Real qualitative behavior of a fourth-order family of iterative methods by using the convergence plane

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Abstract

The real dynamics of a family of fourth-order iterative methods is studied when it is applied on quadratic polynomials. A Scaling Theorem is obtained and the conjugacy classes are analyzed. The convergence plane is used to obtain the same kind of information as from the parameter space, and even more, in complex dynamics.

Keywords: Real dynamics, nonlinear problems, Convergence Plane, iterative methods, basins of attraction, stability

1. Introduction

The application of iterative methods for solving nonlinear problems \( f(x) = 0 \), with \( f : \mathbb{R} \rightarrow \mathbb{R} \), gives rise to rational functions whose dynamics are not well-known. The simplest model is obtained when \( f(x) \) is a quadratic polynomial and the iterative algorithm is Newton’s scheme. This case has been widely studied under the view of complex dynamics (see, for instance \cite{8, 13}). The study of the dynamics of Newton’s method has been extended to other point-to-point iterative schemes (see for example \cite{9, 12, 15, 21}) and to multipoint iterative methods (see for example \cite{1, 2, 10, 11, 20, 22}), for solving nonlinear equations. Nevertheless, the real dynamical analysis is not profusely studied, although some references can be found in the literature (see, for example \cite{3, 4, 6, 14, 16, 17}).

From the numerical point of view, the dynamical properties of the rational function associated with an iterative method give us important information about its stability and reliability. In most of mentioned papers, interesting dynamical planes, including periodical behavior and other anomalies, have been obtained. We are interested in the analysis of the role of the parameter in the stability of the family of iterative methods, which would allows us to select a particular one with good numerical properties.

In this work, the family under study is the class of three-step fourth-order iterative methods for solving nonlinear systems \( F(x) = 0 \), introduced by the authors in \cite{5}, denoted by M4 and whose iterative expression is

\begin{align}
  y_k &= x_k - F'(x_k)^{-1}F(x_k), \\
  z_k &= y_k - \frac{1}{\beta} F'(x_k)^{-1}F(y_k), \\
  x_{k+1} &= z_k - F'(x_k)^{-1} \left( (2 - 1/\beta - \beta) F(y_k) + \beta F(z_k) \right),
\end{align}

where \( \beta \) is an arbitrary complex parameter, \( \beta \neq 0 \). As the authors proved in \cite{5}, the corresponding scheme for \( \beta = 1/5 \) is the unique element of the family with order of convergence five. In this paper, we will analyze the real dynamical behavior of this family of methods applied to the nonlinear equation \( f(x) = 0 \). We will denote by \( G_f(x, \beta) \) the fixed point operator associated to the class of methods (1) on a nonlinear real function \( f(x) \), whose expression is

\[ G_f(x, \beta) = x - \frac{1}{f'(x)} \left[ f(x) + f(y)(2 - \beta) + \beta f \left( y - \frac{1}{\beta} \frac{f(y)}{f'(x)} \right) \right] \]

where \( y = x - \frac{f(x)}{f'(x)} \).

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This research was supported by Ministerio de Ciencia y Tecnología MTM2011-28636-C02-{01, 02}

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The authors in [19] have studied the complex dynamics of the rational function associated to (1), analyzing the conjugacy classes. In it, some complex regions with rich dynamical behavior have been shown providing interesting elements of the family of iterative methods. In this paper, our aim is the study of the real dynamics of this family. Despite what might seem, some numerical results obtained in this paper allow us to conjecture that the real behavior is not included in the complex one. In fact, we will find stable real regions where attracting periodic orbits appear in the complex study, and vice versa.

The dynamical behavior associated to two rational operators (corresponding to a particular iterative method) conjugated by an affine map is qualitatively the same if a Scaling Theorem is satisfied.

**Theorem 1.** Let \( f(x) \) be an analytic function and let \( A(x) = ax + b \) with \( a \neq 0 \) be an affine map. Let \( g(x) = (f \circ A)(x) \). Then, \((A \circ G_g \circ A^{-1})(x, \beta) = G_f(x, \beta)\), that is, \( G_f \) and \( G_g \) are affine conjugated by \( A \).

**Proof:** Let us consider firstly

\[
G_f(A(x), \beta) = T_f(A(x)) - \frac{(2 - \frac{1}{\beta} - \beta) f(N_f(A(x))) + \beta f(T_f(A(x)))}{f'(A(x))},
\]

where \( T_f \) and \( N_f \) are the fixed-point operators of the first and second steps of (1), respectively.

As \( g(x) = (f \circ A)(x) \), it is clear that \( g'(x) = a f'(A(x)) \), \( N_g(x) = x - \frac{A(x) - N_f(A(x))}{a} \) and also \( g(N_g(x)) = f(N_f(A(x))) \). Then,

\[
T_g(x) = x - \frac{1}{a f'(A(x))} \left[ f(A(x)) + \frac{1}{\beta} f(N_f(A(x))) \right]
\]

and \( g(T_g(x)) = f(T_f(A(x))) \).

Finally,

\[
A(G_g(x, \beta)) = a G_g(x, \beta) + b
\]

\[
= a \left[ T_g(x) - \frac{(2 - \frac{1}{\beta} - \beta) g(N_g(x)) + \beta g(T_g(x))}{g'(x)} \right] + b
\]

\[
= G_f(A(x), \beta)
\]

by simply substituting the previous calculations. \( \Box \)

The Scaling Theorem allows us to reduce the study of the dynamics of the iteration function \( G_f \) to the study of specific families of iterations of simpler maps. For example, any real quadratic polynomial \( p(x) = c_2 x^2 + c_1 x + c_0 \) with \( c_2 \neq 0 \) (we may assume that \( c_2 = 1 \)) can be reduced to \( x^2 + c \), by using an affine map. Then, we are going to analyze the following polynomials that correspond to the three different cases: \( p_+(x) = x^2 - 1 \) a particular case of two different real roots, \( p_+(x) = x^2 + 1 \) (a case with two complex roots) and \( p_0(x) = x^2 \), corresponding to multiple roots.

In this paper, we are going to analyze the real dynamics of the set of methods (1) when they are applied to quadratic polynomials, studying the stability of all the real fixed points. The graphic tool used to analyze the stability regions is called the *convergence plane* and it was introduced by Magreñán in [18].

Now, we are going to recall some dynamical concepts that we use in this work (see [7]). Given a rational function \( R : \mathbb{R} \to \mathbb{R} \), the orbit of a point \( z_0 \in \mathbb{R} \) is defined as:

\[
\{ z_0, R(z_0), R^2(z_0), ..., R^n(z_0), ... \}.
\]

We analyze the phase plane of the map \( R \) by classifying the starting points from the asymptotic behavior of their orbits. A \( z_0 \in \mathbb{R} \) is called a fixed point if \( R(z_0) = z_0 \). A periodic point \( z_0 \) of period \( p > 1 \) is a point such that \( R^p(z_0) = z_0 \) and \( R^k(z_0) \neq z_0 \), for \( k < p \). A pre-periodic point is a point \( z_0 \) that is not periodic but there exists a \( k > 0 \) such that \( R^k(z_0) \) is periodic. A critical point \( z_0 \) is a point where the derivative of the rational function vanishes, \( R'(z_0) = 0 \). Moreover, a fixed point \( z_0 \) is called an attractor if \( |R'(z_0)| < 1 \), a superattractor if \( |R'(z_0)| = 0 \), a repulsor if \( |R'(z_0)| > 1 \) and parabolic if \( |R'(z_0)| = 1 \).

The *basis of attraction* of an attractor \( \alpha \) is defined as:

\[
A(\alpha) = \{ z_0 \in \mathbb{R} : R^n(z_0) \to \alpha, \ n \to \infty \}.
\]

Moreover, the *immediate basin of attraction* is the connected component of the basis of attraction that contains the attractor \( \alpha \).
The Fatou set of the rational function \( R, \mathcal{F}(R) \), is the set of points \( z \in \mathbb{R} \) whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in \( \mathbb{R} \) is the Julia set, \( \mathcal{J}(R) \). That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

When an iterative method is applied to solve a generic polynomial equation \( p(z) = 0 \), its roots are fixed points of the associated rational function \( R \). Nevertheless, other fixed points of \( R \) different from the roots of \( p(z) \) can appear. These fixed points are called strange fixed points. Moreover, it is relevant the knowledge of the free critical points (critical points different from the associated to the roots): each invariant Fatou component is associated with, at least, one critical point (see, for example [7]).

The rest of the paper is organized as follows: in Section 2 we analyze the fixed and critical points of the operator \( G_f(x,\beta) \) and the stability of fixed points on quadratic polynomials. The dynamical behavior of the family (1) is analyzed in Section 3, by using the associated convergence plane. We finish the work with some remarks and conclusions.

2. Study of the fixed and critical points

As we have stated in the previous section, we will analyze the behavior of the rational function obtained when the family of iterative methods (1) is applied on the polynomials \( p_0(x), p_+(x) \) and \( p_-(x) \).

In the case of \( p_0(x) \), \( G_{p_0}(x,\beta) = \frac{1+40\beta}{245} x \) is a linear map, whose slope \( a(\beta) \) has a hyperbolic dependence of \( \beta \neq 0 \). Thus, its dynamical behavior as a function of \( \beta \) its trivial, and it is well-known that, when the origin is stable, it is globally stable; when it is unstable (\( \frac{1}{88} < \beta < \frac{1}{108} \)) everything not fixed is divergent. For \( \beta = -\frac{1}{88} \), when \( a(\beta) = 1 \), all the points are fixed, while for \( \beta = \frac{1}{108} \), that is, \( a(\beta) = -1 \) all the points different from the origin are two-periodic.

The polynomial \( p_+(x) \) does not have real roots, therefore it can be proved that the rational operator

\[
G_{p_+}(x,\beta) = -\frac{1 + (3 + 8\beta)x^2 + (3 + 32\beta)x^4 + (1 + 88\beta)x^6 - 64\beta x^7}{64\beta x^6}
\]

has three different strange fixed points, depending on \( \beta \), but all of them are complex numbers for all values of \( \beta \).

Finally, we are going to analyze the dynamics of the operator \( G_f(x,\beta) \) on the polynomial \( p_-(x) \). In this case, the rational function associated to the family is

\[
G_{p_-}(x,\beta) = -\frac{1 + 4x^2 + 8\beta x^2 - 6x^4 - 40\beta x^4 + 4x^6 + 120\beta x^6 - x^8 + 40\beta x^8}{128\beta x^7}.
\]

In addition of the roots of \( p_-(x) \), \( x = 1 \) and \( x = -1 \), the fixed points of the operator are the roots of the polynomial

\[-1 + 3x^2 + 8\beta x^2 - 3x^4 - 32\beta x^4 + x^6 + 88\beta x^6 \].

Nevertheless, four of them are always complex so, the only real strange fixed points are:

\[
e x_1(\beta) = -\left[ \frac{1}{1 + 88\beta} + \frac{32\beta}{3(1 + 88\beta)} - \frac{52\sqrt{2}\beta}{(1 + 88\beta)B_2(\beta)} - \frac{272\sqrt{2}\beta^2}{3(1 + 88\beta)B_2(\beta)} + \frac{2\sqrt{3}B_2(\beta)}{(3(1 + 88\beta))} \right]^{\frac{1}{2}},
\]

\[
e x_2(\beta) = -e x_1(\beta),
\]

where

\[
B_1(\beta) = \sqrt{2\beta^2 + 4720\beta^3 + 203484\beta^4 - 242880\beta^5 + 216832\beta^6}
\]

and

\[
B_2(\beta) = \left( -27\beta + 2034\beta^2 - 2144\beta^3 + 3\sqrt{3}B_1(\beta) \right)^{1/3}.
\]

It is easy to see that \( e x_1(\beta) \) and \( e x_2(\beta) \) only take real values for \( \beta > 0 \) or \( \beta \in (-1/88, 0) \). For \( \beta = -1/88 \), the expression of the rational function is

\[
G_{p_-}(x,\beta) = \frac{11 - 43x^2 + 61x^4 - 29x^6 + 16x^8}{16x^7}
\]

and then, the unique real fixed points are \( x = 1 \) and \( x = -1 \).

In order to analyze the stability of the strange fixed points in the values of the parameter where they are real, we calculate the stability function \( S_i(x,\beta) = |G_{p_-'(x_i,\beta)}|, i = 1,2 \). We conclude that \( S_1(x,\beta) = S_2(x,\beta) > 1 \) for all values of \( \beta \). So, the real strange fixed points are always repulsive.

The previous statements can be summarized in the following result.
Lemma 1. The number of real simple strange fixed points of $G_{p_{-}}(x, \beta)$ is

- zero if $\beta \leq -\frac{1}{88}$,
- two, $ex_{1}(\beta)$ and $ex_{2}(\beta)$, when $\beta > 0$ or $\beta \in (-1/88, 0)$.

Moreover, for all values of parameter $\beta$, the strange fixed points are repulsive.

Now, in order to determine the critical points, we calculate the first derivative of $G_{p_{-}}(x, \beta)$,

$$G'_{p_{-}}(x, \beta) = \frac{(-1 + x^{2})^{3}(-7 + (-1 + 40\beta)x^{2})}{128\beta x^{8}}.$$ 

From $G'_{p_{-}}(x, \beta)$, the free critical points that is, critical points different from the roots, are analyzed in the following result.

**Lemma 2.**

a) If $\beta \leq \frac{1}{40}$ or $\beta = \frac{1}{5}$, then there is no real free critical points.

b) If $\beta > \frac{1}{40}$, the real free critical points are

(i) $cr_{1}(\beta) = -\frac{\sqrt{7}}{\sqrt{41+40\beta}}$

(ii) $cr_{2}(\beta) = -cr_{1}(\beta)$.

In Figure 1 the behavior and relations of the real strange fixed and free critical points are showed, where $ex_{1}(\beta)$ and $ex_{2}(\beta)$ are presented in red and blue, respectively, while the free critical points have been colored in orange and green. It is known that there is at least one critical point associated with each invariant Fatou component. When an attracting (but not superattracting) strange fixed points appears, it is important to know if there exist any critical point in its vicinity, because this fact provides that the fixed point has its own basin of attraction.

As it can be inferred from Lemmas 1 and 2, in the rational function associated to the class of iterative methods (1) there are no attractive strange fixed points that are and, consequently, the dynamical planes for particular values of the parameter would have only two basins of convergence. However, as we will see in the following sections, this is not always the case.

In contrast with the real analysis, when a complex dynamical analysis is made (see [19]), there exist some regions where two of the strange fixed points can be attractive, even superattractive. The fixed points are, in all cases, complex numbers.

3. Convergence Plane for quadratic polynomials

In Section 1, we have seen that the Scaling Theorem reduces the study of quadratic polynomials to the study of the three following polynomials: $p_{0}(x) = x^{2}, p_{+}(x) = x^{2} + 1$ and $p_{-}(x) = x^{2} - 1$, which represent respectively polynomials
with no real roots, polynomials with one double root and polynomials with two different real roots. Specifically, in this section we are going to analyze the dynamics of the operator $G_f(x, \beta)$ applied to those polynomials.

In the complex plane, the parameter space associated with a free critical point is obtained by associating each point of the parameter plane with a complex value of parameter $\beta$ and is used to give rise about the dynamics, but in the real line we can’t use it. Instead of using the parameter spaces we are going to apply the tool called The convergence plane introduced by Magreñán in [18]. This tool gives information about how are the real dynamics for every member of the family and every initial point, so we can find the best choices of the family for each initial point and viceversa. So, it is interesting to find regions of the convergence plane in which the basin of the roots are as much bigger and connected as possible, because these values of the parameter will give us the best members of the family in terms of numerical stability and convergence.

In order to deep in the study of the dynamical behavior of the operator $G_f(x, \beta)$, we are going to analyze now the convergence plane associated to it. The key to obtain this convergence plane is to study the orbits of each initial point (horizontal axis) for every member of the family (vertical axis). We paint a duplet $(x_0, \beta_0)$, after 1000 iterations and with a tolerance of $10^{-6}$, in cyan if the iteration of the member of the family with $\beta = \beta_0$ starting in $x_0$ converges to the fixed point 1, in magenta if it converges to $-1$ and in yellow if the iteration diverges to $\infty$. Moreover, we paint in red the convergence to any of the strange fixed points, in orange the convergence to 2-cycles, in light green the convergence to 3-cycles, in dark red to 4-cycles, in dark blue to 5-cycles, in dark green to 6-cycles, dark yellow to 7-cycles, and in white the convergence to 8-cycles. The areas in black correspond to zones of convergence to other cycles or chaos.

In order to check the results obtained by the convergence plane we will compare them with the ones from two classical tools: Feigenbaum diagrams and Lyapunov exponents.

**Definition 1.** Let be $\{x_0, x_1, \ldots, x_n, \ldots\}$ an orbit. We call Lyapunov exponent

$$h(x_0) = \lim_{n \to \infty} \frac{1}{n} \left( \log |f'(x_0)| + \log |f'(x_1)| + \cdots + \log |f'(x_n)| \right).$$

Note, that this value is the same for every point of the orbit.

The applicability of the Lyapunov exponents resides on the following result (cf [17] or [18]).

**Definition 2.** Let $\{x_0, x_1, x_2, \ldots\}$ be a bounded orbit. The orbit is called chaotic, if

1. it is not asymptotically periodic,
2. its Lyapunov exponent $h(x_0) > 0$.

So, if the associated Lyapunov exponent to $x_1$ is negative, the orbit of $x_1$ is attracted by a fixed point, a strange fixed point or by a cycle, but if the exponent is positive the orbit can diverge to infinity or even have a chaotical behavior.

### 3.1. Convergence Plane for quadratic polynomials without real roots

Now we are going to study the case without real roots, that is, we are going to analyze the dynamics of the operator $G_{p_e}(x, \beta)$.

In Figure 2 appear the Lyapunov exponents and Feigenbaum diagram associated to $G_{p_e}(x, \beta)$. From Figure 2 we can conjecture that this polynomial has a very complicated dynamics but it is not chaotical for all values of $\beta$.

In the left hand of Figure 2 the Feigenbaum diagram associated to the iteration of $G_{p_e}(x, \beta)$ is shown. In order to clarify how to understand Figure 2 one must take into account that:

- The graphic gives the information of only one initial point $x_0$.
- In the vertical axis appears the values of the parameter of the family.
- In the horizontal axis the successive iterations of $x_0$, for every $\beta$, are represented. In this way, one can see if the iteration of the initial value converges to a point, a cycle or if it is chaotic.

If there exists a critical point that serves for a range of $\beta$’s, we generally take that critical point as the initial point.

This tool present two main problems: it only gives the information of one point at each graphic; sometimes the conflictive zones are undetectable if the scale of the graphic is not adequate. One way of avoiding the second problem is using the Lyapunov exponents which appear in the right hand side of Figure 2. In the vertical axis appears the values of the parameter $\beta$ and in the horizontal axis appears the Lyapunov exponent associated to the iteration of the function for that value of $\beta$ and $x_0 = 2$. The main problem that presents this tool is that it only gives the information of one initial guess at each graphic and the behaviour of two near points could be very different. All this problems are solved with the
convergence plane due to the fact that it gives us at least the same information but for every initial point and every value of the parameter and that's the main motivation of using it as a tool.

In Figure 3 we see the convergence plane associated to $G_{p_+}(x, \beta)$. It is clear that there is no convergence to any root, but we have found that there exists non-chaotical regions which corresponds to some attracting cycles. In the right hand of Figure 3 we observe the negative region in which there exist attracting cycles. This information coincide with the one we have seen in Figure 2.

3.2. Convergence Plane for quadratic polynomials with two different real roots

In this section we are going to analyze the dynamics of operator $G_{p_-}(x, \beta)$, which constitutes the richest dynamical case.

First of all, it is easy to see that $G_{p_-}(x, \beta)$ is an odd function and thus the basins are symmetric to the origin. Due to that symmetry we only have to study one of the fixed points related to the roots of the polynomial $p_-(x)$. For the sake of simplicity we will study the positive one $x = 1$.

In Figure 4 we see the convergence plane associated to operator $G_{p_-}(x, \beta)$. The first conclusion at the first glance of the convergence plane is that there exist 3 main zones with different behavior.

- The region with $\beta > 0.0309506 \ldots$
- The region with $-\frac{1}{88} < \beta < 0.0309506 \ldots$
- The region with $\beta < -\frac{1}{88}$. 
The first one is the region with values of $\beta > \beta_1$ where $\beta_1 = 0.0309506 \ldots$ is the value of $\beta$ for which the free critical point $cr_2$ is mapped into the strange fixed point $ex_2$, that is $G_{p_+}(cr_2, \beta_1) = ex_2$. In this zone $G_{p_+}(x, \beta)$ has four fixed points at $x = -1, x = 1$ which are the roots of the polynomial $p_+(x)$ and two strange fixed points $ex_1$ and $ex_2$.

As it is shown in Figure 4 the immediate basin of attraction of $x = 1$ is the interval $[ex_2, +\infty)$. Notice that in Figure 4 it seems that only four intervals of different behavior exists but, if we make a zoom, see Figure 5, we see that there exists many intervals. In concrete there exist infinitely many intervals accumulating from below of the strange fixed point $ex_2$ and from above of the strange fixed point $ex_1$. These intervals correspond to the preimages of the immediate basin of attraction. In Table 1, some of the preimages of $ex_2$ for some values of $\beta$ are shown.

Moreover, for higher values of $\beta$ the qualitative shape of the function changes as it can be seen in Figure 6 but the immediate basin of attraction of $x = 1$ is always $[ex_2, +\infty)$ and the sequence of preimages infinitely many as stated above.

On the other hand, we focus our attention on the region $\beta < \beta_1$. As it has been stated above if $\beta = \beta_1$ a qualitative bifurcation on the basins’ structure occurs due to the free critical point $cr_2$ is mapped into the strange fixed point $ex_2$. For lower values of $\beta$ the immediate basin of $x = 1$ is given by the interval $(ex_2, G_{p_-}^{-1}(ex_2, \beta))$, where $G_{p_-}^{-1}(ex_2, \beta)$ is the
Figure 5: Zooms of the convergence planes associated to the iteration of the operator $G_{p_-}(x, \beta)$.

Figure 6: Changes of the qualitative shape of $G_{p_-}(x, \beta)$ for different values of $\beta$.

Table 1: Some preimages of $ex_2$ for different values of $\beta > \beta_1$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$G_{p_-}^{-1}(ex_2, \beta)$</th>
<th>$G_{p_-}^{-2}(ex_2, \beta)$</th>
<th>$G_{p_-}^{-3}(ex_2, \beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$-0.533431 \ldots$</td>
<td>$0.527124 \ldots$</td>
<td>$-0.527332 \ldots$</td>
</tr>
<tr>
<td>0.5</td>
<td>$-0.415361 \ldots$</td>
<td>$0.413723 \ldots$</td>
<td>$-0.413766 \ldots$</td>
</tr>
<tr>
<td>1</td>
<td>$-0.341057 \ldots$</td>
<td>$0.341223 \ldots$</td>
<td>$-0.341221 \ldots$</td>
</tr>
<tr>
<td>1.5</td>
<td>$-0.289472 \ldots$</td>
<td>$0.289856 \ldots$</td>
<td>$-0.289854 \ldots$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.253137 \ldots$</td>
<td>$0.253411 \ldots$</td>
<td>$-0.253410 \ldots$</td>
</tr>
</tbody>
</table>

rank-1 preimage of the strange fixed point $ex_2$ on its right side. It follows that all the preimages of the immediate basin of $x = 1$ in $(0, ex_2)$ also have a 1-rank preimage on the right side of $x = 1$ and these on their turn also have preimages on the side $x < 0$. This fact explains what occurs in Figures 7 and 8.

Moreover, there also exists convergence to different cycles as it is shown in Figure 8. As a consequence, every point of the plane which is neither cyan nor magenta is not a good choice of $\beta$ in terms of numerical behavior. Moreover, in Table 2 we show some of the values of $\beta$ for which attractive cycles of different orders appear. Let us note that the period of the orbits is even in all the cases.
Figure 7: Convergence plane associated to iteration of the operator $G_p(x, \beta)$ for $\beta \in (-0.02, 0.0325)$.

Figure 8: Convergence plane associated to iteration of the operator $G_p(x, \beta)$ in the region where there exist attracting cycles $\beta \in (1/40, 0.028)$.

Table 2: Values of $\beta$ and the attractive cycle associated to that value.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.025650</td>
<td>${-2755.95, -21.8255, -0.271408, 2037.03, 16.1325, 0.260879}$</td>
</tr>
<tr>
<td>0.025700</td>
<td>${0.271359, -2035.85, -17.3296, -0.271359, 2035.85, 17.3296}$</td>
</tr>
<tr>
<td>0.027690</td>
<td>${-0.49924, 10.0582, 0.508803, -8.18013, -0.497276, 10.4904, 0.513692, -7.34945}$</td>
</tr>
<tr>
<td>0.027691</td>
<td>${-0.499238, 10.0582, 0.508902, -8.16226, -0.497341, 10.4753, 0.513614, -7.36176}$</td>
</tr>
<tr>
<td>0.027710</td>
<td>${0.511141, -7.76619, -0.499043, 10.0924}$</td>
</tr>
<tr>
<td>0.027720</td>
<td>${-0.499656, 9.95647, 0.510672, -7.84291}$</td>
</tr>
</tbody>
</table>

Furthermore, another qualitative bifurcation can be seen in Figure 8 in $\beta = \frac{1}{40}$, which is due to the change of the shape on the function as follows: for $0 < \beta < \frac{1}{40}$, the function becomes negative for every $x$ greater than the preimage of the origin on the right side and for $\beta > \frac{1}{40}$ the function is positive for $x > ex_2$. This fact clearly leads to a different behavior on the preimages of the immediate basins. This change of behavior can be seen in Figure 8.

Moreover, for low values of $\beta$ pairs of $n$-cycles may appear by saddle node bifurcations, having an immediate interval in the region $(0, ex_2)$, and this leads to the possible coexistence of several stable cycles, which are difficult to detect and describe. While the appearance of divergent trajectories (i.e. the yellow regions) can be perfectly explained.

Now, we focus our attention in the zone where parameter $\beta < 0$. For values of $\beta \in (-1/88, 0)$ the strange fixed point $ex_2$ is greater than $x = 1$ and the basin of this fixed point is given by $(G_{p_-1}(ex_2, \beta), ex_2)$ while the initial points in
$(0, G_p^{-1}(e_{x2}, \beta))$ and $(G_p^{-1}(e_{x2}, \beta), \infty)$ have divergent trajectories. Similarly for $x < 0$. An example of this behavior can be seen in Figure 9.

Finally, for values of $\beta < -\frac{1}{88}$ there is no strange fixed points and as a consequence there is no convergence problems, that is, the iteration of every point converges to one root. Moreover, the immediate basin of each fixed points is the semiaxis that contains the fixed point. This behavior can be seen in Figure 10.

A direct conclusion from the study of $G_{p_{-}}(x, \beta)$ is that for values of $\beta < -\frac{1}{88}$ the dynamical behavior seems to be better than for $\beta > \beta_1$ in terms of stability, since Julia set is more complicated in the last case. For the members of the family such that $\beta < -\frac{1}{88}$ we see that there only exist 2 basins of attraction which correspond with the semiplane that contains the root and so, dynamics is trivial. Moreover, we have seen that the main behavior problems reside in the region $\beta \in (-\frac{1}{88}, \beta_1)$ where we find attracting cycles, divergent trajectories, etc.

To sum up this study, we have the following result.

**Proposition 1.** The dynamical behavior of operator $G_{p_{-}}(x, \beta)$ is divided in three different zones:
The one for $\beta \leq -\frac{1}{88}$. For those values of the parameter the dynamical behaviour of operator $G_{p-}(x, \beta)$ is trivial. The basin of attraction of each root is the semiplane that contains that root.

The region with $-\frac{1}{88} < \beta < 0.0309506 \ldots$ For those values of the parameter operator $G_{p-}(x, \beta)$ has the main dynamical problems. We have seen the existence of divergent trajectories, attracting cycles, etc.

For values of $\beta > 0.0309506 \ldots$ operator $G_{p-}(x, \beta)$ has no convergence problems but the Julia set is more intricate than the first region due to the fact that there exist infinitely many intervals accumulating from below of the strange fixed point $e^{x_2}$ and from above of the strange fixed point $e^{x_1}$.

4. Conclusions

In this paper, we have analyzed the real dynamical behavior of the family $M_4$ (introduced in [5]) in terms of convergence and stability, as the complex dynamical behavior was studied in [19]. We have concluded that real is not included in complex dynamics since we have found that in the real line for negative values of $\beta$ there are no cycles although the study of the complex dynamics stated that there exist different cycles. This conclusion would not be possible without the tool developed in [18] which allows to study real dynamics of a uniparametric family for all values of the parameter and all initial points. Moreover, we have showed that values of the parameter close to the origin are not appropriate choices due to for that values we have seen that the behavior is complicated as it it shown in [19]. On the other hand, we have seen that members of the family, with $\beta \notin (-1/88, 0.03)$, the dynamical behavior works properly for both cases but, in terms of stability we can state that the best members of the family are those for which $\beta \leq -1/88$.

5. Acknowledge

The authors strongly want to thank the anonymous referees for the constructive criticism of the paper.

6. References


