Non-linear asymptotic stability for the through-passing flows of inviscid incompressible fluid

A. Morgulis

Department of Mathematics, Mechanics and Computer Science, The South Federal University, Rostov-na-Donu and South Institute of Mathematics, Vladikavkaz Scientific Center RAS, Vladikavkaz, Russia
E-mail: amor@math.rsu.ru

Abstract. The paper addresses the dynamics of inviscid incompressible fluid confined within bounded domain with the inflow and outflow of fluid through certain parts of the boundary. This system is non-conservative essentially since the fluxes of energy and vorticity through the flow boundary are not equal to zero. Therefore, the dynamics of such flows should demonstrate the generic non-conservative phenomena such as the asymptotic stability of the equilibria, the onset of instability or the excitation of the self-oscillations, etc. These phenomena are studied extensively for the flows of the viscous fluids but not for the inviscid ones. In this paper, we prove a sufficient condition for the non-linear asymptotic stability of the inviscid steady flows.

Keywords: asymptotic stability, inviscid fluid, Euler equations

1. Introduction

In this paper, we consider the flows of inviscid incompressible fluid through a given domain thereby assuming that the fluid is allowed to come in and out the flow domain through the prescribed parts of its boundary. Such sort of boundaries and the related fluid flows are referred to as open ones. The relevant examples are the flows through the finite ducts or pipes with the inflow of fluid at one end and outflow on the other end. In the case of open boundaries, the inviscid fluid represent a non-conservative system. The violation of the conservation laws takes place since the inflow and outflow of fluid supplies to and withdraws from the flow both energy and vorticity. The withdrawal can be considered as some sort of dissipation. Therefore, the dynamics of open flows should demonstrate the generic non-conservative phenomena such as the asymptotic stability of the equilibria, the onset of the instability or excitation of the self-oscillations, etc. These features are quite new for the dynamics of inviscid fluid though they are investigated widely for the Navier–Stokes and allied equations.

Generally, the result of the withdrawal-supplying competition depends on boundary conditions at the flow inlet and outlet and the withdrawal dominates sometimes. For example, Morgulis and Yudovich [11] have pointed out wide classes of 2D open flows admitting the decreasing Liapunov functionals. Their construction employs the Arnold stability approach. Later on, Gallaire and Chomaz [7] examined the effect of the boundary conditions on the inviscid swirling flows through the finite pipes. Among other
results they pointed out the case in which the kinetic energy decreases for every solution of the linearized
equations. Such sort of the Liapunov functionals make any instability impossible (with respect to the
related metrics, of course). However, an additional investigation is necessary to decide whether or not
every perturbation decays completely i.e. whether or not the main steady flow possesses the asymptotic
stability? A peculiarity about this problem is that the ‘dissipation’ in the open flows is totally different
from the viscous one. In such circumstances, the simplest mechanism of the asymptotic stability is
the washing of perturbations out of the given flow domain. One can guess that the effect will be most
straightforward provided that the main flow is capable to replace all the current material particles by
the new ones entering the flow domain through the inlet. This ability is referred to as the through-
passing property. With this additional assumption, Morgulis and Yudovich [11] proved that the decrease
of the Arnold functional generically yields the asymptotic stability. However, these results have been
established within the linear approximation only. The present communication partly extends them to the
finite perturbations.

We consider 2D flows through the finite curvilinear channels. We assume that a non-zero normal ve-
locity is specified everywhere on the boundary of the flow domain. This boundary condition prescribes
the inlet and outlet for the flow. In contrast with the case of wholly impermeable boundaries, we have to
add one more boundary condition to set up the problem correctly. The additional boundary condition is
usually imposed at the flow inlet. In this paper, we prescribe the vorticity at the flow inlet (in addition
to the normal velocity). Our result states the non-linear asymptotic stability for the steady flows satisf-
ying the Arnold stability condition (see [5]) and additional smallness restrictions. The peculiar point is
that the Arnold condition can be checked using the boundary data prescribed at the inlet. The Arnold
condition itself implies the global Liapunov stability (relative to the L2-metric for the vorticity perturba-
tions). The smallness is required to get the decay of perturbations. Both conditions are satisfied provided
the vorticity prescribed at the inlet represents a monotonic function which is small enough relative to the
prescribed normal velocity. The result states the local asymptotic stability only. The numerical experi-
ments of Govorukhin et al. [10] showed that the intensive perturbations do not decay but tend to form
rather complex steady configurations.

The paper is organized as follows. In Section 2, we discuss the setting of boundary conditions on
the open boundaries and formulate an initial-boundary value problem governing the open flows. Also,
we formulate our main result and discuss the relevant examples. In Section 3, we review the special
properties of the through-passing steady flows. In Section 4, we mark out the class of boundary condi-
tions admitting the decreasing Liapunov functions. In Section 5, we show that the finite but sufficiently
small perturbations of the through-passing steady flows give rise to unsteady flows possessing the same
property. In Section 6, we complete the proof of main statements concerning the asymptotic stability.

2. General settings and main result

The motion of inviscid incompressible and homogeneous fluid is governed by the Euler equations. They have the form

\[ \mathbf{v}_t + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla P; \quad \text{div } \mathbf{v} = 0. \]  

(2.1)

Here \( \mathbf{v} = \mathbf{v}(z, t) \) and \( P = P(z, t) \) denote the fluid velocity and pressure. In (2.1), the former equation is the equation of motion while the latter one expresses the incompressibility constraint. We restrict
ourselves within the case of two-dimensional flows in a bounded domain $D \subset \mathbb{R}^2$ so that $z \in D$ and $\mathbf{v}(z,t) \in \mathbb{R}^2$. Of course, certain boundary conditions should be imposed at $S = \partial D$. The most common one is the no-flux condition $\mathbf{n}|_{S} = 0$ where $\mathbf{n}$ is the normal field on $S$. It is well known that such setting leads to correct initial-boundary value problem for Eqs (2.1). Here we consider more general case of open boundary consisting of the flow inlet (denoted as $S^+$) the flow outlet (denoted as $S^-$) and of the ‘rigid walls’ which represent the impermeable part of the boundary (denoted as $S^0$). On the open boundaries, it is quite natural to prescribe the sources of fluid setting

$$
\mathbf{v}\mathbf{n}|_{S} = \gamma \neq 0,
$$

where $\gamma$ is given function matching the incompressibility constraint and $\mathbf{n}$ is outward normal field. Then

$$
S^+ = \{ z \in S: \gamma(z) < 0 \},
$$

$$
S^0 = \{ z \in S: \gamma(z) = 0 \},
$$

$$
S^- = \{ z \in S: \gamma(z) > 0 \}.
$$

However, boundary condition (2.2) is insufficient to set up the problem correctly. To understand the necessity of the extra boundary conditions let us consider the vorticity transport in the open flow. Let $\omega = \text{curl} \mathbf{v} = v_x - u_y$ (where $\mathbf{v} = (u,v)$ relative to some Cartesian coordinates $(x,y)$). The Euler equations yields the transport equation for the flow vorticity

$$
\omega_t + \mathbf{v}\nabla \omega = 0.
$$

Consequently, the vorticity is constant in every material particle i.e.

$$
\omega(z,t) = \omega(a,\tau)
$$

provided some integral curve of $\mathbf{v}$ passes through both $(z,t)$ and $(a,\tau)$. If the boundary is wholly impermeable then the flow consists of the same material particles at any time. Therefore, the integral curve passing through $(z,t)$ can be continued up to the point $(a,0)$ where $a = a(z,t)$. Then

$$
\omega(z,t) = \omega(a,0), \quad a = a(z,t) \in D,
$$

for every $z \in D$ and $t \in \mathbb{R}$. Thus the current distribution of vorticity is nothing more than some rearrangement of the initial one at any time. For the open boundaries, this is not the case because some material particles are constantly entering the flow domain while the other ones are leaving it. Therefore, equality (2.4) cannot be applied unless the particle was situated in $D$ initially. Otherwise the particle entered the flow domain through the inlet at some time moment $\tau \in (0,t)$. For such particles,

$$
\omega(z,t) = \omega(a,\tau), \quad a = a(z,t) \in S^+, \quad \tau = \tau(z,t) \in (0,t).
$$

This observation suggests to impose the extra boundary condition as follows

$$
\omega|_{S^+} = \omega^+,
$$
where \( \omega^+ \) is given function and \( S^+ \) is the flow inlet. Such setting of the problem for the two-dimensional Euler equations is known due to [13].

In the sequel, the problem consisting of Eqs (2.1) and boundary conditions (2.2) and (2.6) is referred to as the **Yudovich problem**. In fact, Yudovich was the first who set up general initial-boundary value problem for the inviscid open flows correctly. Moreover, he proved the global existence and uniqueness theorem for the classical solutions. Later on, Chemetov and Starovoitov [6] proved the global existence for the weak solutions. Antontsev and Chemetov [4] established similar global result for more general Navier-slip boundary condition. Also, they showed that the Yudovich boundary condition fits for the setting of initial-boundary value problem for the superconductive vortices (see [3]). We note that the dynamics of open flows can be considered under boundary conditions different from those of Yudovich. One can find the related survey and references in [11]. However, there are no global existence theorems for the open flows except for the mentioned above (even in the case of two dimensions!).

In this paper, we examine the stability of steady solutions of the Yudovich problem. Therefore, we restrict ourselves within time-independent \( \gamma \) and \( \omega^+ \). Additional assumptions regarding the regularity of the flow domain and the boundary data are as follows:

\begin{enumerate}
  \item[(R1)] the flow domain represents a **curvilinear quadrangle**, i.e. \( D \) is bounded 1-connected piecewise smooth domain in \( \mathbb{R}^2 \) whose boundary consists of four smooth arcs and of four singular points (‘vertices’). The singular points are denoted as \( z_k \) \((k = 1, \ldots, 4)\); the interior angles formed by the adjacent arcs at \( \{z_k\}_{k=1}^4 \) belong to \((0, \pi)\);
  \item[(R2)] function \( \gamma \) is continuous on each smooth arc up to its endpoints so that there is a natural partition \( S = S^+ \cup S^- \cup S^0 \cup \{z_k\}_{k=1}^4 \); \( S^+ \) is the single smooth arc with endpoints \( z_1, z_2 \), \( S^- \) is the single smooth arc with endpoints \( z_3, z_4 \) so that \( S^+ \) and \( S^- \) have no common endpoints;
  \item[(R3)] \( \inf_{S^+ \cup S^-} |\gamma| > 0 \);
  \item[(R4)] \( \gamma \in C^\infty(S^+) \cap C^\infty(S^-) \).
\end{enumerate}

Assumptions (R1)–(R3) together outline a channel flow confined within two rigid walls, inlet section and outlet section. The flow inlet and outlet are two opposite ‘sides’ of \( D \); the other pair of the opposite ‘sides’ represents the rigid walls.

Since the flow domain is 1-connected the current distribution of vorticity, the incompressibility constraint and given normal velocity together determine the instant velocity at any time uniquely. It is convenient to resolve the incompressibility constraint setting

\[ v = \nabla^\perp \psi, \tag{2.7} \]

where \( \psi \) is a scalar function and \( \nabla^\perp = (\partial_y, -\partial_x) \) denotes the common skew-gradient. The function introduced in (2.7) is referred to as **stream function**. The use of stream function reduces the reconstruction of the velocity to the Dirichlet problem for the Poisson equation:

\[ -\Delta \psi = \omega; \quad \psi|_S = \psi_0; \quad d\psi_0 = \gamma \, dS. \tag{2.8} \]

Here \( S \) is oriented anti-clockwise so that \( \psi_0 \) increases on \( S^- \) monotonically, and decreases on \( S^+ \) (monotonically as well). On \( S_0, \psi_0 \) is constant (locally). More precisely, \( \psi_0 \) attains its maximum on
one rigid wall and attains its minimum on the other rigid wall. The total increment of \( \psi_0 \) is the total flux of fluid through the channel. For definiteness, let the minimum of \( \psi_0 \) be equal to zero. Then

\[
\psi_0(S^+) = \psi_0(S^-) = (0, Q), \quad \text{where } Q = -\int_{S^+} \gamma \, dS^+ = \int_{S^-} \gamma \, dS^-.
\]

Thus, the Yudovich problem admits a reformulation consisting of the vorticity Eq. (2.3) (where the flow velocity is expressed via the stream function), the Dirichlet problem (2.8) for the stream function, and the extra boundary condition (2.6).

The central result of this paper points out one class of steady flows possessing some sort of the asymptotic stability. In order to formulate it we introduce several auxiliary definitions.

Generally, every open steady flow separates itself into \textit{through flow zone} and \textit{recirculation zone}. The former one consists of the material particles which stay in the flow domain for finite time. The latter one consists of the material particles which stay in the flow domain eternally.

**Definition 1.** An open flow (steady or unsteady) is said to be through-passing if no material particles stay in the flow domain eternally (including those ones that move along the impermeable walls).

It is easy to see that every through-passing \textit{steady flow} \( V \in C(\overline{D}) \) obeys the condition

\[
\inf_D |V| > 0,
\]

and, conversely, every steady flow obeying (2.10) possesses the through-passing property. Clearly, (2.10) excludes any possibility for the separation so that the through flow zone coincides with the whole flow domain (including the rigid walls). Therefore, the steady through-passing flows will be also referred to as \textit{non-separated} ones. We denote as \( t_0^* = t_0^*(V) \) the maximal time during which a material particle stays inside the flow domain.

**Example 1.** Set

\[
D = \{(x, y): 0 < x < l; 0 < y < 1\}
\]

and choose a function \( U \in C^\infty[0, 1] \). Define \( V = (U(y), 0) \). This vector field (together with \( P = \text{const} \)) represents a steady solution to the Euler equations which is referred to as \textit{shear flow}. If \( U(y) > 0 \) for every \( y \in (0, 1) \) then the flow inlet coincides with that side of the rectangle where \( x = 0 \) while the outlet coincides with the opposite side \( x = l \). A shear flow is through-passing provided that the profile of its velocity is strictly positive i.e. \( \min\{U(y) > 0, y \in [0, 1]\} \). Then \( t_0^* = l/\min_{y \in [0, 1]} U(y) \).

Consider now the boundary condition for the stream function:

\[
\psi = \psi_0 \quad \text{on } S, \quad \text{where } d\psi_0 = \gamma \, dS;
\]

(see (2.8) and (2.9)). Let \( \psi^+ \) denote the restriction of \( \psi_0 \) to the inlet. Since \( \psi^+ \) is strictly monotonic the inverse function \( s^+ = s^+(h) \) is well defined. Introduce

\[
f(h) = \omega^+(s^+(h)), \quad 0 < h < Q.
\]
Note that $f$ and its derivatives have well-defined limits when $h \to 0$ or $h \to Q$ (by (R4)).

Let us pass to the formulating of the theorem about stability. We consider a steady flow $V = \nabla \perp \psi$ with vorticity $\Omega$ and its perturbations. We denote as $v = v(z, t), \omega = \omega(z, t)$ and $\psi = \psi(z, t)$ the velocity, vorticity and stream function of the perturbed flow. Also, we set

$$I(t) = \|\omega(\cdot, t) - \Omega\|_2; \quad I_0 = I(0);$$

$$A(t) = \|\omega(\cdot, t) - \Omega\|_\infty; \quad A_0 = A(0).$$

Here $\|\cdot\|_p$ denotes the common norm of $L_p(D), 1 \leq p \leq \infty$. Thus, $I(t)$ and $A(t)$ is the enstrophy and vorticity amplitude for the perturbations and their initial values are $I_0$ and $A_0$, respectively.

Let $D$ represent a curvilinear quadrangle. We denote as $B = B(D)$ the set of admissible boundary data. By definition, $B$ consists of pairs $\gamma, \omega^+$ where $\gamma, \omega^+$ are functions defined on $S = \partial D$ such that $\gamma$ satisfy conditions (R2)–(R5) and $\omega^+ \in C^\infty(\overline{S^+})$.

**Theorem 1.** Given a curvilinear quadrangle $D$ and $(\gamma, \omega^+) \in B(D);$ consider the Yudovich problem

$$\omega_t + v \nabla \omega = 0; \quad v = \nabla \perp \psi; \quad -\Delta \psi = \omega \quad \text{in } D;$$

$$vn = \gamma \quad \text{on } S; \quad \omega = \omega^+ \quad \text{on } S^+.$$

Define $f$ as prescribed in (2.12). Assume that:

(A1) the problem admits a non-separated flow $V \in C(\overline{D});$

(A2) either $\min[0, Q] |f'| > 0$ or $f \equiv \text{const}$ and

$$\max_{[0, Q]} f' < \lambda_1,$$

where $\lambda_1 = \lambda_1(D) > 0$ is the minimal eigenvalue of the Dirichlet problem for $-\Delta$ in $D.$ Then:

(i) there exists $C = C(V, D)$ such that

$$I(t) \leq CI_0 \quad \text{for every } t > 0.$$  

In addition, if

(A3) $\frac{t_0^*(V)\|\nabla \Omega\|_\infty}{\lambda_1^{1/2}} < 1$  

then

(ii) there exist $\varrho_0 = \varrho_0(A_0, V) > 0, t^* = t^*(A_0, V, D) < \infty$ and $q = q(A_0, V, D) < 1$ such that

$$I(t) \leq Cq^{2(t/t^*)}I_0 \to 0, \quad t \to +\infty,$$

provided $I_0 \leq \varrho_0.$ Moreover, if $\omega^+ \equiv \text{const}$ then $q = 0$ so that

$I(t) = 0$ and $v(\cdot, t) \equiv V$ for every $t > t_*.$
**Remark 1.** The smallness restrictions (2.17) and (2.19) are dimensionless. In particular, they both are invariant with respect to the scaling \( v \mapsto Qv, \Omega \mapsto Q\Omega, t \mapsto Q^{-1}t \).

**Remark 2.** Every non-separated flow with constant vorticity satisfies the smallness restriction (2.19).

**Remark 3.** Estimate (2.18) yields the Liapunov stability while (2.20) describes the perturbations decay. The former estimate is global while the latter one is local: the initial perturbation must be small in enstrophy but the threshold depends on the perturbation amplitude. For instance, choose \( z_0 \in D \) and bounded function \( \eta \) supported in the unit disk, and consider the initial perturbation with vorticity \( \eta(h^{-1}(z - z_0)) \). Let \( h \) go to 0. Then \( I_0 \to 0 \) while \( A_0 \) remains constant so that the smallness restriction is satisfied for every sufficiently small \( h \).

**Example 2.** Consider a shear flow \( V = (U(y), 0) \) through a rectangular domain (2.11). Set

\[
U = Q + \tilde{U}; \quad \int_0^1 \tilde{U}(y) \, dy = 0; \quad Q > \max\left(0, -\min_y \tilde{U}(y)\right); \quad (2.21)
\]

\[
\gamma(0, y) = -U(y); \quad \gamma(l, y) = U(y); \quad \gamma(x, 0) = \gamma(x, 1) = 0; \quad (2.22)
\]

\[
\omega^+(y) = -U'(y) = -\tilde{U}'(y). \quad (2.23)
\]

Then the shear flow (2.21) represents the steady solution of the Yudovich problem with data (2.22) and (2.23). Theorem 1 states the Liapunov stability for this shear flow provided \( \tilde{U}''(y) > 0 \) for every \( y \in [0, 1] \) or \( 0 < -\tilde{U}''U^{-1}(y) < \pi^2(1 + l^{-2}) \) for every \( y \in [0, 1] \) or \( \tilde{U}''(y) = 0 \) for every \( y \in [0, 1] \). In addition, Theorem 1 implies the asymptotic stability provided \( Q \) is large enough. Note that the smallness restriction (2.19) implies (2.17) in the case of shear flows.

**Example 3.** Set

\[
D = \{(r, \theta) : 0 < r_0 < r < 1; 0 < \theta < \theta_0 < 2\pi\}, \quad (2.24)
\]

where \( r \) and \( \theta \) denotes the polar coordinates. Choose boundary conditions admitting a rotational ‘shear’ flow \( V = Q(r^{-1} + U(r))e_\theta \) such that \( \Omega = -r^{-1}(rU)_r \) is strictly monotonic on \([r_0, 1]\). Again, Theorem 1 implies the asymptotic stability for \( V \) provided \( Q \) is great enough.

One more application concerns more general flows. It is convenient to restrict our considerations within the normalized data for which \( Q = 1 \) in (2.9). Such normalization always can be achieved with the use of scaling \( \gamma \mapsto Q\gamma, \omega^+ \mapsto Q\omega^+ \) that does not cause any losses in the generality (see Remark 1 to the Theorem 1). We presume such normalization everywhere below unless the contrary is indicated explicitly.

**Corollary 1.** Given with \((\gamma, \omega^+) \in \mathcal{B}(D)\), consider the Yudovich problem (2.15) and (2.16) and define \( f \) as indicated in (2.12). Assume that \( f \) is either strictly monotonic or constant on \([0, 1]\). In addition assume that \( f \) is sufficiently small in the sense that \( \max_{[0, 1]} |f| \leq \varepsilon \) and \( \max_{[0, 1]} |f'| \leq \varepsilon \) where the choice of \( \varepsilon > 0 \) depends on \( \gamma \) and \( D \) only. Then there exists a non-separated flow possessing the asymptotic stability in the sense of Theorem 1.

**Remark 4.** The existence of a non-separated flow under the conditions of Corollary 1 established in [2].
Remark 5. By (R4), the angles at the singular points of the boundary are confined within \((0, \pi)\). This is necessary for the continuity of the velocity fields in the singular points. Then condition \(\inf_{S^+ \cup S^-} |\gamma| > 0\) (see (R5)) is necessary for the matching of the boundary conditions with the through-passing flow.

Remark 6. The required smoothness of the boundary data is not mandatory. For instance, it suffices to have \(\omega^+ \in C^1(\hat{S}^+)\) and \(\tilde{\gamma} \in C^{1,\alpha}(S^+) \cup C^1,\alpha(S^+), \alpha \in (0, 1)\). However, we restrict our considerations within the smooth data in order to avoid exceeding technicalities. Generally, the smooth data produce the solutions which are smooth inside \(D\) up to its boundary except for the singular points of the boundary. In general, the velocity field belongs \(C^{\alpha_0}(\hat{D})\) where \(\alpha_0 = \alpha_0(D)\) depends on the angles at the singular points \(z_k (k = 1, \ldots, 4)\). (See e.g. [12]).

Remark 7. The conditions of Theorems 1 presume that the flow domain is simply-connected. This restriction reflects the matter of fact as the Yudovich problem generically has no steady solutions in the case of multiply-connected domain. Indeed, let the inlet include a closed curve \(\Gamma\). Then

\[
\frac{d}{dt} \int_{\Gamma} v \, dx = -\int_{\Gamma} \omega^+ \gamma \, dS
\]

(see [11]).

3. Through-passing steady flows

Consider the steady version of the Yudovich problem

\[
\nabla \Psi \times \nabla \Omega = 0; \quad -\Delta \Psi = \Omega \quad \text{in } D;
\]

\[
\mathbf{n} \times \nabla \Psi = \gamma \quad \text{on } S; \quad \Omega = \omega^+ \quad \text{on } S^+,
\]

where \(\times\) denotes the common cross-product, \(D\) is a curvilinear quadrangle and \((\gamma, \omega^+) \in \mathcal{B}(D)\). The theorem of Alekseev [1] states that problems (3.1) and (3.2) always has a steady solution. The Alexeev solutions are bounded in vorticity (so that the flow velocity and pressure are continuous) but, generally, they are neither through-passing nor smooth. The weak singularities (e.g. the vorticity jumps) are possible on the separatrix streamlines that join the critical points of the flow stream function. Such singularities are impossible for the non-separated flows. Let us consider more details.

Proposition 1. Assume the steady problems (3.1) and (3.2) has steady solution \((\Psi, \Omega) \in C^1(\hat{D}) \times L_\infty(D)\) for \((\gamma, \omega^+) \in \mathcal{B}(D)\). Assume that \(V = \nabla^\bot \Psi\) represents a non-separated flow. Then \(\Psi\) is smooth in \(D\) up to its boundary except of the vertices \(z_k\) and \((\Psi, \Omega) \in C^{1,\alpha_0}(\hat{D}) \times C^1(\hat{D})\) for some \(\alpha_0(D) > 0\). Moreover, \(\Omega = f(\Psi)\) everywhere in \(D\) with \(f\) determined in (2.12), and

\[
\{\Psi(D)\} = [0, 1];
\]

\[
\{\Omega(D)\} = \{\omega^+(S^+)\} = \{f([0, 1])\};
\]

\[
\inf_{S^+} \frac{|d\omega^+/ds|}{\sup_{S^+} |\gamma|} \leq \left|\frac{df}{d\Psi}(\Psi)\right| \leq \frac{\sup_{S^+} |d\omega^+/ds|}{\inf_{S^+} |\gamma|}.
\]
Proof. Every streamline of a steady flow lies on the level set of its stream function. Inside the recirculation zone, the non-critical level sets of the stream function represent the closed streamlines. In the through flow zone, every streamline represents the non-critical and non-closed level set of the stream function. Such streamline begins at the inlet and ends at the outlet. The origin and the end are uniquely defined for each streamline since the stream function is monotone both on the inlet and on the outlet. Therefore the stream function cannot attain its minimum and maximum inside the through flow zone. In the case of non-separated flow, the through flow zone coincides with the flow domain so that the stream function attains its minimum on one rigid wall while maximum is attained on the other rigid wall. Thus, \( \Psi(\bar{D}) = \psi_0(\bar{S}^+) = [0, 1] \) (where \( \psi_0 \) is determined in (2.8)).

The steady equation for vorticity implies a functional relation between \( \Psi \) and \( \Omega \), e.g.

\[
F(\Psi, \Omega) = 0,
\]

where \( F \) is generally uncertain. The pleasant trait of the Yudovich problem is that the boundary conditions determine \( F(\Psi, \Omega) \) in the through-flow zone though it remains uncertain in the recirculation zone (if any). Namely,

\[
\Omega = f(\Psi),
\]

where \( f \) is determined in (2.12). Here \( f \) is defined on \([0, 1]\) due to the normalization. We remind that \( f \in C^\infty[0, 1] \) by (R4).

In the case of non-separated flow, relation (3.7) is valid everywhere in the flow domain. Consequently, every non-separated flow must satisfy an elliptic problem

\[
-\Delta \Psi = f(\Psi) \quad \text{everywhere in } D, \quad \Psi = \psi_0 \quad \text{on } S
\]

(with \( f(\Psi) \) prescribed in (2.12)) that implies the assertions about the flow regularity (see [12]) as well as (3.4) and (3.5). \( \square \)

We emphasize that the boundary data determine the non-linear term \( f(\Psi) \) of Eq. (3.8) completely. Consequently, there exist \((\gamma, \omega^+) \in \mathcal{B}(D)\) such that problem (3.8) has no solutions. (For example, consider the data leading to \( f(h) = \lambda h^2 \) with sufficiently large \( \lambda > 0 \).) Such data fail to produce a non-separated flow. At the same time, the steady solution does exist (by the Alexeev theorem). Therefore, the separation takes place. Moreover, the vorticity-stream function relation inside the recirculation zone cannot be described using the analytical continuation of \( f \). We have to note here that the separation does not necessarily imply a singularity. For example, set

\[
\omega|_{S^+} = \bar{\Omega} \equiv \text{const}.
\]

Boundary conditions (2.2) and (3.9) always admit a steady flow whose vorticity is the same constant as prescribed in the inlet so that

\[
\mathbf{V} = \nabla \perp \Psi, \quad -\Delta \Psi = \bar{\Omega} \equiv \text{const} \quad \text{in } D; \quad \Psi = \psi_0 \quad \text{on } S.
\]

Problem (3.10) always has only one solution. The related flow is smooth but separated provided that \(|\bar{\Omega}|\) is large enough [9].
Let us introduce an auxiliary quantity associated with the steady flows. Consider the steady problem
\[ \mathbf{V} \nabla \delta_0 = 1 \quad \text{and} \quad \delta_0|_{S^+} = 0, \quad (3.11) \]
where \( \mathbf{V} = \nabla^\perp \Psi \) is the velocity field for some steady flow. Clearly,
\[ \delta_0(z) = \int_{\sigma(z)} \frac{d s}{|\mathbf{V}|}, \quad (3.12) \]
where \( \sigma(z) \) is the part of streamline confined within \( z \) and \( \zeta_0 \in S^+ \) where \( \zeta_0 \) is implicitly determined by equation \( \psi_0(\zeta_0) = \Psi(z) \). The right-hand side in (3.12) is well defined for every non-separated flow. By definition,
\[ t_0^* = \sup_D \delta_0. \quad (3.13) \]
For example, consider the rectilinear duct defined in (2.11) and shear flow \( \mathbf{V} = (U, 0) \) defined in (2.21)–(2.23). Then
\[ \delta_0(x, y) = x/U(y) \quad \text{and} \quad t_0^* = \frac{\max x}{\min y U(y)}. \]
Evidently, \( \delta_0 \) is smooth function up to the boundary. If we set \( U(y) = Q + \tilde{U}(y) \) then \( t_0^* = O(Q^{-1}), \quad Q \to \infty. \)

**Proposition 2.** Given a non-separated flow \( \mathbf{V} = \nabla^\perp \Psi \) in a curvilinear quadrangle \( D \). Let \( (\Psi, \Omega) \in C^1(\bar{D}) \times C^1(\bar{D}) \). Then \( \delta_0 \) and \( \nabla \delta_0 \) are well defined and bounded in \( D \) and
\[ t_0^* \leq c \frac{\max_D |\mathbf{V}| + \max_D |\Omega|}{\min_D |\mathbf{V}|^2}, \quad (3.14) \]
where \( c \) depends on \( D \) only.

**Proof.** Fix \( h \in (0, 1) \) and set
\[ D_h = \{ z \in D : 0 < \Psi(z) < h \}, \quad l_h = \{ z \in D : \Psi(z) = h \}. \]
Clearly, \( D_h \) is confined within streamlines \( l_0 = \{ z \in S^0 : \Psi(z) = 0 \} \) and \( l_h \). By the Stokes formula,
\[ \oint_{\partial D_h} \mathbf{V} \ dz = \int_{D_h} \Omega \ dz. \]
Consequently,
\[ \min_D |\mathbf{V}| \text{length}(l_h) \leq \max_S |\mathbf{V}| \text{length}(S) + \max_D |\Omega| \text{area}(D). \]
Putting this estimate together with definition (3.12) we arrive at (3.14). Further, straightforward calculation shows that
\[ h \nabla \delta_0(z) = \frac{h V(z)}{|V|^2(z)} + (h \times V(z)) \left( \int_{\sigma(z)} (\Omega + 2 \kappa|V|) \frac{ds}{|V|^3} - \frac{s^+ V(\zeta_0)}{\gamma|V|^2(\zeta_0)} \right) \]  
(3.15)
for every \( z \in D \) and \( h \in \mathbb{R}^2 \). Here \( s^+ \) is the unit tangent vector on the inlet such that \( s^+ \times n = 1 \) (\( n \) is outward normal on \( S \)); \( \kappa \) is the curvature of \( \sigma(z) \) so that
\[ \kappa = s \cdot (\nabla \times n), \]
where \( s \) is unit tangent vector on \( \sigma \) directed downstream, and \( n \) is the normal filed on \( \sigma \), such that \( s \times n = 1 \). Let us consider the behavior of \( \nabla \delta_0 \) near the singular points of the boundary. As the angles are confined within \( (0, \pi) \) we have \( \nabla \Psi \in C^{\alpha_0} \) for some \( \alpha_0 \in (0, 1) \). If all the angles in fact belong to \( (0, \pi/2) \) then \( \nabla \Psi \in C^1(\bar{D}) \) and \( \nabla \delta_0 \) is bounded in \( D \). If one of the angles is greater than \( \pi/2 \) then \( \partial^2 \Psi \) is generally unbounded nearby this singular point and
\[ \partial^2 \Psi = O(r^{\alpha-1}), \quad r \to 0, \]  
(3.16)
where \( r \) is the distance to the singularity (for the right angles, the singularity of \( \partial^2 \Psi \) is logarithmic in \( r \)). Putting (3.16) together with (3.15) we again conclude that \( \nabla \delta_0 \) is bounded.  

4. The ‘dissipative’ boundary conditions

The inflow and outflow of fluid through the open boundaries are able to supply to or withdraw from the flow both the kinetic energy and vorticity. In this section, we mark out the class of boundary conditions for which the withdrawal dominates. Under such conditions, there exist the Liapunov functionals decreasing effectively. Originally, Arnold [5] introduced these functionals as constants of motions for certain flows with wholly impermeable boundaries. Here we overview the extension of the Arnold construction to the boundary conditions (2.2) and (2.6) briefly. For further details, the readers are referred to [11].

Let a steady flow admit a relation
\[ \Psi = F(\Omega) \]  
(4.1)
for its stream function \( \Psi \) and vorticity \( \Omega \) with \( F \in C^1(\mathbb{R}) \). For real numbers \( r \) and \( s \), define
\[ \Phi_r(s) = F_0(r + s) - F_0(r) - F'_0(r)s, \quad F'_0 = F. \]  
(4.2)
Consider the perturbed flow and the vorticity perturbation \( \xi = \omega - \Omega \) (where \( \omega \) denotes the vorticity of the perturbed flow). The Arnold functional is written in the form
\[ W(\xi, t) = \frac{1}{2} \int_D \xi G \xi \, dz - \int_D \Phi_{\Omega}(\xi) \, dz, \]  
(4.3)
where \( G \) is the Green operator of the Dirichlet problem for \(-\Delta\) in \( D \).
Lemma 1. Let the Yudovich problem have a steady solution admitting relation (4.1). Then the derivative $\dot{W}$ of functional (4.3) by virtue of the system (2.1), (2.2), (2.6) has the form:

$$\dot{W}(\xi) = \int_{S^-} \gamma \Phi \Omega(\xi) \, ds,$$  \hspace{1cm} (4.4)

where $S^-$ is the outlet of the flow.

Recall that $\gamma|_{S^-} > 0$ by definition. Therefore, the sign of $\dot{W}$ coincides with that of $\Phi \Omega$. The latter one is definite provided the derivative $F''_0 = F'$ is definite in sign (see (4.2)). For example, $W$ is positive while $\dot{W}$ is negative provided $F' < 0$.

Proof of assertion (i) of Theorem 1. Let us assume that the steady flow admits relation (4.1) and

$$0 < c_1 \leq -F'(h) \leq c_2 < \infty$$

for every $h \in \mathbb{R}$. Then

$$\frac{c_1}{2} \|\xi(t)\|_{2,D}^2 \leq - \int_D \Phi \Omega(\xi) \, dz \leq \frac{c_2}{2} \|\xi(t)\|_{2,D}^2.$$

Putting this estimates together with (4.4), we arrive to (2.18) with

$$C = (c_2 + \lambda^{-1}_1(D)) / c_1,$$  \hspace{1cm} (4.6)

where $\lambda_1(D)$ denotes the minimal eigenvalue of $-\Delta$ in $D$. Let now $F' > 0$. Then we assume that

$$\lambda^{-1}_1(D) < c_1 \leq F'(h) \leq c_2 < \infty$$

for every $h \in \mathbb{R}$. This makes $W$ negative while its derivative is positive (by (4.4)) that gives us (2.18) with

$$C = c_2/(c_1 - \lambda^{-1}_1(D)).$$  \hspace{1cm} (4.7)

Putting assumption (A1) together with Proposition 1 we conclude that $\Omega = f(\Psi)$ everywhere in $\bar{D}$ and $f \in C^\infty[0, 1]$. By (A2), $f$ is monotonic so that the inverse function $f^{-1}$ is well defined on $I = f([0, 1])$ and satisfies either inequalities (4.5) or (4.7) therein. We define $F$ as the $C^1$-extension of $f^{-1}$ from $I$ to $\mathbb{R}$. Clearly, we can choose $F$ satisfying either inequalities (4.5) or (4.7).

Consider now the case $f \equiv \text{const.}$ Then $\omega^+ \equiv \bar{\Omega} = \text{const.}$ Then it is easy to see that there is the family of the decreasing Liapunov functionals

$$I_g(t) = \int_D g(\omega - \bar{\Omega}) \, dz,$$

where $g = g(r) > 0$ ($r \in \mathbb{R}$, $r \neq 0$) and $g(0) = 0$. The derivative of $I_g$ has the form

$$\dot{I}_g = - \int_{S^-} \gamma g(\omega - \bar{\Omega}) \, dS \leq 0.$$

(4.10)

Setting $g(r) = r^2$, we arrive at (2.18) with $C = 1$. \qed
We note that the derivatives of the Liapunov functionals concentrate themselves on the outlet in both cases. In particular, both $\dot{I}_g = 0$ and $\dot{W} = 0$ provided the vorticity perturbation vanishes at the outlet. Such singularity in the behavior of the Liapunov functions makes the essential difference with the systems subject to ‘viscous’ dissipation.

5. The unsteady through-passing flows

In this section, we consider the unsteady flows caused by the initial perturbations of the through-passing steady flows. We show that the perturbed flows keeps the through-passing property provided the initial perturbation is small enough and that the conditions of Theorem 1 are satisfied.

We start with some auxiliary definitions. At an instant $t > 0$, consider the material particle of the perturbed flow $v$ whose current coordinate is $z \in D$. There are two possibilities. First, the particle has entered the flow domain at an instant $\tau = \tau(z, t) > 0$ through some point $\zeta = \zeta(z, t) \in S^+$. Second, the particle was inside the flow domain initially. In the latter case, we set $\tau(z, t) = 0$. In particular, $\tau(z, 0) = 0$ everywhere in $D$, $\tau(z, t) = t$ on the inlet, and $0 \leq \tau(z, t) \leq t$. Clearly, $\tau$ is a constant along the particle path. For the same particle, let $\delta = \delta(z, t)$ be the duration of its stay in the flow domain.

$$\delta(z, t) \overset{\text{def}}{=} t - \tau(z, t) \geq 0;$$
$$\delta_t + v \nabla \delta = 1, \quad \delta|_{S^+} = 0;$$
$$\delta|_{t=0} = 0. \quad (5.3)$$

Let us assume that

$$t^* \overset{\text{def}}{=} \sup \{ \delta(z, t)t > 0, z \in D \} < \infty. \quad (5.4)$$

Then the flow possesses the through-passing property. In particular, $\tau(z, t) > 0$ for every $z \in D$ provided that $t > t^*$ so that the flow consists of the new material particles that came through the inlet. For example, consider a shear flow $V = (U, 0)$ defined in (2.21)–(2.23). Then

$$\tau(x, y, t) = \max(0, t - x/U(y)); \quad \delta(x, y, t) = t - \max(0, t - x/U(y)).$$

Also, we note that $\delta(x, y, t) = \delta_0(x, y)$ provided $\tau(x, y, t) > 0$. In particular, $\delta(\cdot, t) \equiv \delta_0$ provided $t > t^*_0$ where $t^*_0 = \sup D \delta_0$.

Again, we set

$$I(t) = \|\omega(\cdot, t) - \Omega\|_2; \quad I_0 = I(0);$$
$$A(t) = \|\omega(\cdot, t) - \Omega\|_{\infty}; \quad A_0 = A(0),$$

where $\Omega$ and $\omega$ denote the vorticities in the main and perturbed flows respectively.

**Proposition 3.** Assume the steady problems (3.1) and (3.2) has steady solution $(\Psi, \Omega) \in C^1(\overline{D}) \times C^1(\overline{D})$ for $(\gamma, \omega^+) \in \mathfrak{B}(D)$. Assume that $V = \nabla^\perp \Psi$ represents a non-separated flow. Consider the perturbed
unsteady flow subject to the same boundary conditions. For every $A_0$, there exists $\varrho = \varrho(A_0, \mathbf{V}) > 0$ such that the perturbed flow obeys (5.4) with some $t^* = t^*(I_0, A_0, \mathbf{V}) < \infty$ provided that $I_0 \leq \varrho_0$. Moreover,

$$t^* \leq t_0^* (1 - L^*)^{-1}, \quad \text{where } L^* = L^*(\mathbf{V}, A_0, I_0) \to 0,$$

(5.5)

when $I_0 \to 0$ for every fixed $A_0$.

**Proof.** By virtue of the Yudovich problem (formulated using variables $\omega$, $\psi$ and Eqs. (2.3), (2.7) and (2.8)), the equations and boundary conditions for the perturbations have the form

$$\xi_t + \mathbf{v}\nabla \xi = -\mathbf{u}\nabla \Omega;$$

(5.6)

$$\mathbf{v} = \mathbf{V} + \mathbf{u}; \quad \mathbf{u} = \nabla^\bot \varphi; \quad -\Delta \varphi = \xi;$$

(5.7)

$$\varphi|_{S} = 0;$$

(5.8)

$$\xi|_{S+} = 0.$$  

(5.9)

Here $\xi$, $\mathbf{u}$ and $\varphi$ denote the perturbations of the vorticity, velocity and stream function, so that $\xi \equiv \omega - \Omega$ and $\varphi \equiv \psi - \Psi$. In addition, define

$$\chi(z, t) = \delta(z, t) - \delta_0(z).$$

(5.10)

By virtue of (5.2) and (5.3), $\chi$ obeys the equation

$$\chi_t + \mathbf{v}\nabla \chi = -\mathbf{u}\nabla \delta_0$$

(5.11)

together with the initial and boundary conditions

$$\chi|_{t=0} = -\delta_0; \quad \chi|_{S+} = 0.$$  

(5.12)

Integration of (5.11) with the use of (5.12) gives us

$$\chi(z, t) = -\delta_0(a(z, t)) - \int_{\tau(z,t)}^{t} (\mathbf{u}\nabla \delta_0)(\bar{z}, s) \, ds,$$

where the integration is performed along the path of the material particle currently situated at the point $z$; $a(z, t)$ denotes the initial point for this path (i.e., the initial position of the material particle situated at the point $z$ currently). The path is parameterized using vector function $\bar{z} = Z(z, t, s)$ which is determined by the Cauchy problem

$$\bar{z}_s = \mathbf{v}(\bar{z}, s), \quad \tau < s < t,$$

(5.13)

$$\bar{z}|_{s=t} = z.$$  

(5.14)

Keeping in mind that $\delta_0$ is everywhere positive we conclude that

$$\delta(z, t) = \delta_0(z) + \chi(z, t) \leq \delta_0(z) + L(t)\delta(z, t),$$

(5.15)

where $L(t) = \sup_{\bar{z},s}|(\mathbf{u}\nabla \delta_0)(\bar{z}, s)|, \quad 0 \leq s \leq t.$  

(5.16)
By virtue of (5.7) and (5.8), we get two estimates
\[
\sup_{z \in \overline{D}} \vert \varphi(z, t) \vert \leq c_1 \sqrt{I(t)}; \quad (5.17)
\]
\[
\vert \nabla \varphi(\cdot, t) \vert_{\alpha; \overline{D}} \leq c_2 A(t). \quad (5.18)
\]
Here \(I(t)\) and \(A(t)\) are defined in (2.13) and (2.14), \([\cdot]_{\alpha; \overline{D}}\) denotes the conventional semi-norm of \(C^\alpha(\overline{D})\) (\(\alpha \in (0, 1)\)), and \(\alpha\), \(c_1\) and \(c_2\) depend only on \(D\). Both estimates are common results in the theory of the second order elliptic equations (see e.g. [8] and [12]). To estimate \(\sup_{\overline{D}} \vert u(z, t) \vert\) in terms of \(I(t)\) and \(A(t)\) we employ the interpolation inequalities for the Hölder norms (e.g. [8], Chapter 6). The particular case we are interested in is
\[
\sup_{\overline{D}} \vert \nabla \varphi(\cdot, t) \vert \leq C(\varepsilon) \sup_{\overline{D}} \vert \varphi \vert + \varepsilon [\nabla \varphi]_{\alpha; \overline{D}}. \quad (5.19)
\]
Putting together (5.19), (5.17) and (5.18) we conclude that
\[
\sup_{\overline{D}} \vert \nabla \varphi(\cdot, t) \vert \leq c_1 C(\varepsilon) \sqrt{I(t)} + \varepsilon c_2 A(t). \quad (5.20)
\]
Let us make the estimate (5.20) uniform in time. For \(I(t)\), we employ the estimate (2.18) (which has been proved in Section 4). To estimate \(A(t)\), we integrate the vorticity Eq. (2.3) that gives us
\[
\omega(z, t) = \omega_0(a(z, t)), \quad (5.21)
\]
where \(\omega_0 = \Omega(z) + \xi(z, 0)\) in \(D\), \(\omega_0 = \omega^+\) on \(S^+\) and \(a(z, t)\) represents the initial position for the material particle situated at the point \(z\) currently. We note two possibilities: first, \(\tau(z, t) = 0\) so that \(a \in D\); second, \(\tau(z, t) > 0\) so that \(a \in S^+\). In the second case \(\omega(z, t) = \Omega(a) = \omega^+(a)\) (by virtue of boundary condition (5.9)). Therefore,
\[
\|\omega(\cdot, t)\|_{\infty; \overline{D}} \leq \|\omega_0\|_{\infty; \overline{D}} \leq M_0 + A_0, \quad (5.22)
\]
where \(M_0 = \sup_{S^+} |\omega^+| = \sup_{\overline{D}} |\Omega|\). (Here we have employed assertion (3.4) of Proposition 1.) Consequently,
\[
\sup_z \vert \nabla \varphi(z, t) \vert \leq c_1 C(\varepsilon) \sqrt{T_0} + \varepsilon c_2 A^* \quad (5.23)
\]
for every \(t > 0\) with \(A^* = 2M_0 + A_0\). Putting this together with (5.16) we get
\[
L(t) \leq L^* = \sup_{\overline{D}} \vert \nabla \delta_0(c_1 C(\varepsilon) \sqrt{T_0} + \varepsilon c_2 A^*). \quad (5.24)
\]
Given with some \(A_0\) one can make \(L^* < 1\) choosing sufficiently small \(\varepsilon\) first and choosing sufficiently small \(T_0\) afterwards. Employing (5.24) together with (5.15) we arrive at (5.5) that completes the proof. \(\Box\)
6. Non-linear asymptotic stability

Proposition 4. Let all the conditions of Theorem 1 be satisfied. Then assertion (ii) of Theorem 1 is true.

We intend to show that the sufficiently small initial perturbations do decay when \( t \to \infty \). The evolution of the perturbation is described by the system (5.6)–(5.9). Integrating (5.6), we get

\[
\xi(z, t) = \xi_0(a(z, t)) - \int_{\tau(z,t)}^{t} (u \nabla \Omega)(\bar{z}, s) \, ds,
\]

where \( \xi_0 \) is the initial vorticity perturbation. Here the integration is performed along the path of the particle which is currently situated at the point \( z \), and \( \bar{z} = \bar{Z}(z, s, t) \) is determined in (5.13) and (5.14). Fix now \( A_0 = ||\xi(\cdot, 0)||_\infty \). By Proposition 3, there exists \( \varrho = \varrho(A_0, V) > 0 \) such that \( \sup_{z,t} \delta = t_* < \infty \) for the perturbed flow provided that \( I_0 \leq \varrho_0 \). We fix \( t > t_* \). Then \( t > \delta(z, t) = t - \tau(z, t) \), so that \( \tau(z, t) > 0 \) for every \( z \in D \). Therefore, \( a \in S^+ \) and \( \xi_0(a(z, t)) = 0 \) for every \( z \in D \) by virtue of the boundary condition (5.9). Thus, the perturbation obeys the integral equation

\[
\xi(z, t) = - \int_{\tau(z,t)}^{t} (u \nabla \Omega)(\bar{z}, s) \, ds, \quad t > t_*.
\]

Consider now \( I(t) = ||\xi(\cdot, t)||_2 \). We have

\[
I(t) = \int_D \left( \int_{\tau(z,t)}^{t} (u \nabla \Omega)(\bar{z}, s) \, ds \right)^2 \, dz \leq ||\nabla \Omega||_\infty^2 \int_D \delta(z, t) \int_{\tau(z,t)}^{t} (\nabla \varphi)^2(\bar{z}, s) \, ds \, dz.
\]

The changing in the order of integration yields

\[
I(t) \leq t_* ||\nabla \Omega||_\infty^2 \int_{t-t_*}^{t} \int_{\tau(z,t) < s} (\nabla \varphi)^2(\bar{z}, s) \, dz \, ds.
\]

For every fixed \( s \in (t - t_*, t) \), consider

\[
E(t, s) = \{ \bar{z} = Z(z, t, s), z: \tau(z, t) > s \}.
\]

Then \( Z: \{ \tau(z, t) < s \} \to E(t, s) \) represents one-to-one mapping. Since the fluid is incompressible, \( \det \partial z/\partial \bar{z} \equiv 1 \) and

\[
I(t) \leq t_* ||\nabla \Omega||_\infty^2 \int_{t-t_*}^{t} \int_{E(t,s)} (\nabla \varphi)^2(\bar{z}, s) \, d\bar{z} \, ds.
\]

Consequently,

\[
I(t) \leq t_* ||\nabla \Omega||_\infty^2 \lambda_1^{-1} \int_{t-t_*}^{t} I(s) \, ds, \quad (6.2)
\]
where $\lambda_1$ is the minimal eigenvalue of the Dirichlet problem for the operator $-\Delta$ in $D$. Let us introduce notation:

$$
\mu_k = \sup_{kt^* \leq t \leq (k+1)t^*} I(t), \quad k = 0, 1, \ldots
$$

By virtue of (6.2),

$$
\mu_k^* \leq q^2 \max(\mu_k, \mu_{k-1}), \quad k = 1, 2, \ldots,
$$

where

$$
q = t^* \|\nabla \Omega\|_\infty \lambda_1^{-1/2}.
$$

Assume now that the main steady flow obeys the condition

$$
q_0 = \frac{t^*_0 \max_D |\nabla \Omega|}{\lambda_1^{1/2}} < 1,
$$

where $t^*_0 = \sup_D \delta_0$. By Proposition 3 (see (5.5)),

$$
q \leq q_0 (1 - L^*)^{-1},
$$

so that $q < 1$ provided $q_0 < 1$ and $I_0$ is small enough. Consequently,

$$
\mu_k \leq q^2 \mu_{k-1} \leq q^{2k} \mu_0, \quad k = 1, 2, \ldots
$$

Here $\mu_0 \leq CI(0)$ by (2.18). Thus,

$$
I(t) \leq Cq^{2(t/t^*)} I_0 \to 0, \quad t \to +\infty,
$$

where $C$ depends only on the unperturbed flow and $[t/t^*]$ denotes the maximal integer number not exceeding $t/t^*$.

**Proof of the Corollary 1.** First, we prove the existence of the non-separated flow. Given with $(\gamma, \omega) \in \mathcal{B}(D)$, define $f$ as shown in (2.12). Clearly, we can choose the extending operator $\mathcal{E} : f \mapsto \tilde{f} \in C^1(\mathbb{R})$ such that

$$
\|\tilde{f}\|_{\infty;\mathbb{R}} \leq c \|f\|_{\infty;[0,1]}, \quad \|\tilde{f}'\|_{\infty;\mathbb{R}} \leq c \|f'\|_{\infty;[0,1]}
$$

with some constant $c$ independent of $f$. Consider the problem

$$
-\Delta \Psi = \tilde{f}(\Psi) \quad \text{everywhere in } D, \quad \Psi = \psi_0 \quad \text{on } S,
$$

(6.4)
where \( \psi_0 \) is defined in (2.8). Every solution of (6.4) gives us a steady flow \( \mathbf{V} = \nabla \perp \psi \) (perhaps separated); moreover, \( \mathbf{V} \mathbf{n} = \gamma \) on \( \Omega \) and \( \Omega = \omega^+ \) on \( S^+ \). Further, the Dirichlet problem

\[
\Delta \psi_i = 0 \quad \text{everywhere in } D, \quad \psi_i = \psi_0 \quad \text{on } S
\]  

(6.5)
determines the stream function \( \psi_i \) for the irrotational component of the flow. Then \( \psi_r = \psi - \psi_i \) is the stream function for the ‘purely vortical’ flow component. For \( \psi_r \), we have the fixed-point problem

\[
\psi_r = G \bar{f}(\psi_i + \psi_r),
\]  

(6.6)
where \( G \) is the Green operator for \( -\Delta \) in \( D \). In (6.6), the right-hand side determines a strictly contracting mapping from \( L_2(D) \) to itself provided \( \max_{[0,1]} |f'| \) is sufficiently small. (Here the smallness restriction depends on \( D \) only.) Hence, there exists unique fixed point \( \psi_r \) and (6.4) has unique solution \( \psi = \psi_r + \psi_i \). The results of [12] imply that both \( \psi_r \) and \( \psi_i \) belong to \( C^{1,\alpha_0}(D) \) where \( \alpha_0 = \alpha_0(D) > 0 \). In particular,

\[
\| \nabla \psi_r \|_\infty \leq c \max_{[0,1]} |f|,
\]  

(6.7)
where \( c \) depends on \( D \) only.

The constructed solution \( \psi = \psi_i + \psi_r \) depends on the extending operator \( f \mapsto \bar{f} \) which is essentially non-unique. However, if the solution possesses the through-passing property then \( \psi \) maps \( \bar{D} \) to \([0, 1]\) that makes the solution independent of the extension. Let us prove that flow \( \mathbf{V} = \nabla \perp \psi \) is non-separated provided \( \max_{[0,1]} |f'| \) is small enough. Since \( \psi_i \) is harmonic in \( D \) it attains its minimal and maximal values on \( S \). Moreover, one rigid wall consists of the minimizers while the other one consists of the maximizers. Further, both the inlet and the outlet consists of the non-critical points. At the same time, a critical point (if any) must be situated on the boundary. However the boundary point lemma (see e.g. [8]) tells us that all the inner points of the rigid walls are non-critical for \( \psi_i \). Since \( \nabla \psi_i \) is continuous in \( D \) the endpoints of the rigid walls are non-critical too (by (R1) and (R4)). Thus, \( \min_D \| \nabla \psi_i \| = c_0(\gamma, D) > 0 \) and

\[
|\nabla \psi(z)| > \left( \min_D |\nabla \psi_i| - c \max_{[0,1]} |f| \right) > 0
\]

provided \( \max_{[0,1]} |f'| \) is small enough (the smallness required depends both on \( \gamma \) and on \( D \)). Putting this together with inequality (3.14) we conclude that

\[
t_0^*(V) \leq c \frac{\max_D |\nabla \psi_i| + c \max_{[0,1]} |f|}{(\min_D |\nabla \psi_i| - c \max_{[0,1]} |f|)^2} < \infty
\]  

(6.8)
provided \( \max_{[0,1]} |f'| \) is small enough. Here \( c \) depends on \( D \) and \( \gamma \) only. Thus the problem has steady solution possessing the through-passing property. Let us check the smallness restrictions (2.17) and (2.19). The former one is always satisfied provided \( \max_{[0,1]} |f'| \) is small enough. To validate the latter one we note that \( \nabla \Omega = f'(\psi) \nabla \psi \) everywhere in \( D \) (since the flow is non-separated) so that

\[
t_0^*(V) \max_D |\nabla \Omega| \leq c \max_{[0,1]} |f'| \left( \frac{\max_D |\nabla \psi_i| + c \max_{[0,1]} |f|}{\min_D |\nabla \psi_i| - c \max_{[0,1]} |f|} \right)^2 < \lambda_1(D)
\]

provided both \( \max_{[0,1]} |f'| \) and \( \max_{[0,1]} |f| \) are small enough. \( \square \)
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