ASYMPTOTIC BEHAVIOR OF THE AVERAGE OF THE ADJACENCIES OF THE TOPOLOGICAL ENTITIES IN SOME SIMPLEX PARTITIONS

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ABSTRACT

In this communication a general kind of simplex partitions are studied. These partitions are associated to some recurrence relations. The study of these equations lead us to set some properties in 2D and 3D about the average adjacencies of the topological entities defining the mesh, when the number of refinements tends to infinite. We prove that this limit is independent of the considered partition in 2D. This is not the case for three dimensions and partitions of tetrahedra. Some simplex partitions in 2D and 3D are studied in detail.

Keywords: partitions, adjacencies, simplicial meshes

1. INTRODUCTION

In the area of numerical methods a considerable effort has been done for designing and implementing a suitable distribution of points or elements into the definition domain in which the problem is going to be solved. Very often we are interested in measuring the goodness of the partition (or triangulation). In this sense several techniques of smoothing or improvement have been developed. So, a better triangulation (in the simplicial case) can be achieved (see, for example references [1][2]). Some regularity measures for simplices (triangles in 2D, tetrahedra in 3D) and for the whole triangulation have been proposed in literature [3][4].

1.1 About the number of neighbors in a Packing

What is the maximum number of unit balls in that can touch one unit ball (without overlapping with each other)? This question caused a dispute between Isaac Newton and David Gregory. Newton conjectured that the answered was 12 while Gregory thought 13 was possible. It took 180 years before the question was answered: Hoppe [5] proved that Newton was right [6].

For a convex body $K$, let $N(K)$ denote the maximum number of congruent copies of $K$ that can touch $K$ without overlapping with each other, call it the Newton number of $K$. The exact value of the Newton number of the $d$-dimensional ball $B^d$ is known only for a few values of $d$. Obviously, $N(B^2) = 6$ and, as mentioned above, $N(B^3) = 12$. Considering the difficulty of the question disputed by Newton and Gregory, it is surprising that $N(B^d)$ could be computed for any $d > 3$ at all. This problem has its own interest in the area of computational geometry but is not in the line of our argument here.

1.2 A measure of topological regularity

The Newton number concept has been used in the last years to define a measure for topological irregularity of simplicial meshes in 2D (triangles) and 3D (tetrahedra) [3][4]:

$$e_r = \frac{1}{n} \sum_{i=0}^{n} |\delta_i - D|$$

where $\delta_i$ represents the degree, or the number of neighboring nodes, connected to the $i$th interior node, and $n$ is the total number of interior nodes in the domain. Thus, in general, as elements become more equilateral, the mesh irregularity approaches 0, but vanishes only when all the nodes have $D$ neighbors, a rare situation. Otherwise, it has a positive value that designates how much the mesh differs from a perfectly regular triangular lattice.

In this paper the asymptotic behavior of the average of the degree of the nodes are studied, when the number of global refinement tends to infinite. And more generally all possible averages of the adjacencies of the topological entities in simplex partitions are studied. Only refinement based on certain partitions of the elements has been considered. These special partitions lead us to a recurrence equation system. A more systematic investigation in this subject is being carried out now [7].

The paper is organized as follows. In the next section some definitions and notations are summarized. The third section is for explaining some of the refinement algorithms in two and three dimension. All the algorithms are based on edge bisection. Finally the asymptotic behavior of the average of the degree of the nodes is studied in 2D and 3D. More
generally, the behavior of the adjacencies of the topological entities, in average and asymptotically is presented for a general class of partitions in 2 and 3 dimensions.

2. DEFINITIONS AND NOTATIONS

Some elementary definitions and notations are summarized here:

Definition 2.1 (m-simplex). Let \( V = \{X_1, X_2, \ldots, X_{m+1}\} \) be a set of \( m+1 \) points in \( R^m \) (\( m \leq n \)) such that \( \{X_iX_j : 2 \leq i \leq m+1\} \) is a linearly independent set of vectors. Then the closed convex hull of \( V \) denoted by \( S = \langle V \rangle = \langle X_1, X_2, \ldots, X_{m+1} \rangle \) is called an \( m \)-simplex in \( R^n \), while the points \( X_1, X_2, \ldots, X_{m+1} \) are called vertices of \( S \), and the number \( m \) is said to be the dimension of \( S \).

Definition 2.2 (conforming triangulation). Let \( \Omega \) be a bounded set in \( R^n \) with non-empty interior, \( \Omega \neq \emptyset \), and polygonal boundary \( \partial \Omega \), and consider a partition of \( \Omega \) into a set \( \tau = \{t_1, t_2, \ldots, t_p\} \) of \( n \)-simplices, such that any adjacent simplex elements share an entire face or edge or a common vertex, i.e. there are no non-conforming nodes in \( \tau \). Then we can say that \( \tau \) is a conforming simplex mesh or a conforming triangulation for \( \Omega \).

Definition 2.3 (skeleton). Let \( \tau \) be an \( n \)-simplicial mesh. The set \( skt(\tau) = \{f : f \) is an \((n-1)\)-face of some \( t)\) will be called the skeleton or the \((n-1)\)-skeleton of \( \tau \) [8]. For instance, the skeleton of a triangulation in three dimensions is comprised of the faces of the tetrahedra, and in two dimensions the skeleton is the set of the edges of the triangles. It should be noted however, that the skeleton can be understood as a new triangulation: if \( \tau \) is a 3-D conforming triangulation in \( R^3 \), \( skt(\tau) \) is a 2-D triangulation embedded in \( R^3 \). Furthermore, if \( \tau \) is conforming, \( skt(\tau) \) is also conforming.

Two (conforming) triangulations \( \tau \) and \( \tau^* \) of the same bounded set \( \Omega \) are said to be nested, and we write \( \tau \prec \tau^* \) if the following condition holds: \( \forall t \in \tau, \exists t_1, \ldots, t_p \in \tau^* \) such that \( t = t_1 \cup t_2 \cup \ldots \cup t_p \). We also say that \( \tau \) is coarser than \( \tau^* \) or that \( \tau^* \) is finer than \( \tau \).

Note that if over some initial mesh \( \tau \) successive refinements by bisection are performed, a sequence of nested triangulations are obtained. In a sequence of nested triangulations, the respective sets of nodes are also nested. This fact enables us to talk about proper nodes and inherited nodes as follows: let \( \tau = \{\tau_1, \tau_2, \ldots, \tau_k\} \) be a sequence of nested grids, where \( \tau_1 \) represents the initial mesh, \( \tau_k \) the finest grid in the sequence, and let \( \tau_j \) be any triangulation of \( T \). One node \( N \in \tau_j \) is called a proper node of \( \tau_j \) or a \( j \)-new node if \( N \) does not belong to any previous mesh. Otherwise, \( N \) is said to be an inherited node in \( \tau_j \). The edges, faces, and elements may be named similarly [9][10].

3. SOME REFINEMENT ALGORITHMS BASED ON EDGE BISECTION

In this section we introduce some of the most common partitions for triangles and tetrahedra used in literature.

3.1 Partitions in 2D

Definition 3.1 (4T Similar partition). The original triangle is divided into four triangles by connecting the midpoints of the father-triangle by straight line segments parallel to the sides. Consequently all the triangles are similar to the original one (see Figure 1(a)).

This is one of the simplest partitions of triangles considered in literature (see for example [11][12]).

Definition 3.2 (single bisection and LE bisection). Single bisection consists in dividing the triangles into two sub-triangles by the midpoint of one of the edges (see Figure 1(b)).

When the longest- edge is chosen to bisect the triangle is said that a generalized bisection, or a longest-edge (LE) bisection has been done.

Definition 3.3 (4T-LE partition). 4T partition bisects the triangle in four triangles as it is shown in Figure 1(c) where the triangle is first subdivided by its longest edge, and then the two resulting triangles are bisected by the midpoint of the common edge with the original triangle.

In the following this partition will be called 4T-LE partition.

Definition 3.4 (Baricentric partition). For any triangle \( t \) the baricentric partition of \( t \) is defined as follows:

1. Put a new-node \( P \) in the baricentric point of \( t \), and put new nodes at the midpoints of the edges.
2. Join the baricentric point \( P \) with the vertices of the edges and with the nodes at the midpoints of the edges. (See Figure 1(d)).

Other triangle partitions can be considered. For example, if the shortest edge is chosen to perform the first bisection, and then we proceed as in the 4T-LE partition we get the 4T-sortest-edge (4T-SE) partition.
Rivara [10][13][14] has proposed several refinement schemes in 2 and 3D. For example, in 2D her 4T algorithm bisects the triangle in four triangles as it is shown in Figure 1(a).

### 3.2 Partitions in 3D

In three dimensions several techniques have been developed in the last five years for refining (and coarsening) tetrahedral meshes by means of bisection of tetrahedra. Algorithms based on the simple longest edge bisection have been developed by Rivara and Levin [14] and by Muthukrishnan et al. [15]. Since each tetrahedron is divided by its longest-edge until the conformity is achieved, it is not known into how many tetrahedra each of the original tetrahedron will be subdivided. So this refinement partition is not considered in this work.

The algorithms of Bänsch [16] and Liu and Joe [2] divide each tetrahedron into eight subtetrahedra by iterative edge bisection. Kossaczky [17] has proposed a recursive approach. His algorithm imposes certain restrictions and preprocessing in the initial mesh. The 3D algorithm is equivalent to that given by Bänsch. Recently, Mukherjee [18] has presented an algorithm equivalent to that by Bänsch and that by Liu and Joe. However, all these algorithms are not applicable to any initial mesh and need some kind of pre-processing.

Recently Plaza and Carey [19][20] have presented a generalization of the 4-T Rivara algorithm to three dimensions. The algorithm has also been studied by Rivara and Plaza [21]. This algorithm works first on the triangular faces of the tetrahedra, the 2-skeleton of the 3D-triangulation, and then subdivides the interior of each tetrahedron in a consistent manner with the subdivision of the skeleton. This idea could be applied to develop similar algorithms in higher dimensions. The algorithm can be applied to any initial tetrahedral mesh without any preprocessing. This is an important feature with respect to other similar algorithms in the literature. As in the refinement case, the coarsening algorithm is based on derefining the 2-skeleton assuring the conformity of the mesh, and then reconstructing the interior of the tetrahedra. For this second task the former 3D (local) refinement patterns can be used in each tetrahedron. The 3D refinement algorithm is also based on the 8T-longest-edge partition [21]:

**Definition 3.5** For any tetrahedron t of unique longest-edge the 8-Tetrahedra LE partition (8T-LE partition) of t is defined as follows:
1. LE-bisection of t producing tetrahedra t1, t2
2. Bisection of ti by the midpoint of the unique edge of ti which is also the longest edge of a common face of ti with the original tetrahedron t, producing tetrahedra tij, for i, j = 1,2
3. Bisection of each tij by the midpoint of the unique edge equal to a edge of the original tetrahedron.

**Theorem 3.1** The 8T-LE-partition of any tetrahedron t produces both a conforming volume triangulation of t and a conforming surface triangulation of t such that:
1. The surface conforming triangulation of t is identical to the surface triangulation obtained by 4-triangles partition of the faces of t.
2. Four different patterns of volume triangulation are obtained (Figure 2) according with the relative position of the longest-edge of the original tetrahedron and the longest edges of the faces.

**Theorem 3.2** The 8T-LE-partition of any tetrahedron t produces both a conforming volume triangulation of t and a conforming surface triangulation of t such that:
1. Only an interior edge $P^*P$ is produced, where $P^*$ and P are respectively the midpoint of the longest-edge of t, and the midpoint of the edge opposite to the longest-edge of t.
2. Eight new internal faces appear inside the tetrahedron t.

**Remark** Other similar tetrahedral refinements [2][15][16] [17] also verify that the global (8T) partition of a single tetrahedron t produces:
1. The 4T division of each triangular face of t.
2. An interior edge.
3. Eight new internal faces inside the tetrahedron t.
4. ADJACENCY OF THE TOPOLOGICAL ELEMENTS DEFINED IN THE MESH

We follow in this section the general definitions given in [22]. Topology provides an unambiguous, shape independent, abstraction of the mesh. Maintaining the relation between the domain and the mesh is simplified, and many operations can be performed more naturally using the mesh’s topological adjacencies.

Each topological entity of dimension \( d \), \( M^d_i \), is bounded by a set of topological entities of dimension \( d-1 \), \( M^{d-1}_i \). A region is a 3D entity with a set of faces bounding it. A face is a 2D entity with a set of edges bounding it. An edge is a 1D entity with two vertices bounding it.

The representation of a general geometric domains requires loop and shell topological entities, and, in the case of non-manifold models, use entities for the vertices, edges, loops, and faces. However, restrictions on the topology of a mesh allow a reduced representation in terms of only the basic 0 to \( d \)-dimensional topological entities. In three dimensions \((d = 3)\) these entities are:

\[
\tau_d = \{M^d_0, M^d_1, M^d_2, M^d_3\}
\]

where \( M^d \), \( d = 0,1,2,3 \) are, respectively, the set of vertices, edges, faces and regions defining the primary topological elements of the mesh domain. Restrictions on the topology of a mesh which allow this reduction are:

1. Regions and faces have no interior holes.
2. Each entity of order \( d_i \) in a mesh, \( M^d_i \), may use a particular entity of lower order, \( M^d_j \), \( d_j < d_i \), at most once.
3. For any entity \( M^d_i \) there is a unique set of entities of order \( d_i - 1 \), \( M^{d_i-1}_i \) that are on the boundary of \( M^d_i \) if at least one member of \( M^{d_i-1}_i \) is classified on \( M^d_j \) where \( d_j > d_i \).

The first restriction, as noted in the same reference, means that regions may be directly represented by the faces that bound them, and faces may be represented by the edges that bound them. The second restriction allows the orientation of an entity to be defined in terms of its boundary entities (without the introduction of entity uses). For example, the orientation of an edge, \( M^1_i \) bounded by vertices \( M^0_j \) and \( M^0_k \) is uniquely defined as going from \( M^0_j \) to \( M^0_k \) only if \( j < k \).

The third restriction means that an interior entity as noted in [19] is uniquely specified by its bounding entities. This allows an implementation using a reduced representation for interior entities. This condition only applies to interior entities, entities on the boundary of the model may have a non-unique set of boundary entities.

Beal and Shephard [22] use the study of the different topological elements defined in the mesh to propose a data structure considering the trade-offs between the storage space required and the time to access to various adjacency information. We only take from them the relationship between the numbers of the various entities in the mesh as in the following table. Our attention is restricted to tetrahedral mesh, although the argument we are going to follow in the next section could be applied to a general polygonal partition of the space.

<table>
<thead>
<tr>
<th>Tetrahedral mesh</th>
<th>( M^3_i )</th>
<th>( M^2_i )</th>
<th>( M^1_i )</th>
<th>( M^0_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M^3_i )</td>
<td>4 ( N^3_M )</td>
<td>6 ( N^3_M )</td>
<td>4 ( N^3_M )</td>
<td></td>
</tr>
<tr>
<td>( M^2_i )</td>
<td>4 ( N^2_M )</td>
<td>3 ( N^2_M )</td>
<td>3 ( N^2_M )</td>
<td></td>
</tr>
<tr>
<td>( M^1_i )</td>
<td>6 ( N^1_M )</td>
<td>3 ( N^1_M )</td>
<td>2 ( N^1_M )</td>
<td></td>
</tr>
<tr>
<td>( M^0_i )</td>
<td>4 ( N^0_M )</td>
<td>3 ( N^0_M )</td>
<td>2 ( N^1_M )</td>
<td></td>
</tr>
</tbody>
</table>

The relationship between the numbers of the various entities in the mesh is shown in Table I. The tetrahedral mesh is assumed to be infinite with all equilateral tetrahedra (note that this is actually impossible since equilateral tetrahedra do not close pack). These values were checked against real meshes, to check the equilateral assumption gave reasonable results, giving the following ranges: \( 2.02 < N^3_M/N^1_M < 2.19 \), \( 1.2 < N^3_M/N^1_M < 1.45 \), \( 0.18 < N^0_M/N^3_M < 0.27 \), showing reasonably good agreement.

<table>
<thead>
<tr>
<th>Tetrahedral mesh</th>
<th>( M^3_i )</th>
<th>( M^2_i )</th>
<th>( M^1_i )</th>
<th>( M^0_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M^3_i )</td>
<td>4 ( M^{col} )</td>
<td>6 ( M^{col} )</td>
<td>4 ( M^{col} )</td>
<td></td>
</tr>
<tr>
<td>( M^2_i )</td>
<td>2 ( M^{col} )</td>
<td>3 ( M^{col} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( M^1_i )</td>
<td>5 ( M^{col} )</td>
<td>2 ( M^{col} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( M^0_i )</td>
<td>23 ( M^{col} )</td>
<td>35 ( M^{col} )</td>
<td>14 ( M^{col} )</td>
<td></td>
</tr>
</tbody>
</table>

Table II shows the average number of adjacencies of each entity type of each other type on a per entity basis. For example, each face is in two tetrahedra, each edge is in 5 tetrahedra in average, and a node is in 23 tetrahedra in average. Note that some of the relations noted in the table are trivial since every topological element verifies it.
5. Generation Functions and Number of Topological Entities

When the global partition of the reference element (triangle, quadrilateral, tetrahedron, hexahedron) follows some pattern, the relations between the number on topological entities in the refined mesh and the number of topological entities in the unrefined mesh are related by certain recurrence equations. For example, in 2D case, and in the refinement of Bank, and also for the Rivara 4T algorithm the recurrence relations are the following:

\[ N_n = N_{n-1} + E_{n-1} \] \hspace{1cm} (1)
\[ E_n = 2E_{n-1} + 3T_{n-1} \] \hspace{1cm} (2)
\[ T_n = 4T_{n-1} \] \hspace{1cm} (3)

where \( N_0, E_0, T_0 \) are respectively the number of nodes, edges and triangles in the initial input mesh. Note that for two different partitions there are the same recurrence associated equations. So, the recurrence equations although associated to each particular partition do not characterize the partitions.

Since our goal is the study of the asymptotic behavior of the adjacency relations between the topological entities in the mesh, when the number of global refinements tends to infinity, we have to calculate or estimate the numbers \( N_n, E_n, T_n \). That is, we need to solve the recurrence equations (1), (2), (3). This task can be accomplished by means of generation functions (see for example [23][24]). Let \( N(x), E(x), \) and \( T(x) \) be the generation functions for \( N_n, E_n, T_n \) respectively. This means that the numbers \( N_n, E_n, T_n \) are precisely the coefficients of the variable \( x \) in the Taylor expansion of the corresponding generation function:

\[ N(x) = \sum_{n \geq 0} N_n x^n, E(x) = \sum_{n \geq 0} E_n x^n, T(x) = \sum_{n \geq 0} T_n x^n. \]

6. Asymptotic Results in 2D

We summarized here the asymptotic results regarding the average numbers of the adjacencies for triangular grids.

Lemma 6.1 Let \( \tau \) be a triangular mesh in which some global refinement is applied. Let \( N_n, E_n, T_n \) be respectively the total number of nodes, edges and triangles after the \( n \)-th refinement application. Then the average number of triangles per node \( \mu_0^2 \{ M_i^0 \} \) is:

\[ \mu_0^2 = \mu(M_i^0) = \frac{2E_n}{N_n} \] \hspace{1cm} (4)

Besides, the average number of edges per node is:

\[ \mu_1^0 = \mu(M_i^1) = \frac{2E_n}{N_n} \]

and these two numbers are the same.

Proof: Let us see first the average of triangles per node. Since we are calculating an average per node, the denominator has to be \( N_n \). About the numerator, note that it should be \( \sum_{k=1}^{N_n} M_k^2 \{ M_i^0 \} \), that is we have to add the number of triangles per each node, but this sum is equal to the sum of the number of nodes per triangle, so \( \sum_{k=1}^{N_n} M_k^2 \{ M_i^0 \} = 3T_n \).

Note that for the average number of edges per node the reasoning is the same. This follows from the fact that the average number of triangles per edge is:

\[ \mu_1^1 = \mu(M_i^1) = \frac{2}{E_n} \]

so \( 3T_n = 2E_n \), and the proof is complete. \( \square \)

Theorem 6.1 Let \( \tau \) be a triangular mesh in which some global refinement is applied. Let \( N_n, E_n, T_n \) be respectively the total number of nodes, edges and triangles after the \( n \)-th refinement application. Then the asymptotic number of topological entities are independent of the particular partition of each triangle and these numbers are those of the following table.

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Table III. Asymptotic average number of adjacencies per entity \( M_i^r \{ M_i^d \}, \mu_i^r \)

We summarized here the asymptotic results regarding the average numbers of topological entities in the mesh. When the number of global refinements tends to infinity, we have to calculate or estimate the numbers \( N_n, E_n, T_n \). That is, we need to solve the recurrence relations (1), (2), (3). This task can be accomplished by means of generation functions (see for example [23][24]). Let \( N(x), E(x), \) and \( T(x) \) be the generation functions for \( N_n, E_n, T_n \) respectively. This means that the numbers \( N_n, E_n, T_n \) are precisely the coefficients of the variable \( x \) in the Taylor expansion of the corresponding generation function:

\[ N(x) = \sum_{n \geq 0} N_n x^n, E(x) = \sum_{n \geq 0} E_n x^n, T(x) = \sum_{n \geq 0} T_n x^n. \]

Proof: Let us see first the average of triangles per node. Since we are calculating an average per node, the denominator has to be \( N_n \). About the numerator, note that it should be \( \sum_{k=1}^{N_n} M_k^2 \{ M_i^0 \} \), that is we have to add the number of triangles per each node, but this sum is equal to the sum of the number of nodes per triangle, so \( \sum_{k=1}^{N_n} M_k^2 \{ M_i^0 \} = 3T_n \).

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<tbody>
<tr>
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</tbody>
</table>

Table III. Asymptotic average number of adjacencies per entity \( M_i^r \{ M_i^d \}, \mu_i^r \).
\[
a = \text{number of triangles per triangle} \\
b = \text{number of edges per edge} \\
c = \text{number of internal edges per triangle} \\
d = \text{number of nodes per edge} \\
e = \text{number of internal nodes per triangle}
\]

However there are some relations between these parameters. For example, \(b = d - 1\), \(e = 1 - a + c\). Last equation comes from the Euler relation for the vertices, edges and triangles applied to the first partition of a single triangle. Finally,

\[
c = \frac{3(a - b)}{2} \quad (10)
\]

The last relation comes from the counting of the edges of the son-triangles after a partition has been applied to a single triangle. The total number of edges is on one hand equal to \(3a\), and on the other hand this number is also equal to \(3b + 2c\).

Taking into account the relations between the parameters of the recurrence equations, and solving these equations the values for all the asymptotic values of the adjacencies can be found. Table III shows the asymptotic average number of adjacencies of each entity type of each other type on a per entity basis in 2D. It is worthy to be noted that these values are independent of the particular partition of the triangles considered.

\section*{6.1 An example}

Some of the adjacency relation numbers founded in below can also be founded by geometric arguments. For example, about the number of triangles per vertex in the refinement pattern associated to the Bank algorithm, note that all internal new node will be of degree 6. Since the number of internal new nodes is of order \(O(n^2)\), where \(n\) is the number of external nodes.

\section*{7. ASYMPTOTIC RESULTS IN 3D}

In 3D the situation about the asymptotic behavior of the adjacency relations between the topological elements in the mesh is quite different from the 2D case. Now we obtain different number for the average limit in some of the adjacencies depending on the particular partition considered. Following the procedure outlined below, we present in this section some study cases showing this fact.

\subsection*{7.1 3D-8T partitions}

We summarized here the asymptotic results regarding the average numbers of the adjacencies for tetrahedral grids. The results showed in Table IV are valid for the 8T-LE partition or other partition equivalent (Theorem 3.2 and Remark). They are based on the solution of the associated system of recurrence equations:

\[
N_n = N_{n-1} + E_{n-1} \quad (11)
\]

\[
E_n = 2E_{n-1} + 3F_{n-1} + T_{n-1} \quad (12)
\]

\[
F_n = 4F_{n-1} + 8T_{n-1} \quad (13)
\]

\[
T_n = 8T_{n-1} \quad (14)
\]

Table IV. Asymptotic average number of adjacencies per entity \(M^\text{row}_i\{M^\text{col}_i\}, \mu^\text{col}_i\)

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</tr>
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<tbody>
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<td>4</td>
<td>6</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>(M^2_i)</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>(M^1_i)</td>
<td>36/7</td>
<td>36/7</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(M^0_i)</td>
<td>24</td>
<td>36</td>
<td>14</td>
<td></td>
</tr>
</tbody>
</table>

Table IV shows the average number of adjacencies of each entity type of each other type on a per entity basis in 3D.

\subsection*{7.2 3D-4T partition}

This second example corresponds to the partition of the tetrahedra in which only one interior node is introduced at the baricenter point of each tetrahedron. The partition is then realized by joining that node with the former four nodes of the initial tetrahedron. In this way each tetrahedron is divided in four sub-tetrahedra. The recurrence equations associated with this partition are the following:

\[
N_n = N_{n-1} + T_{n-1} \quad (15)
\]

\[
E_n = E_{n-1} + 4T_{n-1} \quad (16)
\]

\[
F_n = F_{n-1} + 6T_{n-1} \quad (17)
\]

\[
T_n = 4T_{n-1} \quad (18)
\]

Table IV shows the average number of adjacencies of each entity type of each other type on a per entity basis in 3D.
Table V. Asymptotic average number of adjacencies per entity $M^\text{row}_i$, $M^\text{col}_i$, $F^\text{row}_i$.

<table>
<thead>
<tr>
<th>Tetrahedral mesh</th>
<th>$M^3_i$</th>
<th>$M^2_i$</th>
<th>$M^1_i$</th>
<th>$M^0_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^3_i$</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$M^2_i$</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$M^1_i$</td>
<td>9/2</td>
<td>9/2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$M^0_i$</td>
<td>12</td>
<td>18</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

7.3 3D-Baricentric partition

This last example corresponds to the baricentric partition of the tetrahedra in which an interior node is located at the baricenter point of each tetrahedron, and also at the baricentric points of the faces and edges (midpoints of the edges). The baricentric partition is illustrated in Fig.4.

Figure 4. Baricentric partition in 3D

The associated equations to this partition are:

\[ N_n = N_{n-1} + E_{n-1} + F_{n-1} + T_{n-1} \]  \hspace{1cm} (19)
\[ E_n = 2E_{n-1} + 6F_{n-1} + 14T_{n-1} \]  \hspace{1cm} (20)
\[ F_n = 6F_{n-1} + 36T_{n-1} \]  \hspace{1cm} (21)
\[ T_n = 24T_{n-1} \]  \hspace{1cm} (22)

Solving the equations we can obtain the asymptotic average number for each adjacency.

8. CONCLUDING REMARKS

The asymptotic averages proved in this communication are in complete agreement with the experimental data, in 2D and 3D. The study of the asymptotic behavior of the partitions based on recurrence equation system could be the clue in the proof of the non-degeneracy of this kind of refinement algorithms in 3D and higher dimensions. Further statistical analysis could also help in this direction. The method presented can be applied to other polyhedral or polygonal partitions of the plane, not only simplicial partitions.

REFERENCES


