrambling matrices that were introduced in [1], Wolfowitz [2] gave a condition under which all the chains driven from a finite set of stochastic matrices are strongly ergodic. In his elegant work [3], Shen gave geometric interpretations and provided some generalizations of the results in [1] by considering vector norms other than \( \| \cdot \|_\infty \), which was originally used in [1] to measure the scrambleness of a matrix.

The study of backward product of row-stochastic matrices, however, was motivated by different applications all of which were in search of a form of a consensus between a set of processors, individuals, or agents. DeGroot [4] studied such a product (for...
a homogeneous chain) as a tool for reaching consensus on a distribution of a certain unknown parameter among a set of agents. Later, Chatterjee and Seneta [5] provided a theoretical framework for reaching consensus by studying the backward product of an inhomogeneous chain of stochastic matrices. Motivated by the theory of inhomogeneous Markov chains, they defined the concepts of weak and strong ergodicity in this context and showed that those two properties are equivalent. Furthermore, they developed the theory of coefficients for ergodicity. Motivated by some distributed computational problems, in [6], Tsitsiklis and Bertsekas studied such a product from the dynamical system point of view. In fact, they considered a dynamics that enables an exogenous input as well as delays in the system. Through the study of such dynamics, they gave a more practical conditions for a chain to ensure the consensus. The work in [6] had a great impact on the subsequent studies of distributed estimation and control problems.

The common ground in the study of both forward and backward products of stochastic matrices are the chains of doubly stochastic matrices. By transposing the matrices in such a chain, forward product of matrices can be transformed into backward product of the transposes of the matrices. However, the transposition of a row-stochastic matrix is not necessarily a row-stochastic matrix, unless the matrix is doubly stochastic. Therefore, in the case of doubly stochastic matrices, any property of backward products translates to the same property for forward products.

Here, we study backward product of a chain of stochastic matrices in general, and then focus on doubly stochastic matrices in particular. We start by introducing a concept of absolute infinite flow property as a refinement of infinite flow property, proposed in our earlier work in [7]. We demonstrate that absolute infinite flow property is more stringent than infinite flow property, by showing that a sequence of permutation matrices cannot have absolute infinite flow property, while it may have infinite flow property. We introduce a concept of rotational transformation, which plays a central role in our exploration of the relations between absolute infinite flow property and ergodicity of a stochastic chain. In fact, using the properties of rotational transformation, we show that absolute infinite flow property is necessary for ergodicity of any stochastic chain. However, even though this property requires a rich structure for a stochastic chain, it is still not sufficient for ergodicity, which we illustrate on an example. In pursue of chains for which this property is sufficient for ergodicity, we come to identify a class of decomposable chains, which have a particularly simple test for absolute infinite flow property. Upon exploring decomposable chains, we show that doubly stochastic chains are decomposable by using Birkhoff-von Neumann theorem. Finally, we show that ergodicity and absolute infinite flow property are equivalent for doubly stochastic chains. We conclude the study, by providing a convergence rate result for ergodic doubly stochastic chains and by establishing a result on limiting behavior of doubly stochastic chain that need not be ergodic.

This work is a continuation of our earlier work in [7, 8] where we have studied random backward products for independent, not necessarily identical, random matrix processes with row-stochastic matrices. In [7], we have introduced infinite flow property and shown that this property is necessary for ergodicity of any chain of row-stochastic matrices. Moreover, we have shown that this property is also sufficient for ergodicity of such chains under some additional conditions on the matrices. In [8], we have extended
some of our results in [7] to a larger class of row-stochastic matrices by using some approximations of chains. The setting in [7, 8] has been within random matrix processes with row-stochastic matrices. Unlike our work in [7, 8], the work in this present paper is focused on deterministic sequences of row-stochastic matrices. Furthermore, the new concept of absolute infinite flow property is demonstrated to be significantly stronger than infinite flow property. Finally, our line of analysis in this paper relies on the development of some new concepts, such as rotational transformation and decomposable chains. Our result on convergence rate for ergodic doubly stochastic chains extends the rate result in [9] to a larger class of doubly stochastic chains, such as those that may not have uniformly bounded positive entries or uniformly bounded diagonal entries.

The main new results of this work include: (1) Formulation and development of absolute infinite flow property. We show that this property is necessary for ergodicity of any stochastic chain, despite the fact that it is much stronger than infinite flow property. (2) Introduction and exploration of rotational transformation of a stochastic chain with respect to a permutation chain. We establish that rotational transformation preserves several properties of a chain, such as ergodicity and absolute infinite flow property. (3) Establishment of equivalence between ergodicity and absolute infinite flow property for doubly stochastic chains. We accomplish this through the use of the Birkhoff-von Neumann decomposition of doubly stochastic matrices and properties of rotational transformation of a stochastic chain. (4) Development of a rate of convergence for ergodic doubly stochastic chains. (5) Establishment of the limiting behavior for doubly stochastic chains. We prove that a product of doubly stochastic matrices is convergent up to a permutation sequence.

The structure of this paper is as follows: in Section 1.1, we introduce the notation that we use throughout the paper. In Section 2, we discuss the concept of ergodicity and introduce absolute infinite flow property. In Section 3, we introduce and investigate the concept of rotational transformation. Using the properties of this transformation, we show that this property is necessary for ergodicity of any stochastic chain. Then, in Section 4, we introduce decomposable chains and study their properties. In Section 5, we study the product of doubly stochastic matrices by exploring ergodicity, rate of convergence, and the limiting behavior in the absence of ergodicity. We summarize the development in Section 6.

1.1 Notation and Basic Terminology

We view all vectors as columns. For a vector $x$, we write $x_i$ to denote its $i$th entry, and we write $x \geq 0$ ($x > 0$) to denote that all its entries are nonnegative (positive). We use $x^T$ to denote the transpose of a vector $x$. We write $\|x\|$ to denote the standard Euclidean vector norm i.e., $\|x\| = \sqrt{\sum_i x_i^2}$. We use $e_i$ to denote the vector with the $i$th entry equal to 1 and all other entries equal to 0, and we write $e$ for the vector with all entries equal to 1. For a given set $C$ and a subset $S$ of $C$, we write $S \subset C$ to denote that $S$ is a proper subset of $C$. A set $S \subset C$ such that $S \neq \emptyset$ is referred to as a nontrivial subset of $C$. We write $[m]$ to denote the integer set $\{1, \ldots, m\}$. For a set $S \subset [m]$, we let $|S|$ be the cardinality of the set $S$ and $\bar{S}$ be the complement of $S$ with respect to $[m]$, i.e., $\bar{S} = \{i \in [m] \mid i \notin S\}$. 
We denote the identity matrix by $I$. For a matrix $A$, we use $A_{ij}$ to denote its $(i,j)$th entry, $A_i$ and $A^j$ to denote its $i$th row and $j$th column vectors, respectively, and $A^T$ to denote its transpose. We write $\|A\|$ for the matrix norm induced by the Euclidean vector norm. A matrix $A$ is row-stochastic when its entries are nonnegative and the sum of a row entries is 1 for each row. Since we deal exclusively with row-stochastic matrices, we refer to such matrices simply as stochastic. A matrix $A$ is doubly stochastic when both $A$ and $A^T$ are stochastic. We often refer to a matrix sequence as a chain.

For an $m \times m$ matrix $A$, we use $\sum_{i<j} A_{ij}$ to denote the summation of the entries $A_{ij}$ over all $i,j \in [m]$ with $i<j$. Given a nonempty index set $S \subseteq [m]$ and a matrix $A$, we write $A_S$ to denote the following summation:

$$A_S = \sum_{i \in S, j \in S} A_{ij} + \sum_{i \in \bar{S}, j \in S} A_{ij}.$$ 

Note that $A_S$ satisfies $A_S = \sum_{i \in S, j \in \bar{S}} (A_{ij} + A_{ji})$.

An $m \times m$ matrix $P$ is a permutation matrix if it contains exactly one entry equal to 1 in each row and each column. Given a permutation matrix $P$, we use $P(S)$ to denote the image of an index set $S \subseteq [m]$ under the permutation $P$; specifically $P(S) = \{i \in [m] \mid P_{ij} = 1 \text{ for some } j \in S\}$. We note that a set $S \subseteq [m]$ and its image $P(S)$ under a permutation $P$ have the same cardinality, i.e., $|S| = |P(S)|$. Furthermore, for any permutation matrix $P$ and any nonempty index set $S \subseteq [m]$, the following relation holds:

$$\sum_{i \in P(S)} e_i = P \sum_{j \in S} e_j.$$ 

We denote the set of $m \times m$ permutation matrices by $\mathcal{P}_m$. Since there are $m!$ permutation matrices of size $m$, we may assume that the set of permutation matrices is indexed, i.e., $\mathcal{P}_m = \{P(\xi) \mid 1 \leq \xi \leq m!\}$. Also, we say that $\{P(k)\}$ is a permutation sequence if $P(k) \in \mathcal{P}_m$ for all $k \geq 0$. The sequence $\{I\}$ is the permutation sequence $\{P(k)\}$ with $P(k) = I$ for all $k$, and it is referred to as the trivial permutation sequence.

## 2 Ergodicity and Absolute Infinite Flow

In this section, we discuss the concepts of ergodicity and infinite flow property, and we introduce a more restrictive property than infinite flow, which will be a central concept in our later development.

### 2.1 Ergodic Chain

Here, we define ergodicity for a backward product of a chain $\{A(k)\}$ of $m \times m$ stochastic matrices $A(k)$. For $k > s \geq 0$, let $A(k : s) = A(k-1)A(k-2) \cdots A(s)$. We use the following definition for ergodicity.

**Definition 1.** We say that a chain $\{A(k)\}$ is ergodic if

$$\lim_{k \to \infty} A(k : s) = ev^T(s) \quad \text{for any } s \geq 0,$$

where $v(s)$ is a stochastic vector for any $s \geq 0$. 

In other words, a chain \( \{A(k)\} \) is ergodic if the infinite backward product \( \cdots A(s+2)A(s+1)A(s) \) converges to a matrix with identical rows, which must be stochastic vectors since the chain \( \{A(k)\} \) is stochastic.

To give an intuitive understanding of the backward product \( A(k:s) \) of a stochastic chain \( \{A(k)\} \), consider the following simple model for opinion dynamics for a set \([m] = \{1, \ldots, m\} \) of \( m \) agents. Suppose that at time \( s \), each of the agents has a belief about a certain issue that can be represented by a scalar \( x_i(s) \in \mathbb{R} \). Now, suppose that agents’ beliefs evolve in time by the following dynamics: at any time \( k \geq s \), the agents meet and discuss their beliefs about the issue and, at time \( k+1 \), they update their beliefs by taking convex combinations of the agents’ beliefs at the prior time \( k \), i.e.,

\[
x_i(k+1) = \sum_{j=1}^{m} A_{ij}(k)x_j(k) \quad \text{for all } i \in [m],
\]

with \( \sum_{j=1}^{m} A_{ij}(k) = 1 \) for all \( i \in [m] \), i.e., \( A(k) \) is stochastic. In this opinion dynamics model, we are interested in properties of the chain \( \{A(k)\} \) that will ensure the \( m \) agents reach a consensus on their beliefs for any starting time \( s \geq 0 \) and any initial belief profile \( x(s) \in \mathbb{R}^m \). Formally, we want to determine some restrictions on the chain which will guarantee that \( \lim_{k \to \infty} (x_i(k) - x_j(k)) = 0 \) for any \( i, j \in [m] \) and any \( s \geq 0 \). As proven in \[5\], this limiting behavior is equivalent to ergodicity of \( \{A(k)\} \) in the sense of Definition \[1\].

One can visualize the dynamics in Eq. \[1\] using the trellis graph associated with a given stochastic chain. The trellis graph of a stochastic chain \( \{A(k)\} \) is an infinite directed weighted graph \( G = (V, E, \{A(k)\}) \), with the vertex set \( V \) equal to the infinite grid \([m] \times \mathbb{Z}^+ \) and the edge set

\[
E = \{((j, k), (i, k+1)) \mid j, i \in [m], k \geq 0\}.
\]

In other words, we consider a copy of the set \([m]\) for any time \( k \geq 0 \) and we stack these copies over time, thus generating the infinite vertex set \( V = \{(i, k) \mid i \in [m], k \geq 0\} \). We then place a link from each \( j \in [m] \) at time \( k \) to every \( i \in [m] \) at time \( k+1 \), i.e., a link from each vertex \((j, k) \in V\) to each vertex \((i, k+1) \in V\). Finally, we assign the weight \( A_{ij}(k) \) to the link \(((j, k), (i, k+1))\). Now, consider the whole graph as an information tunnel through which the information flows: we inject a scalar \( x_i(0) \) at each vertex \((i, 0)\) of the graph. Then, from this point on, at each time \( k \geq 0 \), the information is transferred from time \( k \) to time \( k+1 \) through each edge of the graph that acts as a communication link. Each link attenuates the in-vertex’s value with its weight, while each vertex sums the information received through the incoming links. One can observe that the resulting information evolution is the same as the dynamics given in Eq. \[1\]. As an example, consider the \( 2 \times 2 \) static chain \( \{A(k)\} \) with \( A(k) \) defined by:

\[
A(k) = \begin{bmatrix}
\frac{1}{3} & \frac{3}{4} \\
\frac{3}{4} & \frac{7}{4}
\end{bmatrix}
\quad \text{for } k \geq 0.
\]

The trellis graph of this chain and the resulting dynamics is depicted in Figure \[1\].
2.2 Infinite Flow and Absolute Infinite Flow

Here, we discuss infinite flow property, as introduced in [7], and a more restrictive version of this property which turned out to be necessary for ergodicity of stochastic chains, as shown later on in Section 3.1.

We start by recalling the definition of the infinite flow property from [7].

Definition 2. A stochastic chain \( \{A(k)\} \) has infinite flow property if

\[
\sum_{k=0}^{\infty} A_S(k) = \infty \quad \text{for any nonempty } S \subset [m],
\]

where \( A_S(k) = \sum_{i \in S, j \in \bar{S}} (A_{ij}(k) + A_{ji}(k)). \)

Graphically, the infinite flow property requires that in the trellis graph of a given model \( \{A(k)\} \), the weights on the edges between \( S \times \mathbb{Z}^+ \) and \( \bar{S} \times \mathbb{Z}^+ \) sum up to infinity.

To illustrate the infinite flow property, consider the opinion dynamics in Eq. (1). One can interpret \( A_{ij}(k) \) as the credit that agent \( i \) gives to agent \( j \)'s opinion at time \( k \). Therefore, the sum \( \sum_{i \in S, j \in \bar{S}} A_{ij}(k) \) can be interpreted as the credit that the agents’ group \( S \subset [m] \) gives to the opinions of the agents that are outside of \( S \) (the agents in \( \bar{S} \)) at time \( k \). Similarly, \( \sum_{i \in \bar{S}, j \in S} A_{ij}(k) \) is the credit that the agents’ group \( \bar{S} \) gives to the agents in group \( S \). The intuition behind the concept of infinite flow property is that without having infinite accumulated credit between agents’ groups \( S \) and \( \bar{S} \), we cannot have an agreement among the agents for any starting time \( s \) of the opinion dynamic and for any opinion profile \( x(s) \) of the agents. In other words, the infinite flow property ensures that there is enough information flow between the agents as time passes by.

We have established in [7] that infinite flow property is necessary for ergodicity of a stochastic chain, as restated below for later use.

Theorem 1. [7] Infinite flow property is a necessary condition for ergodicity of any stochastic chain.

Although infinite flow property is necessary for ergodicity, this property alone is not strong enough to disqualify some stochastic chains from being ergodic, such as permutation sequences. As a concrete example consider a static chain \( \{A(k)\} \) of permutation matrices \( A(k) \) given by

\[
A(k) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{for } k \geq 0.
\]
Figure 2: The trellis graph of the permutation chain in Eq. (4). For the regular sequence \( \{ S(k) \} \), with \( S(k) = \{ 1 \} \) if \( k \) is even and \( S(k) = \{ 2 \} \) otherwise, the vertex set \( \{(i,k) \mid i \in S(k), k \geq 0 \} \) is marked by black vertices. The flow \( F(\{ A(k) \}; \{ S(k) \} ) \), as defined in (6), corresponds to the summation of the weights on the dashed edges.

As a remedy for this situation, in [7] we have imposed some additional conditions, including some feedback-type conditions, on the matrices \( A(k) \) that eliminate permutation matrices. Here, we take a different approach.

Specifically, we will require a stronger infinite flow property by letting the set \( S \) in Definition 2 vary with time. In order to do so, we will consider sequences \( \{ S(k) \} \) of index sets \( S(k) \subset [m] \) with a form of regularity in the sense that the sets \( S(k) \) have the same cardinality for all \( k \). In what follows, we will reserve notation \( \{ S(k) \} \) for the sequences of index sets \( S(k) \subset [m] \). Furthermore, for easier exposition, we define the notion of regularity for \( \{ S(k) \} \) as follows.

**Definition 3.** A sequence \( \{ S(k) \} \) is regular if the sets \( S(k) \) have the same (nonzero) cardinality, i.e., \( |S(k)| = |S(0)| \) for all \( k \geq 0 \) and \( |S(0)| \neq 0 \).

The nonzero cardinality requirement in Definition 3 is imposed only to exclude the trivial sequence \( \{ S(k) \} \) consisting of empty sets.

Graphically, a regular sequence \( \{ S(k) \} \) corresponds to the subset \( \{(i,k) \mid i \in S(k), k \geq 0 \} \) of vertices in the trellis graph associated with a given chain. As an illustration, let us revisit the 2 × 2 chain given in Eq. (4). Consider the regular \( \{ S(k) \} \) defined by

\[
S(k) = \begin{cases} 
\{ 1 \} & \text{if } k \text{ is even}, \\
\{ 2 \} & \text{if } k \text{ is odd}.
\end{cases}
\]

The vertex set \( \{(i,k) \mid i \in S(k), k \geq 0 \} \) associated with \( \{ S(k) \} \) is shown in Figure 2.

Now, let us consider a chain \( \{ A(k) \} \) of stochastic matrices \( A(k) \). Let \( \{ S(k) \} \) be any regular sequence. At any time \( k \), we define the flow associated with the entries of the matrix \( A(k) \) across the index sets \( S(k+1) \) and \( S(k) \) as follows:

\[
A_{S(k+1),S(k)}(k) = \sum_{i \in S(k+1), j \in S(k)} A_{ij}(k) + \sum_{i \in S(k+1), j \in S(k)} A_{ij}(k) \quad \text{for } k \geq 0.
\]

The flow \( A_{S(k+1),S(k)}(k) \) could be viewed as an instantaneous flow at time \( k \) induced by the corresponding elements in the matrix chain and the index set sequence. Accordingly, we define the total flow of a chain \( \{ A(k) \} \) over \( \{ S(k) \} \), as follows:

\[
F(\{ A(k) \}; \{ S(k) \}) = \sum_{k=0}^{\infty} A_{S(k+1),S(k)}(k).
\]

(6)
We are now ready to extend the definition of infinite flow property to time varying index sets $S(k)$.

**Definition 4.** A stochastic chain $\{A(k)\}$ has absolute infinite flow property if

$$F(\{A(k)\}; \{S(k)\}) = \infty \quad \text{for every regular sequence} \ \{S(k)\}.$$

Note that the absolute infinite flow property of Definition 4 is more restrictive than the infinite flow property of Definition 2. In particular, we can see this by letting the set sequence $\{S(k)\}$ be static, i.e., $S(k) = S$ for all $k$ and some nonempty $S \subset [m]$. In this case, the flow $A_{S(k+1),S(k)}(k)$ across the index sets $S(k+1)$ and $S(k)$ as defined in Eq. (5) reduces to $A_S(k)$, while the flow $F(\{A(k)\}; \{S(k)\})$ in Eq. (6) reduces to $\sum_{k=0}^{\infty} A_S(k)$. This brings us to the quantities that define the infinite flow property (Definition 2). Thus, the infinite flow property requires that the flow across a trivial regular sequence $\{S\}$ is infinite for all nonempty $S \subset [m]$, which is evidently less restrictive requirement than that of Definition 4. In light of this, we see that if a stochastic chain $\{A(k)\}$ has absolute infinite property, then it has infinite flow property.

The distinction between absolute infinite flow property and infinite flow property is actually much deeper. Recall our example of the chain $\{A(k)\}$ in Eq. (4), which demonstrated that a permutation chain may posses infinite flow property. Now, for this chain, consider the regular sequence $\{S(k)\}$ with $S(2k) = \{1\}$ and $S(2k+1) = \{2\}$ for $k \geq 0$. The trellis graph associated with $\{A(k)\}$ is shown in Figure 2 where $\{S(k)\}$ is depicted by black vertices. The flow $F(\{A(k)\}; \{S(k)\})$ corresponds to the summation of the weights on the dashed edges in Figure 2 which is equal to zero in this case. Thus, the chain $\{A(k)\}$ does not have absolute flow property.

In fact, while some chains of permutation matrices may have infinite flow property, it turns out that no chain of permutation matrices has absolute infinite flow property. In other words, absolute infinite flow property is strong enough to filter out the chains of permutation matrices in our search for necessary and sufficient conditions for ergodicity which is a significant distinction between absolute infinite flow property and infinite flow property. To formally establish this property, we turn our attention to an intimate connection between a regular index sequence and a permutation sequence that can be associated with the index sequence.

Specifically, an important feature of a regular sequence $\{S(k)\}$ is that it can be obtained as the image of the initial set $S(0)$ under a certain permutation sequence $\{P(k)\}$. To see this, note that we can always find a one-to-one matching between the indices in $S(k)$ and $S(k+1)$ since $|S(k)| = |S(k+1)|$. The complements $\bar{S}(k)$ and $\bar{S}(k+1)$ of the sets $S(k)$ and $S(k+1)$, respectively, also have the same cardinality, so there is a one-to-one matching between $\bar{S}(k)$ and $\bar{S}(k+1)$ as well. Thus, we have a matching for the indices in $[m]$ which is one-to-one mapping between $S(k)$ to $S(k+1)$, and also one-to-one mapping between $\bar{S}(k)$ and $\bar{S}(k+1)$. Therefore, we can define an $m \times m$ matrix $P(k)$ as the incidence matrix corresponding to this matching, as follows: for every $j \in S(k)$ we let $P_{ij}(k) = 1$ if index $j$ is matched with the index $i \in S(k+1)$ and $P_{ij}(k) = 0$ otherwise; similarly, for every $j \in \bar{S}(k)$ we let $P_{ij}(k) = 1$ if index $j$ is matched with the index $i \in S(k+1)$ and $P_{ij}(k) = 0$ otherwise. The resulting matrix $P(k)$ is a permutation matrix and the set $S(k+1)$ is the image of set $S(k)$ under the
permutation matrix $P(k)$, i.e., $S(k + 1) = P(k)(S(k))$. Continuing in this way, we can see that $S(k)$ is just the image of $S(k-1)$ under some permutation matrix $P(k-1)$, i.e., $S(k) = P(k-1)(S(k-1))$ and so on. As a result, any set $S(k)$ is an image of a finitely many permutations of the initial set $S(0)$; formally $S(k) = P(k-1)\cdots P(1)P(0)(S(0))$. Therefore, we will refer to the set $S(k)$ as the image of the set $S(0)$ under $\{P(k)\}$ at time $k$. Also, we will refer to the sequence $\{S(k)\}$ as the trajectory of the set $S(0)$ under $\{P(k)\}$.

In the next lemma, we show that no chain of permutation matrices has absolute infinite property.

**Lemma 1.** For any permutation chain $\{P(k)\}$, there exists a regular sequence $\{S(k)\}$ for which $F(\{P(k)\}, \{S(k)\}) = 0$.

**Proof.** Let $\{P(k)\}$ be an arbitrary permutation chain and let $\{S(k)\}$ be the trajectory of a nonempty set $S(0) \subset [m]$ under the permutation $\{P(k)\}$. Note that $\{S(k)\}$ is regular, and we have

$$P_{S(k+1),S(k)}(k) = \sum_{i \in S(k+1), j \in \overline{S(k)}} P_{ij}(k) + \sum_{i \in \overline{S(k+1)}, j \in S(k)} P_{ij}(k) = 0,$$

which is true since $P(k)$ is a permutation matrix and $S(k+1)$ is the image of $S(k)$ under $P(k)$. Q.E.D.

Hence, by this lemma none of the permutation sequences $\{P(k)\}$ has absolute infinite flow property.

### 3 Necessity of Absolute Infinite Flow for Ergodicity

As discussed in Theorem 1, infinite flow property is necessary for ergodicity of a stochastic chain. In this section, we show that the absolute infinite flow property is actually necessary for ergodicity of a stochastic chain despite the fact that this property is much more restrictive than infinite flow property. We do this by considering a stochastic chain $\{A(k)\}$ and a related chain, say $\{B(k)\}$, such that the flow of $\{A(k)\}$ over a trajectory translates to a flow of $\{B(k)\}$ over an appropriately defined trajectory. The technique that we use for defining the chain $\{B(k)\}$ related to a given chain $\{A(k)\}$ is developed in the following section. Then, in Section 3.2 we prove the necessity of absolute infinite flow for ergodicity.

#### 3.1 Rotational Transformation

Rotational transformation is a process that takes a chain and produces another chain through the use of a permutation sequence $\{P(k)\}$. Specifically, we have the following definition of the rotational transformation with respect to a permutation chain.
Definition 5. Given a permutation chain \( \{P(k)\} \), the rotational transformation of an arbitrary chain \( \{A(k)\} \) with respect to \( \{P(k)\} \) is the chain \( \{B(k)\} \) given by

\[
B(k) = P^T(k+1:0)A(k)P(k:0) \quad \text{for } k \geq 0,
\]

where \( P(0:0) = I \). We say that \( \{B(k)\} \) is the rotational transformation of \( \{A(k)\} \) by \( \{P(k)\} \).

The rotational transformation has some interesting properties for stochastic chains which we discuss in the following lemma. These properties play a key role in the subsequent development, while they may also be of interest in their own right.

Lemma 2. Let \( \{A(k)\} \) be an arbitrary stochastic chain and \( \{P(k)\} \) be an arbitrary permutation chain. Let \( \{B(k)\} \) be the rotational transformation of \( \{A(k)\} \) by \( \{P(k)\} \). Then, the following statements are valid:

(a) The chain \( \{B(k)\} \) is stochastic. Furthermore,

\[
B(k:s) = P^T(k:0)A(k:s)P(s:0) \quad \text{for any } k > s \geq 0,
\]

where \( P(0:0) = I \).

(b) The chain \( \{A(k)\} \) is ergodic if and only if the chain \( \{B(k)\} \) is ergodic.

(c) For any \( S \subset [m] \) and \( k \geq 0 \), we have \( A_{S(k+1),S(k)}(k) = B_S(k) \), where \( S(k) \) is the image of \( S \) under \( \{P(k)\} \) at time \( k \), i.e., \( S(k) = P(k:0)(S) \).

Proof. (a) By the definition of \( B(k) \), we have \( B(k) = P^T(k+1:0)A(k)P(k:0) \). Thus, \( B(k) \) is stochastic as the product of finitely many stochastic matrices is a stochastic matrix.

The proof of relation \( B(k:s) = P^T(k:0)A(k:s)P(s:0) \) proceeds by induction on \( k \) for \( k > s \) and an arbitrary but fixed \( s \geq 0 \). For \( k = s + 1 \), by the definition of \( B(s) \) (see Definition 3), we have \( B(s) = P^T(s+1:0)A(s)P(s:0) \), while \( B(s+1:s) = B(s) \) and \( A(s+1:s) = A(s) \). Hence, \( B(s+1,s) = P^T(s+1:0)A(s+1:s)P(s:0) \) which shows that \( B(k,s) = P^T(k:0)A(k:s)P(s:0) \) for \( k = s + 1 \), thus implying that the claim is true for \( k = s + 1 \).

Now, suppose that the claim is true for some \( k > s \), i.e., \( B(k,s) = P^T(k:0)A(k:s)P(s:0) \) for some \( k > s \). Then, for \( k + 1 \) we have

\[
B(k+1:s) = B(k)B(k:s) = B(k) \left( P^T(k:0)A(k:s)P(s:0) \right),
\]

where the last equality follows by the induction hypothesis. By the definition of \( B(k) \), we have \( B(k) = P^T(k+1:0)A(k)P(k:0) \), and by replacing \( B(k) \) by \( P^T(k+1:0)A(k)P(k:0) \) in Eq. (7), we obtain

\[
B(k+1:s) = \left( P^T(k+1:0)A(k)P(k:0) \right) \left( P^T(k:0)A(k:s)P(s:0) \right)
= P^T(k+1:0)A(k) \left( P(k:0)P^T(k:0) \right) A(k:s)P(s:0)
= P^T(k+1:0)A(k)A(k:s)P(s:0),
\]
where the last equality follows from $P^TP = I$ which is valid for any permutation matrix $P$, and the fact that the product of two permutation matrices is a permutation matrix. Since $A(k)A(k : s) = A(k + 1 : s)$, it follows that

$$B(k + 1 : s) = P^T(k + 1 : 0)A(k + 1 : s)P(s : 0),$$

thus showing that the claim is true for $k + 1$.

(b) Let the chain $\{A(k)\}$ be ergodic and fix an arbitrary starting time $t_0 \geq 0$. Then, for any $\epsilon > 0$, there exists a sufficiently large time $N_\epsilon \geq t_0$, such that the rows of $A(k : t_0)$ are within $\epsilon$-vicinity of each other; specifically $\|A_i(k : t_0) - A_j(k : t_0)\| \leq \epsilon$ for any $k \geq N_\epsilon$ and all $i, j \in [m]$. We now look at the matrix $B(k : t_0)$ and its rows. By part (a), we have for all $k > t_0$,

$$B(k : t_0) = P^T(k : 0)A(k : t_0)P(t_0 : 0).$$

Furthermore, the $i$th row of $B(k : t_0)$ can be represented as $e_i^TB(k : t_0)$. Therefore, the norm of the difference between the $i$th and $j$th row of $B(k : t_0)$ is given by

$$\|B_i(k : t_0) - B_j(k : t_0)\| = \|(e_i - e_j)^TB(k : t_0)\| = \|(e_i - e_j)^TP^T(k : 0)A(k : t_0)P(t_0 : 0)\|.$$ 

Letting $e_{i(k)} = P(k : 0)e_i$ for any $i \in [m]$, we further have

$$\|B_i(k : t_0) - B_j(k : t_0)\| = \|(e_{i(k)} - e_{j(k)})^TA(k : t_0)P(t_0 : 0)\|
= \|(A_i(k : t_0) - A_j(k : t_0))P(t_0 : 0)\|
= \|A_i(k : t_0) - A_j(k : t_0)\|, \quad (8)$$

where the last inequality holds since $P(t_0 : 0)$ is a permutation matrix and $\|Px\| = \|x\|$ for any permutation $P$ and any $x \in \mathbb{R}^m$. Choosing $k \geq N_\epsilon$ and using $\|A_i(k : t_0) - A_j(k : t_0)\| \leq \epsilon$ for any $k \geq N_\epsilon$ and all $i, j \in [m]$, we obtain

$$\|B_i(k : t_0) - B_j(k : t_0)\| \leq \epsilon \quad \text{for any } k \geq N_\epsilon \text{ and all } i, j \in [m].$$

Therefore, it follows that the ergodicity of $\{A(k)\}$ implies the ergodicity of $\{B(k)\}$.

For the reverse implication we note that $A(k : t_0) = P(k : 0)B(k : t_0)P^T(t_0 : 0)$, which follows by part (a) and the fact $PP^T = PP^T = I$ for any permutation $P$. The rest of the proof follows a line of analysis similar to the preceding case, where we exchange the roles of $B(k : t_0)$ and $A(k : t_0)$.

(c) By the definition of $B(k)$, we have $B(k) = P^T(k + 1 : 0)A(k)P(k : 0)$. Therefore, for any $i, j \in [m]$,

$$B_{ij}(k) = e_i^TB(k)e_j = e_i^TP^T(k + 1 : 0)A(k)P(k : 0)e_j.$$

Now, let $e_{i(k)} = P(k : 0)e_i$ for any $i \in [m]$. In this notation, we have $P(k : 0)e_j = e_{j(k)}$ and $e_i^TP^T(k + 1 : 0)$ is equal to $e_i^T_{i(k + 1)}$. Hence,

$$B_{ij}(k) = e_i^T_{i(k + 1)}A(k)e_{j(k)} = A_{i(k + 1), j(k)}(k) \quad \text{for any } i, j \in [m]. \quad (9)$$
Now, given the set \( S \), consider \( B_S(k) \) for which we have

\[
B_S(k) = \sum_{i \in S, j \in S} B_{ij}(k) + \sum_{i \in S, j \in \bar{S}} B_{ij}(k) = \sum_{i \in S, j \in \bar{S}} A_{i(k+1),j(k)}(k) + \sum_{i \in S, j \in \bar{S}} A_{i(k+1),j(k)}(k),
\]

where the last equality follows by Eq. (9). Since \( S(k) = P(k : 0)S \) for all \( k \), where \( P(0 : 0) = I \), it follows that \( i(k) \in S(k) \) for \( i \in S \). Therefore,

\[
\sum_{i \in S, j \in S} A_{i(k+1),j(k)}(k) + \sum_{i \in S, j \in \bar{S}} A_{i(k+1),j(k)}(k) = \sum_{\ell \in S(k+1), p \in S(k)} A_{\ell p}(k) + \sum_{\ell \in S(k+1), p \in \bar{S}} A_{\ell p}(k) = A_{S(k+1),S(k)}(k),
\]

thus showing that \( B_S(k) = A_{S(k+1),S(k)}(k) \). Q.E.D.

As listed in Lemma 2, the rotational transformation has some interesting properties: it preserves ergodicity and it preserves absolute infinite flow property, interchanging flows. We will use these properties intensively in the development in this section and the rest of the paper.

### 3.2 Necessity of Absolute Infinite Flow Property

In this section, we establish the necessity of absolute infinite flow property for ergodicity of stochastic chains. The proof of this result relies on Lemma 2.

**Theorem 2.** The absolute infinite flow property is necessary for ergodicity of any stochastic chain.

**Proof.** Let \( \{A(k)\} \) be an ergodic stochastic chain. Let \( \{S(k)\} \) be any regular sequence. Then, there is a permutation sequence \( \{P(k)\} \) such that \( \{S(k)\} \) is the trajectory of the set \( S(0) \subset [m] \) under \( \{P(k)\} \), i.e., \( S(k) = P(k : 0)(S(0)) \) for all \( k \), where \( P(0 : 0) = I \). Let \( \{B(k)\} \) be the rotational transformation of \( \{A(k)\} \) by the permutation sequence \( \{P(k)\} \). Then, by Lemma 2(a), the chain \( \{B(k)\} \) is stochastic. Moreover, by Lemma 2(b), the chain \( \{B(k)\} \) is ergodic. Now, by the necessity of infinite flow property (Theorem 1), the chain \( \{B(k)\} \) should have infinite flow property, i.e.,

\[
\sum_{k=0}^{\infty} B_S(k) = \infty \quad \text{for any } S \subset [m]. \tag{10}
\]

Therefore, in particular, we must have \( \sum_{k=0}^{\infty} B_S(k) = \infty \) for \( S = S(0) \). By Lemma 2(c), Eq. (10) implies

\[
\sum_{k=0}^{\infty} A_{S(k+1),S(k)}(k) = \infty,
\]

thus showing that \( \{A(k)\} \) has absolute infinite flow property. Q.E.D.
The converse statement of Theorem 2 is not true generally, namely absolute infinite flow need not be sufficient for ergodicity of a chain. We reinforce this statement later in Section 4 (Corollary 1). Thus, even though absolute infinite flow property requires a lot of structure for a chain \( \{A(k)\} \), by requiring that the flow of \( \{A(k)\} \) over any regular sequence \( \{S(k)\} \) be infinite, this is still not enough to guarantee ergodicity of the chain. However, as we will soon see, it turns out that this property is sufficient for ergodicity of the doubly stochastic chains.

4 Decomposable Stochastic Chains

In this section, we consider a class of stochastic chains, termed decomposable, for which verifying absolute infinite flow property can be reduced to showing that the flows over some specific regular sequences are infinite. We explore some properties of this class which will be also used in later sections.

Here is the definition of a decomposable chain.

**Definition 6.** A chain \( \{A(k)\} \) is decomposable if \( \{A(k)\} \) can be represented as a non-trivial convex combination of a permutation chain \( \{P(k)\} \) and a stochastic chain \( \{\tilde{A}(k)\} \), i.e., there exists a \( \gamma > 0 \) such that

\[
A(k) = \gamma P(k) + (1 - \gamma) \tilde{A}(k) \quad \text{for all } k \geq 0.
\]

(11)

We refer to \( \{P(k)\} \) as a permutation component of \( \{A(k)\} \) and to \( \gamma \) as a mixing parameter for \( \{A(k)\} \).

An example of a decomposable chain is a chain \( \{A(k)\} \) with uniformly bounded diagonal entries, i.e., with \( A_{ii}(k) \geq \gamma \) for all \( k \geq 0 \) and some \( \gamma > 0 \). Instances of such chains have been studied in [10, 11, 12, 13, 14, 15, 16]. In this case, \( A(k) = \gamma I + (1 - \gamma) \tilde{A}(k) \) where \( \tilde{A}(k) = \frac{1}{\gamma}(A(k) - \gamma I) \). Note that \( A(k) - \gamma I \geq 0 \), where the inequality is to be understood component-wise, and \((A(k) - \gamma I)e = (1 - \gamma)e\), which follows from the stochasticity of \( A(k) \). Therefore, \( \tilde{A}(k) \) is a stochastic matrix for any \( k \geq 0 \) and the trivial permutation \( \{I\} \) is a permutation component of \( \{A(k)\} \). Later, we will show that any doubly stochastic chain is decomposable.

We have some side remarks about decomposable chains. The first remark is an observation that a permutation component of a decomposable chain \( \{A(k)\} \) need not to be unique. An extreme example is the chain \( \{A(k)\} \) with \( A(k) = \frac{1}{m}ee^T \) for all \( k \geq 0 \). Since \( \frac{1}{m}ee^T = \frac{1}{m} \sum_{\xi=1}^{m} P(\xi) \), any sequence of permutation matrices is a permutation component of \( \{A(k)\} \). Another remark is about a mixing coefficient \( \gamma \) of a chain \( \{A(k)\} \). Note that mixing coefficient is independent of the permutation component. Furthermore, if \( \gamma > 0 \) is a mixing coefficient for a chain \( \{A(k)\} \), then any \( \xi \in (0, \gamma] \) is also a mixing coefficient for \( \{A(k)\} \), as it can be seen from the decomposition in Eq. (11).

An interesting property of any decomposable chain is that if they are rotationally transformed with respect to their permutation component, the resulting chain has trivial permutation component \( \{I\} \). This property is established in the following lemma.
Lemma 3. Let \( \{A(k)\} \) be a decomposable chain with a permutation component \( \{P(k)\} \) and a mixing coefficient \( \gamma \). Let \( \{B(k)\} \) be the rotational transformation of \( \{A(k)\} \) with respect to \( \{P(k)\} \). Then, the chain \( \{B(k)\} \) is decomposable with a trivial permutation component \( \{I\} \) and a mixing coefficient \( \gamma \).

Proof. Note that by the definition of a decomposable chain (Definition 6), we have
\[
A(k) = \gamma P(k) + (1 - \gamma) \tilde{A}(k) \quad \text{for any} \quad k \geq 0,
\]
where \( P(k) \) is a permutation matrix and \( \tilde{A}(k) \) is a stochastic matrix. Therefore,
\[
A(k)P(k : 0) = \gamma P(k)P(k : 0) + (1 - \gamma) \tilde{A}(k)P(k : 0).
\]
By noticing that \( P(k)P(k : 0) = P(k + 1 : 0) \) and by using left-multiplication with \( P^T(k + 1 : 0) \), we obtain
\[
P^T(k + 1 : 0)A(k)P(k : 0) = \gamma P^T(k + 1 : 0)P(k + 1 : 0) + (1 - \gamma) P^T(k + 1 : 0) \tilde{A}(k)P(k : 0).
\]
By the definition of the rotational transformation (Definition 5), we have \( B(k) = P^T(k + 1 : 0)A(k)P(k : 0) \). Using this and the fact \( P^T P = I \) for any permutation matrix \( P \), we further have
\[
B(k) = \gamma I + (1 - \gamma) P^T(k + 1 : 0) \tilde{A}(k)P(k : 0).
\]
Define \( \tilde{B}(k) = P^T(k + 1 : 0) \tilde{A}(k)P(k : 0) \) and note that each \( \tilde{B}(k) \) is a stochastic matrix. Hence,
\[
B(k) = \gamma I + (1 - \gamma) \tilde{B}(k), \tag{12}
\]
thus showing that the chain \( \{B(k)\} \) is decomposable with the trivial permutation component and a mixing coefficient \( \gamma \). Q.E.D.

In the next lemma, we prove that absolute infinite flow property and infinite flow property are one and the same for decomposable chains with a trivial permutation component.

Lemma 4. For a decomposable chain with a trivial permutation component, infinite flow property and absolute infinite flow property are equivalent.

Proof. By definition, absolute infinite flow property implies infinite flow property for any stochastic chain. For the reverse implication, let \( \{A(k)\} \) be decomposable with a permutation component \( \{I\} \). Also, assume that \( \{A(k)\} \) has infinite flow property. We claim that \( \{A(k)\} \) has absolute infinite flow property. To see this, let \( \{S(k)\} \) be any regular sequence. If \( S(k) \) is constant after some time \( t_0 \), i.e., \( S(k) = S(t_0) \) for \( k \geq t_0 \) and some \( t_0 \geq 0 \), then
\[
\sum_{k=t_0}^{\infty} A_{S(k+1),S(k)}(k) = \sum_{k=t_0}^{\infty} A_{S(t_0)}(k) = \infty,
\]
where the last equality holds since \( \{A(k)\} \) has infinite flow property and \( \sum_{k=0}^{t_0} A_{S(t_0)}(k) \) is finite. Therefore, if \( S(k) = S(t_0) \) for \( k \geq t_0 \), then we must have \( \sum_{k=0}^{\infty} A_{S(k+1),S(k)}(k) = \infty \).

If there is no \( t_0 \geq 0 \) with \( S(k) = S(t_0) \) for \( k \geq t_0 \), then we must have \( S(k_r+1) \neq S(k_r) \) for an increasing time sequence \( \{k_r\} \). Now, for an \( i \in S(k_r) \setminus S(k_r+1) \neq \emptyset \), we have \( A_{S(k_r+1),S(k_r)}(k_r) \geq A_{ii}(k_r) \) since \( i \in S(k_r+1) \). Furthermore, \( A_{ii}(k) \geq \gamma \) for all \( k \) since \( \{A(k)\} \) has the trivial permutation sequence \( \{I\} \) as a permutation component with a mixing coefficient \( \gamma \). Therefore,

\[
\sum_{k=0}^{\infty} A_{S(k+1),S(k)}(k) \geq \sum_{r=0}^{\infty} A_{S(k_r+1),S(k_r)}(k_r) \geq \gamma \sum_{r=0}^{\infty} 1 = \infty.
\]

All in all, \( F(\{A(k)\}, \{S(k)\}) = \infty \) for any regular sequence \( \{S(k)\} \) and, hence, the chain \( \{A(k)\} \) has absolute infinite flow property. Q.E.D.

Lemma 4 shows that absolute infinite flow property may be easier to verify for the chains with a trivial permutation component, by just checking infinite flow property. This result, together with Lemma 3 and the properties of rotational transformation established in Lemma 2, provide a basis to show that a similar reduction of absolute infinite flow property is possible for any decomposable chain.

**Theorem 3.** Let \( \{A(k)\} \) be a decomposable chain with a permutation component \( \{P(k)\} \). Then, the chain \( \{A(k)\} \) has absolute infinite flow property if and only if \( F(\{A(k)\}, \{S(k)\}) = \infty \) for any trajectory \( \{S(k)\} \) under \( \{P(k)\} \), i.e., for all \( S(0) \subset [m] \) and its trajectory \( \{S(k)\} \) under \( \{P(k)\} \).

**Proof.** Since, by definition absolute, infinite flow property implies \( F(\{A(k)\}, \{S(k)\}) = \infty \) for any regular sequence \( \{S(k)\} \), it suffice to show that \( F(\{A(k)\}, \{S(k)\}) = \infty \) for any trajectory \( \{S(k)\} \) under \( \{P(k)\} \). To show this, let \( \{B(k)\} \) be the rotational transformation of \( \{A(k)\} \) with respect to \( \{P(k)\} \). Since \( \{A(k)\} \) is decomposable, by Lemma 3, it follows that \( \{B(k)\} \) has the trivial permutation component \( \{I\} \). Therefore, by Lemma 4 \( \{B(k)\} \) has absolute infinite flow property if and only if it has infinite flow property, i.e.,

\[
\sum_{k=0}^{\infty} B_S(k) = \infty \quad \text{for all nonempty } S \subset [m]. \tag{13}
\]

By Lemma 4, we have \( B_S(k) = A_{S(k+1),S(k)}(k) \), where \( S(k) \) is the image of \( S(0) = S \) under the permutation \( \{P(k)\} \) at time \( k \). Therefore, Eq. (13) holds if and only if

\[
\sum_{k=0}^{\infty} A_{S(k+1),S(k)}(k) = \infty,
\]

which in view of \( F(\{A(k)\}, \{S(k)\}) = \sum_{k=0}^{\infty} A_{S(k+1),S(k)}(k) \) shows that \( F(\{A(k)\}, \{S(k)\}) = \infty \). Q.E.D.
In light of Theorem 3, verification of absolute infinite flow property for decomposable chains is considerably simpler than for an arbitrary stochastic chain. For decomposable chains, it suffice to verify 

\[ F(\{A(k)\};\{S(k)\}) = \infty \]

only for the trajectory \( \{S(k)\} \) of \( S(0) \) under a permutation component \( \{P(k)\} \) of \( \{A(k)\} \) for any \( S(0) \subseteq [m] \).

Another direct consequence of Lemma 4 is that absolute infinite flow property is not generally sufficient for ergodicity.

**Corollary 1.** Absolute infinite flow property is not a sufficient condition for ergodicity.

**Proof.** Consider the following static chain:

\[
A(k) = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 1
\end{bmatrix} \quad \text{for } k \geq 0.
\]

It can be seen that \( \{A(k)\} \) has infinite flow property. Furthermore, it can be seen that \( \{A(k)\} \) is decomposable and has the trivial permutation sequence \( \{I\} \) as a permutation component. Thus, by Lemma 4, the chain \( \{A(k)\} \) has absolute infinite flow property. However, \( \{A(k)\} \) is not ergodic. This can be seen by noticing that the vector \( v = (1, \frac{1}{2}, 0)^T \) is a fixed point of the dynamics \( x(k+1) = A(k)x(k) \) with \( x(0) = v \), i.e., \( v = A(k)v \) for any \( k \geq 0 \). Hence, \( \{A(k)\} \) is not ergodic. Q.E.D.

Although absolute infinite flow property is a stronger necessary condition for ergodicity than infinite flow property, Corollary 1 demonstrates that absolute infinite flow property is not yet strong enough to be equivalent to ergodicity. However, using the developed results so far, we will show that absolute infinite flow property is in fact equivalent to ergodicity for doubly stochastic chains, as discussed in the following section.

## 5 Doubly Stochastic Chains

In this section, we focus on the class of the doubly stochastic chains. We first show that this class is a subclass of the decomposable chains. Using this result and the results developed in the preceding sections, we establish that absolute infinite flow property is equivalent to ergodicity for doubly stochastic chains.

We start our development by proving that a doubly stochastic chain is decomposable. The key ingredient in this development is the Birkhoff-von Neumann theorem ([17] page 527) stating that a doubly stochastic matrix is a convex combination of permutation matrices. More precisely, by the Birkhoff-von Neumann theorem, a matrix \( A \) is doubly stochastic if and only if \( A \) is a convex combination of permutation matrices, i.e.,

\[
A = \sum_{\xi=1}^{m!} q_{\xi} P(\xi),
\]

where \( \sum_{\xi=1}^{m!} q_{\xi} = 1 \) and \( q_{\xi} \geq 0 \) for \( \xi \in [m!] \). Here, we assume that all the permutation matrices are indexed in some manner (see notation in Section 1).
Now, consider a sequence \( \{A(k)\} \) of doubly stochastic matrices. By applying Birkhoff-von Neumann theorem to each \( A(k) \), we have

\[
A(k) = \sum_{\xi=1}^{m!} q_\xi(k) P(\xi),
\]

where \( \sum_{\xi=1}^{m!} q_\xi(k) = 1 \) and \( q_\xi(k) \geq 0 \) for all \( \xi \in [m!] \) and \( k \geq 0 \). Since \( \sum_{\xi=1}^{m!} q_\xi(k) = 1 \) and \( q_\xi(k) \geq 0 \) for all \( \xi \in [m!] \) and \( k \geq 0 \), there exists a scalar \( \gamma \geq \frac{1}{m!} \) such that for every \( k \geq 0 \), we can find \( \xi(k) \in [m!] \) satisfying \( q_\xi(k) \geq \gamma \). Therefore, for any time \( k \geq 0 \), there is a permutation matrix \( P(k) = P(\xi(k)) \) such that

\[
A(k) = \gamma P(k) + \sum_{\xi=1}^{m!} \alpha_\xi(k) P(\xi) = \gamma P(k) + (1 - \gamma) \tilde{A}(k),
\]

where \( \gamma > 0 \) is a time-independent scalar and \( \tilde{A}(k) = \frac{1}{1-\gamma} \sum_{\xi=1}^{m!} \alpha_\xi(k) P(\xi) \).

The decomposition of \( A(k) \) in Eq. (15) fits the description in the definition of decomposable chains (Definition 6). Therefore, we have established the following result.

**Lemma 5.** Any doubly stochastic chain is a decomposable chain.

In the light of Lemma 5, all the results developed in Section 4 are applicable to doubly stochastic chains. In particular, Theorem 3 is the most relevant, which states that verifying absolute infinite flow property for decomposable chains can be reduced to verifying infinite flow along particular sequences of index sets. Another result that we use is the special instance of Theorem 6 in [7] as applied to doubly stochastic chains. Any doubly stochastic chain that has the trivial permutation component \( \{I\} \) (i.e., Eq. (15) holds with \( P(k) = I \)) fits the framework of Theorem 6 in [7]. We restate this theorem for convenience.

**Theorem 4.** Let \( \{A(k)\} \) be a doubly stochastic chain with a permutation component \( \{I\} \). Then, the chain \( \{A(k)\} \) is ergodic if and only if it has infinite flow property.

Now, we are ready to deliver our main result of this section, showing that ergodicity and absolute infinite flow are equivalent for doubly stochastic chains. We accomplish this by combining Theorem 3 and Theorem 4.

**Theorem 5.** A doubly stochastic chain \( \{A(k)\} \) is ergodic if and only if it has absolute infinite flow property.

**Proof.** Let \( \{P(k)\} \) be a permutation component for \( \{A(k)\} \) and let \( \{B(k)\} \) be the rotational transformation of \( \{A(k)\} \) with respect to its permutation component. By Lemma 3, \( \{B(k)\} \) has the trivial permutation component \( \{I\} \). Moreover, since \( B(k) = P^T(k+1:0)A(k)P(k:0) \), where \( P^T(k+1:0) \), \( A(k) \) and \( P(k:0) \) are doubly stochastic matrices, it follows that \( \{B(k)\} \) is a doubly stochastic chain. Therefore, by Theorem 4, it follows that \( \{B(k)\} \) is ergodic if and only if it has infinite flow property. Then, by Lemma 3, the chain \( \{B(k)\} \) has infinite flow property if and only if \( \{A(k)\} \) has absolute infinite flow property. Q.E.D.
Theorem 5 provides an alternative characterization of ergodicity for doubly stochastic chains, under only requirement to have absolute infinite flow property. We note that Theorem 5 does not impose any other specific conditions on matrices $A(k)$ such as uniformly bounded diagonal entries or uniformly bounded positive entries, which have been typically assumed in the existing literature (see for example [10, 11, 12, 13, 14, 15, 16]).

We observe that absolute infinite flow typically requires verifying that infinite flow exist along every regular sequence of index sets. However, to use Theorem 5 we do not have to check infinite flow for every regular sequence. This reduction in checking absolute infinite flow property is due to Theorem 3 which shows that in order to assert absolute infinite flow property for doubly stochastic chains, it suffice that the flow over some specific regular sets is infinite. We summarize this observation in the following corollary.

**Corollary 2.** Let $\{A(k)\}$ be a doubly stochastic chain with a permutation component $\{P(k)\}$. Then, the chain is ergodic if and only if $F(\{A(k)\}; \{S(k)\}) = \infty$ for all trajectories $\{S(k)\}$ of subsets $S(0) \subset [m]$ under $\{P(k)\}$.

### 5.1 Rate of Convergence

Here, we explore a rate of convergence result for an ergodic doubly stochastic chain $\{A(k)\}$. In the development, we adopt a dynamical system point of view for the chain and we consider a Lyapunov function associated with the dynamics. The final ingredient in the development is the establishment of another important property of rotational transformation related to the invariance of the Lyapunov function.

Let $\{A(k)\}$ be a doubly stochastic chain and consider the dynamic system driven by this chain. Specifically, define the following dynamics, starting with any initial point $x(0) \in \mathbb{R}^m$,

$$x(k + 1) = A(k)x(k) \quad \text{for } k \geq 0. \quad (16)$$

With this dynamics, we associate a Lyapunov function $V(\cdot)$ defined as follows:

$$V(x) = \sum_{i=1}^{m} (x_i - \bar{x})^2 \quad \text{for } x \in \mathbb{R}^m, \quad (17)$$

where $\bar{x} = \frac{1}{m}e^T x$. This function has often been used in studying ergodicity of doubly stochastic chains (see for example [15, 18, 16, 17]).

We now consider the behavior of the Lyapunov function under rotational transformation of the chain $\{A(k)\}$, as given in Definition 5. It emerged that the Lyapunov function $V$ is invariant under the rotational transformation, as shown in Lemma 6. We emphasize that the invariance of the Lyapunov function $V$ holds for arbitrary stochastic chain; the doubly stochasticity of the chain is not needed at all.

**Lemma 6.** Let $\{A(k)\}$ be a stochastic chain and $\{P(k)\}$ be an arbitrary permutation chain. Let $\{B(k)\}$ be the rotational transformation of $\{A(k)\}$ by $\{P(k)\}$. Let $\{x(k)\}$
and \( \{y(k)\} \) be the dynamics obtained by \( \{A(k)\} \) and \( \{B(k)\} \), respectively, with the same initial point \( y(0) = x(0) \) where \( x(0) \in \mathbb{R}^m \) is arbitrary. Then, for the function \( V(\cdot) \) defined in Eq. (17) we have

\[
V(x(k)) = V(y(k)) \quad \text{for all } k \geq 0.
\]

**Proof.** Since \( \{y(k)\} \) is the dynamic obtained by \( \{B(k)\} \), there holds for any \( k \geq 0, \)

\[
y(k) = B(k - 1)y(k - 1) = \ldots = B(k - 1) \cdots B(1)B(0)y(0) = B(k : 0)y(0).
\]

By Lemma 2, we have \( B(k : 0) = P^T(k : 0)A(k : 0)P(0 : 0) \) with \( P(0 : 0) = I \), implying

\[
y(k) = P^T(k : 0)A(k : 0)y(0) = P^T(k : 0)A(k : 0)x(0) = P^T(k : 0)x(k), \tag{18}
\]

where the second equality follows from \( y(0) = x(0) \) and the last equality follows from the fact that \( \{x(k)\} \) is the dynamic obtained by \( \{A(k)\} \). Now, notice that the function \( V(\cdot) \) of Eq. (17) is invariant under any permutation, that is \( V(Px) = V(x) \) for any permutation matrix \( P \). In view of Eq. (18), the vector \( y(k) \) is just a permutation of \( x(k) \). Hence, \( V(y(k)) = V(x(k)) \) for all \( k \geq 0 \). \( \text{Q.E.D.} \)

Consider an ergodic doubly stochastic chain \( \{A(k)\} \) with a trivial permutation component \( \{I\} \). Let \( t_0 = 0 \) and for any \( \delta \in (0, 1) \) recursively define \( t_q \), as follows:

\[
t_{q+1} = \arg \min_{t \geq t_{q+1}} \min_{S \subseteq [m]} \sum_{t = t_q}^{t-1} A_S(k) \geq \delta, \tag{19}
\]

where the second minimum in the above expression is taken over all nonempty subsets \( S \subset [m] \). Basically, \( t_q \) is the first time \( t > t_{q-1} \) when the accumulated flow from \( t = t_{q-1} + 1 \) to \( t = t_q \) exceeds \( \delta \) over every nonempty \( S \subset [m] \). We refer to the sequence \( \{t_q\} \) as *accumulation times* for the chain \( \{A(k)\} \). We observe that, when the chain \( \{A(k)\} \) has infinite flow property, then \( t_q \) exists for any \( q \geq 0 \), and any \( \delta > 0 \).

Now, for the sequence of time instances \( \{t_q\} \), we have the following rate of convergence result.

**Lemma 7.** Let \( \{A(k)\} \) be an ergodic doubly stochastic chain with a trivial permutation component \( \{I\} \) and a mixing coefficient \( \gamma > 0 \). Also, let \( \{x(k)\} \) be the dynamics driven by \( \{A(k)\} \) starting at an arbitrary point \( x(0) \). Then, for any \( q \geq 1 \), we have

\[
V(x(t_q)) \leq \left( 1 - \frac{\gamma \delta (1 - \delta)^2}{m(m-1)^2} \right) V(x(t_{q-1})),
\]

where \( t_q \) is defined in (19).

The proof of Lemma 7 is based on the proof of Theorem 10 in [9] and Theorem 6 in [7], and it is provided in Appendix.

Using the invariance of the Lyapunov function under rotational transformation and the properties of rotational transformation, we can establish a result analogous to
Lemma 7 for an arbitrary ergodic chain of doubly stochastic matrices \( \{A(k)\} \). In other words, we can extend Lemma 7 to the case when the chain \( \{A(k)\} \) does not necessarily have trivial permutation component \( \{I\} \). To do so, we appropriately adjust the definition of the accumulation times \( \{t_q\} \) for this case. In particular, we let \( \delta > 0 \) be arbitrary but fixed, and let \( \{P(k)\} \) be a permutation component of an ergodic chain \( \{A(k)\} \).

Next, we let \( t_0 = 0 \) and for \( q \geq 1 \), we define \( t_q \) as follows:

\[
t_{q+1} = \arg \min_{t \geq t_q + 1} \min_{S(0) \subset [m]} \sum_{t = t_q}^{t-1} A_{S(k+1)S(k)}(k) \geq \delta, \tag{20}
\]

where \( \{S(k)\} \) is the trajectory of the set \( S(0) \) under \( \{P(k)\} \).

We have the following convergence result.

**Theorem 6.** Let \( \{A(k)\} \) be an ergodic doubly stochastic chain with a permutation component \( \{P(k)\} \) and a mixing coefficient \( \gamma > 0 \). Also, \( \{x(k)\} \) be the dynamics driven by \( \{A(k)\} \) starting at an arbitrary point \( x(0) \). Then, for any \( q \geq 1 \), we have

\[
V(x(t_q)) \leq \left( 1 - \frac{\gamma \delta (1 - \delta)^2}{m(m - 1)^2} \right) V(x(t_q - 1)),
\]

where \( t_q \) is defined in (20).

**Proof.** Let \( \{B(k)\} \) be the rotational transformation of the chain \( \{A(k)\} \) with respect to \( \{P(k)\} \). Also, let \( \{y(k)\} \) be the dynamics driven by chain \( \{B(k)\} \) with the initial point \( y(0) = x(0) \). By Lemma 3, \( \{B(k)\} \) has the trivial permutation component \( \{I\} \). Thus, by Lemma 7, we have for all \( q \geq 1 \),

\[
V(y(t_q)) \leq \left( 1 - \frac{\gamma \delta (1 - \delta)^2}{m(m - 1)^2} \right) V(y(t_q - 1)).
\]

Now, by Lemma 2, we have \( A_{S(k+1)S(k)}(k) = B_{S(k)}(k) \). Therefore, the accumulation times for the chain \( \{A(k)\} \) are the same as the accumulation times for the chain \( \{B(k)\} \). Furthermore, according Lemma 6, we have \( V(y(k)) = V(x(k)) \) for any \( k \geq 0 \) and, hence, for all \( q \geq 1 \),

\[
V(x(t_q)) \leq \left( 1 - \frac{\gamma \delta (1 - \delta)^2}{m(m - 1)^2} \right) V(x(t_q - 1)).
\]

Q.E.D.

### 5.2 Doubly Stochastic Chains without Absolute Flow Property

So far we have been concerned with doubly stochastic chains with absolute infinite flow property. In this section, we turn our attention to the case when absolute flow property is absent. In particular, we are interested in characterizing the limiting behavior of backward product of a doubly stochastic chain that does not have absolute infinite flow.

Since a doubly stochastic chain is decomposable, Theorem 3 is applicable, so by this theorem when the chain \( \{A(k)\} \) does not have absolute infinite flow property, then
$F(\{A(k)\}; \{S(k)\}) < \infty$ for some $S(0) \subset [m]$ and its trajectory under a permutation component $\{P(k)\}$ of $\{A(k)\}$. This permutation component will be important so we denote it by $\mathcal{P}$. For this permutation component, let $G^\infty_\mathcal{P} = ([m], E^\infty_\mathcal{P})$ be the undirected graph with the edge set $E^\infty_\mathcal{P}$ given by

$$E^\infty_\mathcal{P} = \left\{ \{i, j\} \mid \sum_{k=0}^{\infty} A_{i(k+1),j(k)}(k) = \infty \right\},$$

where $\{i(k)\}$ and $\{j(k)\}$ are the trajectories of the sets $S(0) = \{i\}$ and $S(0) = \{j\}$, respectively, under the permutation component $\{P(k)\}$; formally, $e_{i(k)} = P(k : 0)e_i$ and $e_{j(k)} = P(k : 0)e_j$ for all $k$ with $P(0 : 0) = I$. We refer to the graph $G^\infty_\mathcal{P} = ([m], E^\infty_\mathcal{P})$ as the infinite flow graph of the chain $\{A(k)\}$. When the permutation component $\{P(k)\}$ is trivial, we use $G^\infty$ to denote $G^\infty_\mathcal{P}$.

By Theorem 3 in \cite{8}, we have that $G^\infty$ is connected if and only if the chain has absolute infinite flow property. Theorem 6 in \cite{8} shows that for a chain with a trivial permutation component $\{I\}$, the connectivity of $G^\infty$ is closely related to the limiting matrices of the product $A(k : t_0)$, as $k \to \infty$ (in the case of the trivial permutation component $\{I\}$, we have $i(k) = i$ for any $i \in [m]$ and $k \geq 0$). The following result is just an implication of Theorem 6 in \cite{8} for the special case of doubly stochastic chains that have trivial permutation component.

**Theorem 7.** Let $\{A(k)\}$ be a doubly stochastic chain with a trivial permutation component $\{I\}$. Then, for any starting time $t_0$, the limit $A^\infty(t_0) = \lim_{k \to \infty} A(k : t_0)$ exists. Moreover, the $i$th row and the $j$th row of $A^\infty(t_0)$ are identical for any $t_0 \geq 0$, i.e., $\lim_{k \to \infty} \|A_i(k : t_0) - A_j(k : t_0)\| = 0$, if and only if $i$ and $j$ belong to the same connected component of $G^\infty$.

Using Lemma 2, Lemma 3, and Theorem 7 we prove a similar result for the case of a general doubly stochastic chain.

**Theorem 8.** Let $\{A(k)\}$ be a doubly stochastic chain with a permutation component $\{P(k)\}$. Then, for any starting time $t_0 \geq 0$, the product $A(k : t_0)$ converges up to a permutation of its rows; i.e., there exists a permutation sequence $\{Q(k)\}$ such that $\lim_{k \to \infty} Q(k)A(k : t_0)$ exists for any $t_0 \geq 0$. Moreover, for the trajectories $\{i(k)\}$ and $\{j(k)\}$ of $S(0) = \{i\}$ and $S(0) = \{j\}$, respectively, under the permutation component $\{P(k)\}$, we have

$$\lim_{k \to \infty} \|A_{i(k)}(k : t_0) - A_{j(k)}(k : t_0)\| = 0 \quad \text{for any starting time $t_0$},$$

if and only if $i$ and $j$ belong to the same connected component of $G^\infty$.

**Proof.** Let $\{B(k)\}$ be the rotational transformation of $\{A(k)\}$ by the permutation component $\{P(k)\}$. As proven in Lemma 3, the chain $\{B(k)\}$ has a trivial permutation component. Hence, by Theorem 7 the limit $B^\infty(t_0) = \lim_{k \to \infty} B(k : t_0)$ exists for any $t_0 \geq 0$. On the other hand, by Lemma 2(a), we have

$$B(k : t_0) = P^T(k : 0)A(k : t_0)P(t_0 : 0) \quad \text{for all $k > t_0$}.$$
Multiplying by $P^T(t_0 : 0)$ from the right, and using $PP^T = I$ which is valid for any permutation matrix $P$, we obtain

$$B(k : t_0)P(t_0 : 0)^T = P^T(k : 0)A(k : t_0)$$ for all $k > t_0$.

Therefore, $\lim_{k \to \infty} B(k : t_0)P(t_0 : 0)^T$ always exists for any starting time $t_0$ since $B(k : t_0)P^T(t_0 : 0)$ is obtained by a fixed permutation of the columns of $B(k : t_0)$. Therefore, if we let $Q(k) = P^T(k : 0)$, then $\lim_{k \to \infty} Q(k)A(k : t_0)$ exists for any $t_0$ which proves the first part of the theorem.

For the second part, by Theorem 7 we have $\lim_{k \to \infty} \|B_i(k : t_0) - B_j(k : t_0)\| = 0$ for any $t_0 \geq 0$ if and only if $i$ and $j$ belong to the same connected component of the infinite flow graph of $\{B(k)\}$. By the definition of the rotational transformation, we have $B(k : t_0) = P^T(k : 0)A(k : t_0)P(t_0 : 0)$. Therefore, for the $i$th and $j$th row of $B(k : t_0)$, we have according to Eq. (8):

$$\|B_i(k : t_0) - B_j(k : t_0)\| = \|A_i(k : t_0) - A_j(k : t_0)\|,$$

where $e_i(k) = P(k : 0)e_i$ and $e_j(k) = P(k : 0)e_j$ for all $k$. Therefore, $\lim_{k \to \infty} \|B_i(k : t_0) - B_j(k : t_0)\| = 0$ if and only if $\lim_{k \to \infty} \|A_i(k : t_0) - A_j(k : t_0)\| = 0$. Thus, $\lim_{k \to \infty} \|A_i(k : t_0) - A_j(k : t_0)\| = 0$ for any $t_0 \geq 0$ if and only if $i$ and $j$ belong to the same connected component of the infinite flow graph of $\{B(k)\}$. The last step is to show that the infinite flow graph of $\{B(k)\}$ and $\{A(k)\}$ are the same. This however, follows from Eq. (9), according to which we have

$$\sum_{k=0}^{\infty} B_{ij}(k) = \sum_{k=0}^{\infty} A_{i(k+1),j(k)}(k).$$

Q.E.D.

By Theorem 8, for any doubly stochastic chain $\{A(k)\}$ and any fixed $t_0$, the sequence consisting of the rows of $A(k : t_0)$ converges to an ordered set of $m$ points in the probability simplex of $\mathbb{R}^m$, as $k$ approaches to infinity. In general, this is not true for an arbitrary stochastic chain. For example, consider the stochastic chain

$$A(2k) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A(2k + 1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ for all $k \geq 0$. (21)

For this chain, we have $A(2k : 0) = A(2k)$ and $A(2k + 1 : 0) = A(2k + 1)$. Hence, depending on the parity of $k$, the set consisting of the rows of $A(k : 0)$ alters between $\{(1,0,0),(0,1,0),(0,0,1)\}$ and $\{(1,0,0),(0,0,1),(0,0,1)\}$ and, hence, never converges to a unique ordered set of 3 points in $\mathbb{R}^3$.

## 6 Conclusion

In this paper, we studied backward product of a chain of stochastic and doubly stochastic matrices. We introduced the concept of absolute infinite flow property and studied its
relation to ergodicity of the chains. In our study, a rotational transformation of a chain and the properties of the transformation played important roles in the development. We showed that this transformation preserves many properties of the original chain. Based on the properties of rotational transformation, we proved that absolute infinite flow property is necessary for ergodicity of a stochastic chain. Moreover, we showed that this property is also sufficient for ergodicity of doubly stochastic chains. Then, we developed a rate of convergence for ergodic doubly stochastic chains. Finally, we characterized limiting behavior of a doubly stochastic chain in the absence of ergodicity.

Proof of Lemma 7.

Proof. We prove the assertion for \( q = 1 \). The case of an arbitrary \( q \geq 0 \) follows similarly. Let \( x(0) \in \mathbb{R}^m \) be an arbitrary starting point and let \( \{x(k)\} \) be the dynamics driven by the doubly stochastic chain \( \{A(k)\} \). Without loss of generality, assume that \( x_1(0) \leq \ldots \leq x_m(0) \). By the definition of \( t_1 \), we have \( \sum_{k=0}^{t_1-1} A_S(k) \geq \delta \) for any \( S \subset [m] \). Thus by Lemma 12 in [7], we have

\[
\sum_{k=0}^{t_1-1} \sum_{i<j} (A_{ij}(k) + A_{ji}(k)) (x_i(k) - x_j(k))^2 \geq \frac{\delta(1-\delta)^2}{x_m(0) - x_1(0)} \sum_{\ell=1}^{m-1} (x_{\ell+1}(0) - x_\ell(0))^3.
\] (22)

On the other hand, by Lemma 11 in [7], we have

\[
V(x(0)) = \sum_{i=1}^{m} (x_i(0) - \bar{x}(0))^2 \leq m(x_m(0) - x_1(0))^2 \leq m(m-1) \sum_{\ell=1}^{m-1} (x_{\ell+1}(0) - x_\ell(0))^2 \leq m(m-1)^2 \frac{1}{x_m(0) - x_1(0)} \sum_{\ell=1}^{m-1} (x_{\ell+1}(0) - x_\ell(0))^3.
\] (23)

Therefore, using Eq. (22) and Eq. (23), we obtain

\[
\sum_{k=0}^{t_1-1} \sum_{i<j} (A_{ij}(k) + A_{ji}(k)) (x_i(k) - x_j(k))^2 \geq \frac{\delta(1-\delta)^2}{m(m-1)^2} V(x(0)).
\] (24)

Now, by Lemma 4 in [9], we have that

\[
V(x(k)) = V(x(k-1)) - \sum_{i<j} H_{ij}(k-1)(x_i(k-1) - x_j(k-1))^2,
\]

where \( H(k) = A^T(k)A(k) \). Thus, it follows

\[
V(x(t_1)) = V(x(0)) - \sum_{k=0}^{t_1-1} \sum_{i<j} H_{ij}(k)(x_i(k) - x_j(k))^2 \\
\geq V(x(0)) - \gamma \sum_{k=0}^{t_1-1} \sum_{i<j} (A_{ij}(k) + A_{ji}(k)) (x_i(k) - x_j(k))^2.
\] (25)
In the last inequality we use the following relation:

\[ H_{ij}(k) = \sum_{\ell=1}^{m} A_{\ell i}(k) A_{\ell j}(k) \geq A_{ii}(k) A_{ij}(k) + A_{jj}(k) A_{ji}(k) \geq \gamma (A_{ij}(k) + A_{ji}(k)), \]

where the last inequality holds since \( \{A(k)\} \) has the trivial permutation component. Therefore, using Eq. (24) and Eq. (25), we conclude that

\[ V(x(t_1)) \leq V(x(0)) - \frac{\gamma \delta (1 - \delta)^2}{m(m-1)^2} V(x(0)) = \left(1 - \frac{\gamma \delta (1 - \delta)^2}{m(m-1)^2}\right) V(x(0)). \]

Q.E.D.

References


