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Title: $L^p$ error estimates for projection approximations

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Abstract

We provide an error estimate for the local mean projection approximation in $L^p([0, \tau_*])$ for $p \in [1, +\infty[$, in terms of the regularity of the underlying grid, and we apply it to the corresponding projection approximation of weakly singular Fredholm integral equations of the second kind.

Keywords: Projection approximations; Lebesgue spaces; Error bounds; Integral equations

1. Local mean projections

Let $p \in [1, +\infty[$. We recall that the $L^p$-oscillation of $x \in L^p([0, \tau_*])$ relative to $\delta > 0$ is defined by

$$
\omega_p(x, \delta) := \text{esssup}_{s \in [0, \delta]} \left( \int_0^{\tau_*} |x(\tau + s) - x(\tau)|^p \, d\tau \right)^{1/p},
$$

where $x$ is extended by zero outside $[0, \tau_*]$.

We shall build a particular sequence of projections in $L^p([0, \tau_*])$. For this purpose, let $(\tau_{n,j})_{j=0}^n$ be a grid on $[0, \tau_*]$ such that

$$
0 =: \tau_{n,0} < \tau_{n,1} < \cdots < \tau_{n,n-1} < \tau_{n,n} := \tau_*,
$$

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and set
\[ h_{n,j} := \tau_{n,j} - \tau_{n,j-1} \quad \text{for } j \in [1, n], \quad h_n := \max\{h_{n,j} : j \in [1, n]\}, \]
\[ \mu_n := \min\{h_{n,j} : j \in [1, n]\}, \quad r_n := \mu_n / h_n. \]

The ratio \( r_n \in [0, 1] \) measures the regularity of the grid. For quasi-uniform grids, there exists a constant \( r \) independent of \( n \) such that, for all \( n, r \leq r_n \), and for uniform grids, \( r_n = 1 \) for all \( n \). We recall that a projection \( \Pi_n \) of finite rank \( n \) is defined by
\[ \Pi_n x := \sum_{j=1}^{n} \langle x, e_{n,j}^* \rangle e_{n,j}, \quad x \in X, \]
where \( (e_{n,j})_{j=1}^n \) is an ordered basis of the range of \( \Pi_n \), and \( (e_{n,j}^*)_{j=1}^n \) some adjoint of it. We define, for \( x \in L^p([0, \tau_n]) \) and for \( \tau \in [0, \tau_n] \),
\[ \langle x, e_{n,j}^* \rangle := \frac{1}{h_{n,j}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} x(\tau') \mathrm{d}\tau', \quad e_{n,j}(\tau) := \begin{cases} 1 & \text{if } \tau \in [\tau_{n,j-1}, \tau_{n,j}), \\ 0 & \text{otherwise}. \end{cases} \tag{2} \]

It is clear that \( \langle e_{n,j}, e_{n,i}^* \rangle = \delta_{i,j} \) for \( i, j \in [1, n] \).

**Theorem 1.** If \( \lim_{n \to \infty} h_n = 0 \), then, for all \( x \in L^p([0, \tau_n]) \), \( \lim_{n \to \infty} \| (I - \Pi_n)x \|_p = 0. \)

**Proof.** The pointwise convergence of \( \Pi_n x \) to \( x \), for each \( x \in L^p([0, \tau_n]) \), may be established by proving it for \( x \in C^0([0, \tau_n]) \) (this will be done using the uniform continuity of \( x \) on \([0, \tau_n]\) and using the Intermediate Value Theorem for integrals on each subinterval \([\tau_{n,j-1}, \tau_{n,j})\), \( j \in [1, n] \)). Next, use the density of \( C^0([0, \tau_n]) \) in \( L^p([0, \tau_n]) \) and apply the Banach–Steinhaus theorem to the sequence \((\Pi_n)\) which is bounded:
\[ \| \Pi_n \|_p = \sup\{ \| \Pi_n x \|_p : x \in L^p([0, \tau_n]), \| x \|_p = 1 \} \leq 1. \]

Indeed,
\[ \| \Pi_n x \|_p^p = \int_{0}^{\tau_n} \left[ \sum_{j=1}^{n} \frac{1}{h_{n,j}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} x(\tau') \mathrm{d}\tau' e_{n,j}(\tau) \right]^p \mathrm{d}\tau, \]
but, for \( \tau \in [\tau_{n,i-1}, \tau_{n,i}] \),
\[ \left\| \sum_{j=1}^{n} \frac{1}{h_{n,j}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} x(\tau') \mathrm{d}\tau' e_{n,j}(\tau) \right\|_{L^p} \leq \frac{1}{h_{n,i}} \left[ \int_{\tau_{n,i}}^{\tau_{n,i}} |x(\tau')|^p \mathrm{d}\tau' \right]^{1/p} h_{n,i}^{1/q}, \]
where \( 1/p + 1/q = 1 \). Hence, always for \( \tau \in [\tau_{n,i-1}, \tau_{n,i}] \),
\[ \left\| \sum_{j=1}^{n} \frac{1}{h_{n,j}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} x(\tau') \mathrm{d}\tau' e_{n,j}(\tau) \right\|_{L^p} \leq \frac{1}{h_{n,i}} \int_{\tau_{n,i}}^{\tau_{n,i}} |x(\tau')|^p \mathrm{d}\tau'. \]
Hence
\[ \int_{\tau_{n,i-1}}^{\tau_{n,i}} \left\| \sum_{j=1}^{n} \frac{1}{h_{n,j}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} x(\tau') \mathrm{d}\tau' e_{n,j}(\tau) \right\|_{L^p} \mathrm{d}\tau \leq \int_{\tau_{n,i-1}}^{\tau_{n,i}} |x(\tau')|^p \mathrm{d}\tau'. \]
Since
\[
\left| \int_0^{\tau_n} \frac{1}{h_{n,j}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} x(\tau') \, d\tau' e_{n,j}(\tau) \right|^p \leq \int_0^{\tau_n} |x(\tau')|^p \, d\tau'
\]
and the bound is proved. □

Theorem 2. For all \( x \in L^p([0, \tau_s]) \), \( \|(I - \Pi_n)x\|_p \leq \left( \frac{2}{\tau_n} \right)^{1/p} \omega_p(x, h_n) \).

Proof. For \( x \in L^p([0, \tau_s]), i \in [1, n] \), and almost all \( s \in [\tau_{n,i-1}, \tau_{n,i}] \),
\[
(I - \Pi_n)x(s) = \frac{1}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} (x(s) - x(t)) \, dt.
\]
Since
\[
\left| \int_{\tau_{n,i-1}}^{\tau_{n,i}} (x(s) - x(t)) \, dt \right| \leq \int_{\tau_{n,i-1}}^{\tau_{n,i}} |x(s) - x(t)| \, dt
\]
\[
\leq h_{n,i}^{1/q} \left[ \int_{\tau_{n,i-1}}^{\tau_{n,i}} |x(s) - x(t)|^p \, dt \right]^{1/p},
\]
where \( 1/p + 1/q = 1 \),
\[
\int_{\tau_{n,i-1}}^{\tau_{n,i}} |(I - \Pi_n)x(s)|^p \, ds = \frac{1}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} |x(s) - x(t)|^p \, dt \, ds
\]
\[
= \frac{2}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} |x(s) - x(t)|^p \, ds \, dt
\]
\[
= \frac{2}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} \int_0^{\tau_{n,i}-t} |x(\tau + t) - x(t)|^p \, d\tau \, dt
\]
\[
\leq \frac{2}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} \int_0^{h_{n,i}} |x(\tau + t) - x(t)|^p \, d\tau \, dt,
\]
and hence
\[
\|(I - \Pi_n)x\|_p \leq \frac{2}{\mu_n} \int_0^{r_n} \int_0^{r_n} |x(\tau + t) - x(t)| \, d\tau \, dt \leq \frac{2}{r_n} \omega_p(x, h_n)^p,
\]
as we wanted to prove. □

2. Projection approximation of weakly singular integral operators

Any complex Lebesgue space \( X := L^p([0, \tau_s]), p \in [1, +\infty[, \) can be used as a theoretical framework for the integral operator \( T : X \to X \) defined by
\[
(Tx)(\tau) := \int_0^{\tau_s} g(|\tau - \tau'|)x(\tau') \, d\tau', \quad \tau \in [0, \tau_s],
\]
the kernel \( g \) being such that
\[
g \in C^0([0, \tau_s]) \cap L^q([0, \tau_s]), \quad \text{where } 1/p + 1/q = 1,
\]
where \( \mu_n \) and \( r_n \) are defined as in (1).
\[ g \text{ is a decreasing function on } [0, \tau_*], \quad (5) \]
\[ g(\tau) \geq 0 \text{ for all } \tau \in [0, \tau_*]. \quad (6) \]

When \( g(0^+) = +\infty \), the operator \( T \) is said to be weakly singular. The choice \( X := L^1([0, \tau_*]) \) has been studied in [1] (as well as the choice \( X := C^0([0, \tau_*]) \)), and it is well known that, for \( 1 \leq p < +\infty \), \( T \) is a compact operator from \( L^p([0, \tau_*]) \) into itself (cf. [2]).

For \( z \neq 0 \) in the resolvent set of \( T \), we consider the Fredholm equation of the second kind
\[ T \varphi = z \varphi + f. \quad (7) \]
We recall that a bounded linear finite rank operator \( T_n : X \to X \) can be written as
\[ T_n := \sum_{j=1}^{n} \langle \cdot, \ell_{n,j} \rangle e_{n,j}, \]
where \( n \geq 1 \), \( \ell_{n,j} \in X^* \), the adjoint space of \( X \), and \( e_{n,j} \in X \) for \( j \in [1, n] \). The resolution of the approximate equation
\[ T_n \varphi_n = z \varphi_n + f, \quad (8) \]
where \( z \) belongs to the resolvent set of \( T_n \), leads to an \( n \)-dimensional linear system since Eq. (8) reads as
\[ \sum_{j=1}^{n} \langle \varphi_n, \ell_{n,j} \rangle e_{n,j} - z \varphi_n = f, \]
and applying \( \ell_{n,i} \) we get the system
\[ (A - zI)x = b, \]
where
\[ A(i, j) := \langle e_{n,j}, \ell_{n,i} \rangle, \quad b(i) := \langle f, \ell_{n,i} \rangle \quad x(j) := \langle \varphi_n, \ell_{n,j} \rangle. \]

Once this system is solved, the solution of (8) is built through
\[ \varphi_n = \frac{1}{z} \left( \sum_{j=1}^{n} x(j)e_{n,j} - f \right). \]

Assume both \( T - zI \) and \( T_n - zI \) are invertible, and set
\[ R(z) := (T - zI)^{-1}, \quad R_n(z) := (T_n - zI)^{-1}. \]
Then,
\[ \varphi_n - \varphi = (R_n(z) - R(z))f = R_n(z)(T - T_n)R(z)f = R_n(z)(T - T_n)\varphi. \]
Hence, if the sequence \( \| R_n(z) \|_p \) is bounded, the absolute error will be dominated by
\[ \| \varphi_n - \varphi \|_p \leq C \| (T - T_n)\varphi \|_p. \quad (9) \]
We are interested in approximations of the form \( T_n := \Pi_n T \), where \( \Pi_n \) is a sequence of projections with finite rank \( n \), pointwise convergent to the identity operator \( I \). The corresponding projection approximation of \( T \) reads
\[ T_n = \Pi_n T = \sum_{j=1}^{n} \langle \cdot, \ell_{n,j} \rangle e_{n,j}, \text{ where } \ell_{n,j} := T^*e_{n,j}. \quad (10) \]
Since $T$ is compact, $\lim_{n \to \infty} \|(I - \Pi_n) T\|_p = 0$, and hence there exist an integer $n_0$ and a constant $C > 0$ such that, for all integers $n$,

$$n \geq n_0 \implies R_n(z) \text{ exists, and } \|R_n(z)\|_p \leq C.$$ 

The error bound (9) becomes

$$\|\varphi_n - \varphi\|_p \leq C\|(I - \Pi_n) T\varphi\|_p,$$

and the following result follows:

**Theorem 3.** Let $\varphi$ be the solution of (7) with $T$ defined by (3). Let $\varphi_n$ be the solution of (8) with $T_n$ defined by (10) using (1) and (2). Then, there exists a constant $C > 0$ such that, for $n$ large enough,

$$\frac{\|\varphi_n - \varphi\|_p}{\|\varphi\|_p} \leq C \left( \frac{2}{r_n} \right)^{1/p} (2^{1/q} \omega_q(g, h_n) + 2\|g\chi_{[0, h_n]}\|_q),$$

where for $\delta > 0$, $\chi_{[0, \delta]}$ denotes the characteristic function of $[0, \delta]$, and $1/p + 1/q = 1$.

**Proof.** The preceding remarks and Theorem 2 give

$$\|\varphi_n - \varphi\|_p \leq C \left( \frac{2}{r_n} \right)^{1/p} \omega_p(T\varphi, h_n).$$

Let $\delta > 0$ be given. We shall estimate

$$\omega_p(T\varphi, \delta)^p := \text{esssup}_{0 \leq s \leq \delta} \left[ \int_0^{\tau_s} \left| \int_0^{\tau_s} [g(|\tau + s - t|) - g(|\tau - t|)]\varphi(t) \, dt \right|^p \, d\tau.\right.$$

Let $q$ be such that $1/p + 1/q = 1$. Since, for almost all $s \in [0, \delta],$

$$\int_0^{\tau_s} |g(|\tau + s - t|) - g(|\tau - t|)| \varphi(t) \, dt \leq \|\varphi\|_p \left[ \int_0^{\tau_s} |g(|\tau + s - t|) - g(|\tau - t|)|^q \, dt \right]^{1/q},$$

and

$$\int_0^{\tau_s} |g(|\tau + s - t|) - g(|\tau - t|)|^q \, dt \leq 2 \int_0^{\tau_s} |g(s + t) - g(t)|^q \, dt + 2^q \int_0^{\tau_s} |g(t)|^q \, dt \leq 2\omega_q(g, \delta)^q + 2^q\|g\chi_{[0, \delta]}\|_q^q.$$

Hence,

$$\omega_p(T\varphi, \delta) \leq \tau_s^{1/p}(2^{1/q} \omega_q(g, \delta) + 2\|g\chi_{[0, \delta]}\|_q)\|\varphi\|_p,$$

which proves the estimate.  

**Remark 1.** This estimate should be compared to the more pessimistic one derived by the authors in [1] for the $L^1([0, \tau_s])$ context:

$$\frac{\|\varphi - \varphi_n\|_1}{\|\varphi\|_1} \leq C \left( \frac{1}{r_n} \left( 1 + \frac{1}{r_n} \right) \int_0^{h_n} g(\tau) \, d\tau + \frac{\|(I - \pi_n)f\|_1}{\|f\|_1} \right).$$
References
