Note—Pricing and Inventory Control for a Perishable Product

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In this note, we study the concurrent determination of pricing and inventory replenishment decisions for a perishable product in an infinite horizon. Demands in consecutive periods are independent and influenced by prices charged in each period. In particular, we treat price as a decision variable to maximize the total discounted profit. We analyze the optimal solution-structure of a two-period lifetime problem and from insights gained in numerical experiments, develop a base-stock/list-price heuristic policy for products with arbitrary fixed lifetimes. Experiments show this policy to be effective.

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1. Introduction

We study concurrent pricing and inventory control in a dynamic setting for a perishable product with a fixed lifetime, i.e., the product perishes after \( m \geq 2 \) periods in stock. The planning horizon is infinite, and at the start of each period, decisions are made on pricing and replenishment assuming a zero lead time. Expired items are discarded at a per-unit cost, incurring holding and backordering costs. Inventory is consumed in a first-in-first-out (FIFO) manner, as commonly assumed in early perishable inventory control literature (Nahmias 1982). FIFO occurs primarily when (1) customers do not discriminate between old and new items (Tsiros and Heilman 2005), e.g., “best by” labels for cereals and snacks; (2) customers cannot differentiate between the old and new products, e.g., when expiration dates are unavailable; or (3) the seller manipulates search cost by positioning to deplete old inventory first (Cachon et al. 2005), e.g., by forcing old items to be accessed first such as is achieved when positioning milk cartons on sloping trays.

For inventory control of fixed-lifetime perishable products, Nahmias and Pierskalla (1973) is the earliest study that assumes a two-period lifetime with backlogging or lost sales. Early related work includes Nahmias (1975a), Fries (1975), and Nahmias (1975b). Because optimal solutions were difficult to obtain, later work, including Nahmias (1976), Cohen (1976), and Nandakumar and Morton (1993), focused on heuristic policies. See Nahmias (1982) for a review of early literature on perishable inventory control. In recent years, combined pricing and inventory control has received much attention. Petruzzi and Dada (1999), Monahan et al. (2004), and Cachon and Kök (2007) re-examined the newsvendor problem by incorporating price as a decision variable. Federgruen and Heching (1999, 2002) extended the classical dynamic inventory control problem to include price as a decision variable.

In the case when perishable products are not issued in a FIFO manner, Ferguson and Koenigsberg (2007) studied joint pricing and inventory control decisions in a two-period setting. They considered the potential competition between old (perceived as lower quality
by customers) and new (high quality) units and characterized conditions, where it is better to carry all, some, or none of leftover inventory from the first period. Deniz et al. (2004) considered the substitution effect between old and new units of perishable products, i.e., separate demand streams for items of different ages and various substitution costs. They characterized optimal conditions for adopting particular substitutions.

This work treats price as a decision variable in fixed-lifetime perishable inventory control. In §2, we study a two-period lifetime and its optimal solution-structure. From insight gained in numerical experiments for the two-period problem, we propose a base-stock/list-price heuristic for arbitrary fixed-lifetime products in §3.

2. Two-Period Lifetime Problem

Consider a product with a two-period lifetime. As with Federgruen and Heching (1999), we make the following assumptions. Demands in consecutive periods are independent and non-negative, and demand $D_t$ in period $t$ depends on the prevailing price given by a stochastic demand function $D_t(p_t, \xi_t)$, where $p_t$ is the price charged in period $t$ and $\xi_t$ is a random term with known distribution. We take $D_t$ as nonincreasing with price and bounded if price is bounded. We assume that $D_t(p_t, \xi_t)$ is linear in price $p_t$, and the expected demand $ED_t(p_t, \xi_t)$ is finite and decreasing in $p_t$. In the following notation, subscripts $t$ are suppressed because we address the problem in an infinite horizon.

$\delta =$ discount factor, $\delta \in (0, 1]$.
$c, h, b =$ unit purchasing cost, unit holding cost, and unit back logging cost, respectively.
$\theta =$ unit disposal cost incurred for disposing of expired products.
$x =$ inventory level at the start of a period, before ordering, i.e., the amount of expiring items if $x \geq 0$, or the backlogged demand if $x < 0$.
$y =$ inventory level at the start of a period, after ordering.
$p =$ price charged in a period, $p \in [\underline{p}, \bar{p}]$.

Seeking to maximize total discounted profit, we formulate the problem as a Markov decision process, i.e., the maximum expected discounted profit starting with initial inventory $x$ is given by $v^*(x) = cx + \max_{p \in [\underline{p}, \bar{p}], y \geq x} \Pi(v^*, x, y, p)$, where, with $L(y, D(p)) := hE(y - D)^+ + bE(D - y)^+$,

$$\Pi(v^*, x, y, p) = pED - \theta E(x - D)^+ - cy - L(y, D(p)) + \delta E v^*(y - x - (D - x)^+).$$

Defining $V^*(x) := v^*(x) - cx$, we rewrote the recursive equation as

$$V^*(x) = \max_{p \in [\underline{p}, \bar{p}], y \geq x} \Pi(V^*, x, y, p), \quad (1)$$

where $\Pi$ is now expressed by

$$\Pi(V^*, x, y, p) = pED + (\delta - 1)cy - \delta cx - L(y, D(p)) - [\theta E(x - D)^+ + \delta c E(D - x)^+] + \delta EV^*(y - x - (D - x)^+). \quad (2)$$

2.1. Optimal Solution-Structure

In view of (2), for a given value function $V(x)$ let

$$\Pi(V, x, y, p) = pED + (\delta - 1)cy - \delta cx - L(y, D(p)) - [\theta E(x - D)^+ + \delta c E(D - x)^+] + \delta EV^*(y - x - (D - x)^+),$$

and define the optimal operator $T$ by

$$TV(x) = \max_{p \in [\underline{p}, \bar{p}], y \geq x} \Pi(V, x, y, p).$$

Define $\widetilde{V}(x) := [V(x); V(x)$ as concave and nonincreasing in $x$, and $d(V(x)) / dx \geq -(\theta + \delta c)$], where the inequality condition specifies that one more unit of starting inventory will incur a loss of $(\theta + \delta c)$ in the worst case. The following lemma shows that $\widetilde{V}(x)$ is closed under the operator $T$, i.e., the properties that define $\widetilde{V}(x)$ are preserved under maximization.

**Lemma 1.** If $V \in \widetilde{V}$, then $TV \in \widetilde{V}$.

We prove the lemma by showing that (1) $\Pi(V, x, y, p)$ is jointly concave in $(x, y, p)$; (2) $TV(x)$ is nonincreasing in $x$; (3) the optimal order-up-to level $y^*(x)$ is nondecreasing, and the optimal price $p^*(x)$ is nonincreasing in $x$; (4) the optimal order quantity $z^*(x) = y^*(x) - x$ is nonincreasing in $x$; (5) $TV(x)$ is concave in $x$; and (6) $dTV(x) / dx \geq -(\theta + \delta c)$.

Theorem 1 derived from Lemma 1, provides insight into the structure of optimal policy. To prove Theorem 1, we need to show convergence of sequence $T^n V$. From the assumption that expected demand is finite and decreasing in price, we know that the one-step reward in the recursion is bounded above. Hence, using Theorem 6.6 and 6.8 (Feinberg 2002), we
can say that $T^*V$ converges and, assuming proper initial conditions, $\lim_{n \to \infty} T^*V = V^*$. Thus, $V^*(x)$ is concave and nonincreasing in $x$, which gives (a) below, while (b) and (c) are immediate from the proof of Lemma 1.

**Theorem 1.** (a) $\Pi(V^*, x, y, p)$ is jointly concave in $x$, $y$, and $p$, and $V^*(x)$ is concave and nonincreasing in $x$; (b) the optimal order-up-to level $y^*(x)$ is nonincreasing in $x$, and the optimal price $p^*(x)$ is nonincreasing in $x$; (c) the order quantity $z^*(x) = y^*(x) - x$ is nonincreasing in $x$.

The fact that the optimal order-up-to level is nondecreasing and the order quantity is nonincreasing compares with similar results in Nahmias (1982). Although we consider the infinite-horizon case here, these results also apply to a finite horizon. (Please see the online supplement.)

### 2.2. Numerical Experiment

Theorem 1 tells us that optimal replenishment and pricing decisions depend on initial inventory level. This is consistent with the perishable inventory control literature (Nahmias and Pierskalla 1973) but differs from the parallel problem for nonperishable products, for which a base-stock/list-price policy is optimal (Federgruen and Heching 1999). Figure 1 shows typical changes of the optimal order-up-to level and price with the initial inventory level. When initial inventory levels are low, the changes are relatively small, i.e., the influence of perishability is not significant. Further, because the order quantity $z^*(x)$ is nonincreasing with $x$, the initial inventory, i.e., inventory leftover from the previous period, is typically low. This suggests that base-stock/list-price heuristic policy may be well suited for the problem, at least for lower initial inventory levels. In the next section, we propose a heuristic with a fixed base-stock level and a fixed price for products with arbitrary fixed lifetimes. The motivation for the choice of the fixed base-stock level and fixed price is provided.

### 3. Arbitrary Lifetime Problem

#### 3.1. A Base-Stock/List-Price Heuristic Policy

Consider a product with a fixed $m$-period lifetime, $m \geq 2$. Let $x_i$ denote the amount of inventory that is $m - i$ periods old, $i = 1, \ldots, m - 1$, and $x = \sum_{i=1}^{m-1} x_i$.

With the notation for the two-period lifetime case and using $x = (x_1, x_2, \ldots, x_{m-1})$, we can write the total expected profit for the generic case with the following recursive equation:

$$v^*(x) = \max_{y \geq x, p \in [p^*, \bar{p}]} \left\{ pE[D] - c(y - x) - L(y, D(p)) - \theta E(x_1 - D)^+ + \delta E v^*(s(x, y, D)) \right\},$$

where $s(x, y, D)$ is a transfer function defined, for demand $d$ in any period, by

$$s(x, y, d) = (s_1(x, y, d), s_2(x, y, d), \ldots, s_{m-1}(x, y, d))$$

where

$$s_i(x, y, d) = \left[ x_{i+1} - \left( d - \sum_{j=1}^{i} x_j \right)^+ \right]^+, \quad 1 \leq i \leq m - 2$$

$$s_{m-1}(x, y, d) = y - x - (d - x)^+.$$

We now seek to approximate $s$ using bounds provided in Nahmias (1976), who studied a myopic base-stock policy for the fixed-lifetime perishable problem. In this study, for $x, y$ as above and taking $O_{y-x}$ to denote the amount of newly ordered inventory $(y - x)$ that will perish within $m$ periods, $E(O_{y-x})$ was bounded by $\int_0^{y-x} \mathcal{F}^m(u) \, du \leqslant E(O_{y-x}) \leqslant \int_0^{y-x} \mathcal{F}^m(u + x) \, du = H(y) - H(x)$, where $\mathcal{F}^m$ is the $m$-fold convolution of demand distribution, and $H(d) = \int_0^d \mathcal{F}^m(u) \, du$. $H(y) - H(x)$ is a good approximation of $E(O_{y-x})$ when initial inventory level is low.
For a base-stock/list-price policy, fixing price at $p$, and assuming a base-stock level $y$, we can approximate the total common inventory level at the beginning of a period can be approximated by

$$
\sum_{i=1}^{m} s_i(x, y, p) = y - D(p) + \text{[expired inventory in previous period]}
\approx y - D(p) - H(y) + H(y - D(p)).
$$

If we suppose adoption of a base-stock/list-price policy $(y, p)$, the expected total profit for this policy can be written as

$$
v_{y, p}(x) = cx - \theta E[O_1 + \delta O_2 + \cdots + \delta^{m-2} O_{m-1}] + p E[D(p)] - cy - L(y, D(p)) - \delta^{m-1} \theta E[O_{m-1}]
+ \delta c \sum_{i=1}^{m-1} s_i(x, y, D(p)) + \delta p E[D(p)] - cy
- L(y, D(p)) - \delta^{m-1} \theta E[O_{m-1}]
+ \delta c \sum_{i=1}^{m-1} s_i(x, y, D(p)), y, D(p)) + \cdots.
$$

Replacing the initial total inventory level $x$, transfer functions and expected outdates with the approximations and after rearrangement, we get

$$
v_{y, p}(x) \approx \frac{1}{1 - \delta} \left\{ (p - c) E[D(p)] - L(y, D(p)) - (c + \theta)[H(y) - EH(y - D(p))] \right\}.
$$

We note here that the terms related to the initial state, including $x, O_1, \ldots, O_{m-1}$, must also be approximated because these together with price will affect final profit, which differs when only inventory replenishment decisions are considered.

This motivates the choice of $(y^*, p^*)$ for a heuristic base-stock/list-price policy, which can be obtained by solving

$$
\max_{p \in [\hat{p}, \bar{p}]} \{(p - c) E[D(p)] - \min_{y \geq 0} [L(y, D(p)) + (c + \theta)[H(y) - EH(y - D(p))]]\}.
$$

We now establish Theorem 2 for a Polya frequency of order 2 (PF$_2$) distribution. A distribution $\Phi$ with density function $\phi$ is PF$_2$ if $\phi(x) / (\Phi(x + y) - \Phi(x))$ increases in $x$ for every fixed $y > 0$; examples of PF$_2$ distributions include exponential, uniform, Erlang, and normal distributions (Porteus 2002). Such distributions lead to the convexity of $L(y, D(p)) + (c + \theta)[H(y) - EH(y - D(p))]$.

**Theorem 2.** If the demand is a PF$_2$ random variable for any given price, then $W(y) := L(y, D(p)) + (c + \theta)[H(y) - EH(y - D(p))]$ is quasi-convex. (See online supplement for proof.)

Using Theorem 2, for any given $p$, the optimal $y$ can be found by solving $W'(y) = 0$, which after rearrangement is given by

$$(h + b)\tilde{\mathcal{F}}(y) + (c + \theta)[\mathcal{F}^{m*}(y) - \mathcal{F}^{(m+1)*}(y)] - b = 0.
$$

Since $W(y)$ has a single sign change, a binary search reveals the optimal $y$, and the problem can be solved by enumerating $p \in [\hat{p}, \bar{p}]$.

### 3.2. Numerical Experiments

We tested the $(y^*, p^*)$ policy in experiments using two types of demand functions—the additive model, i.e., $D(p) = \alpha - \beta p + \xi$, and the multiplicative model, i.e., $D(p) = (\alpha - \beta p)\xi$, where $\alpha$ and $\beta$ are linear coefficients and the $\xi$’s are normally distributed and truncated to ensure that demand realizations are not negative. We adapted the test case parameters from Federgruen and Heching (1999). Specifically, $[\alpha, \beta, c, h, \hat{p}, \bar{p}, \delta] = [174, 3, 22.15, 0.22, 25, 44, 0.95]$. In addition, $\theta$ is chosen from 1 and 5, representing a low and high disposal cost, respectively, and $b$ is between 1.98 and 21.78, representing a low and high penalty cost, respectively, for backordering. For the additive demand case, $\xi$ has a mean equal to zero and standard deviation $c_v(\alpha - \beta \bar{p})$, where $c_v$ is a specified variance coefficient. For the multiplicative case, $\xi$ has a mean equal to one and standard deviation equal to $c_r$. For both cases, $c_r$ varies from 0.2 to 1.2. We measured the performance of the heuristic in two ways, using average (avg) and maximum (max) errors given by

$$
\max = \max_{x: 0 \leq x \leq y^*} \frac{V^*(x) - \tilde{V}^*(x)}{V^*(x)},
\text{avg} = \frac{\sum_{x: 0 \leq x \leq y^*} ((V^*(x) - \tilde{V}^*(x))/V^*(x))}{[y: 0 \leq x \leq y^*]}.
$$


where \( y^* \) is the base-stock level of the \((y^*, p^*)\) policy, 
\[ V^*(x) = v^*(x) - cx, \quad \tilde{V}^*(x) = \tilde{v}^*(x) - cx, \] 
and \( \tilde{V}^*(x) \) is the total profit for the \((y^*, p^*)\) policy. Only initial inventory levels from zero to \( y^* \) are used because if the \((y^*, p^*)\) policy is adopted, total initial inventory \( x \) will never exceed \( y^* \), and 
\[ V^*(x) = V^*(0), \quad \tilde{V}^*(x) = \tilde{V}^*(0) \] 
for all \( x \) such that \( x < 0 \).

The \((y^*, p^*)\)-policy performs well. Table 1 shows its performance for products with a two- and three-period lifetime. Both the average and the maximum errors are about 1%. In fact, comparison for the two-period case between the heuristic \((y^*, p^*)\) policy and the optimal base-stock/list-price policy obtained by enumeration showed that the heuristic \((y^*, p^*)\) policy is optimal or nearly optimal in most cases.

### Table 1 Performance of the \((y^*, p^*)\) Policy

<table>
<thead>
<tr>
<th>Demand function</th>
<th>( b/(b+h) )</th>
<th>Low</th>
<th>90%</th>
<th>99%</th>
<th>High</th>
<th>90%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D(p) = a - \beta p + \xi )</td>
<td>0.2</td>
<td>Avg 0.82 0.74 0.53 0.51 0.84 0.75 0.58 0.52</td>
<td>Max 1.36 1.35 1.42 1.58 1.51 1.51 1.72 1.88</td>
<td>0.4</td>
<td>Avg 0.68 1.51 0.52 0.77 0.63 1.52 0.56 0.81</td>
<td>Max 1.04 2.33 1.21 2.05 1.08 2.60 1.46 2.52</td>
<td>0.8</td>
</tr>
<tr>
<td>( D(p) = (a - \beta p) \xi )</td>
<td>0.2</td>
<td>Avg 0.93 1.06 0.41 0.61 0.96 1.07 0.36 0.63</td>
<td>Max 1.38 1.66 1.16 1.62 1.55 1.85 1.31 1.94</td>
<td>0.4</td>
<td>Avg 0.51 1.38 0.43 0.45 0.52 1.41 0.53 0.52</td>
<td>Max 0.60 1.84 0.96 1.32 0.63 2.03 1.18 1.68</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Note. Errors are shown in percentages.

### Electronic Companion
An electronic companion to this paper is available on the Manufacturing & Service Operations Management website (http://msom.pubs.informs.org/ecompanion.html).

### References


