The $\mathcal{H}_2$ Control Problem for Decentralized Systems with Delays

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Abstract

This paper gives a new solution to the output feedback $\mathcal{H}_2$ problem for communication delay patterns specified by a strongly connected graph. A characterization of all stabilizing controllers satisfying the delay constraints is given and the decentralized $\mathcal{H}_2$ problem is cast as a convex model matching problem. The main result shows that the model matching problem can be reduced to a finite-dimensional quadratic program. This quadratic program can be reformulated as a finite-horizon decentralized LQG problem. A recursive state-space method for computing the optimal controller based on vectorization is given.

I. INTRODUCTION

Decentralized control problems arise when inputs to a dynamic system are chosen by multiple controllers with access to different information. In decentralized control with delays, local measurements are passed to the various controllers over a communication network with delays. As a result of the delays, some controllers will have access to measurements before others. This paper provides a new solution to the $\mathcal{H}_2$ optimal control problem, subject to communication delays, based on the Youla parametrization and and vectorization.

A. Contributions

This paper solves the decentralized $\mathcal{H}_2$ problem for a class of delay patterns arising from strongly-connected communication networks. The main insight in the paper is that for such delay patterns, the $\mathcal{H}_2$ problem can be reformulated as a finite-horizon decentralized control problem. Thus, finite-dimensional optimization methods can be used to solve the original infinite-horizon optimal control problem.

To derive the finite-horizon optimal control problem, a Youla parametrization framework developed for sparsity problems, [1], is adapted to communication delay patterns. The parametrization is then used to characterize all stabilizing controllers that satisfy a given delay pattern. The corresponding decentralized model matching problem is also derived. As is typical, the model matching problem depends on the choice of a doubly-coprime factorization. For a doubly-coprime factorization based on the centralized LQG controller, it is shown that the model matching problem is, in fact, finite-dimensional.
B. Related Work

Output feedback $\mathcal{H}_2$ problems with various communication delay patterns have been previously solved using approaches based on vectorization [2] and linear matrix inequalities (LMIs) [3], [4]. The primary difference between those methods and the method in this paper, is that the infinite-horizon decentralized problems are reduced to infinite-horizon centralized problems with state dimensions that grow with the size of the delay. This paper, on the other hand, reduces the infinite-horizon problem to a decentralized finite-horizon problem of the same state-dimension, with horizon growing with the size of the delay.

Most work on decentralized optimal control with delays is based on dynamic programming. For the special case known as the one-step delay information sharing pattern, the output feedback $\mathcal{H}_2$ problem was solved in the 1970s by dynamic programming [5], [6], [7]. For more complex delay patterns, dynamic programming has extensions to decentralized state feedback [8], [9], [10], but output feedback is difficult because appropriate sufficient statistics for dynamic programming are not obvious [11], [12], [13]. Recently, methods based on POMDPs have been developed for output feedback control of nonlinear systems with general delay patterns [14], [15]. It would be interesting to see how these recent dynamic programming methods can be adapted to the $\mathcal{H}_2$ problem studied in this paper.

In contrast to dynamic programming approaches, this paper uses operator theoretic calculations to reduce the decentralized model matching problem to a quadratic program. It is an extension of [16], which uses spectral factorization to derive a similar quadratic program. This paper, on the other hand, relies on a well-chosen doubly-coprime factorization to simplify the model matching problem and remove the need for spectral factorization. Furthermore, the method in this paper applies to unstable systems, while the work in [16] is restricted to stable systems.

More broadly, this paper is influenced by recent developments in decentralized optimal control with sparsity constraints. As mentioned above, the Youla parametrization method in this paper is based on the parametrization of sparse controllers from [1]. Many of the operator theoretic calculations are modified from spectral factorization methods for sparsity constraints such as [17], [18], [19].

C. Overview

The paper is structured as follows. Section II defines the general problem studied in this paper, the decentralized $\mathcal{H}_2$ problem with a strongly-connected delay pattern. Section III gives a parametrization of all stabilizing controllers that satisfy a given delay pattern, and presents the corresponding model matching problem. In Section IV, the decentralized $\mathcal{H}_2$ problem is reduced to a quadratic program, and this program is solved by vectorization. Numerical results are given in Section V and finally conclusions are given in VI.

II. Problem

This section introduces the basic notation and the model matching problem of interest. Subsection II-C describes how common delayed information sharing patterns can be cast in the framework of this paper.
A. Preliminaries on \( \mathcal{H}_2 \)

Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) be the unit disc of complex numbers and let \( \overline{D} \) be its closure. A function \( G : (\mathbb{C} \cup \{ \infty \}) \setminus \overline{D} \to \mathbb{C}^{p \times q} \) is in \( \mathcal{H}_2 \) if it can be expanded as

\[
G(z) = \sum_{i=0}^{\infty} \frac{1}{z^i} G_i,
\]

where \( G_i \in \mathbb{C}^{p \times q} \) and \( \sum_{i=0}^{\infty} \text{Tr}(G_i G_i^*) < \infty \). Define the conjugate of \( G \) by

\[
G(z)^\sim = \sum_{i=0}^{\infty} z^i G_i^*.
\]

For a real rational transfer matrix, \( G = \begin{bmatrix} A & B_1 \\ C_1 & D_{12} \\ B_2 & C_2 & D_{21} \end{bmatrix} \), the conjugate is given by

\[
(C(zI - A)^{-1} B + D)^\sim = B^T \left( \frac{1}{z} I - A^T \right)^{-1} C^T + D^T.
\]

The space \( \mathcal{H}_2 \) is a Hilbert space with inner product defined by

\[
\langle G, H \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Tr} \left( G(e^{j\theta}) H(e^{j\theta})^\sim \right) d\theta
\]

\[
= \sum_{i=0}^{\infty} \text{Tr}(G_i H_i^*)
\]

where the second equality follows from Parseval’s identity.

If \( M \) is a subspace of \( \mathcal{H}_2 \), denote the orthogonal projection onto \( M \) by \( P_M \).

A function \( G : (\mathbb{C} \cup \{ \infty \}) \setminus \overline{D} \to \mathbb{C}^{p \times q} \) is in \( \mathcal{H}_\infty \) if it is analytic, bounded, and has a well-defined limit \( G(e^{j\theta}) \in \mathbb{C}^{p \times q} \) almost everywhere on the unit circle.

Let \( \mathcal{R}_p \) denote the space of proper real rational transfer matrices. Furthermore, denote \( \mathcal{R}_p \cap \mathcal{H}_2 \) and \( \mathcal{R}_p \cap \mathcal{H}_\infty \) by \( \mathcal{R}\mathcal{H}_2 \) and \( \mathcal{R}\mathcal{H}_\infty \), respectively. Note that \( \mathcal{R}\mathcal{H}_2 = \mathcal{R}\mathcal{H}_\infty \), since both correspond to transfer matrices with no poles outside of \( D \).

B. Formulation

This subsection introduces the generic problem of interest. Let \( G \) be a discrete-time plant given by

\[
G = \begin{bmatrix} A & B_1 \\ C_1 & 0 & D_{12} \\ B_2 & C_2 & D_{21} & 0 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},
\]

with inputs of dimension \( p_1, p_2 \) and outputs of dimension \( q_1, q_2 \).

For the existence of solutions of the appropriate Riccati equations, as well as simplicity of formulas, assume that

- \((A, B_1, C_1)\) is stabilizable and detectable,
- \((A, B_2, C_2)\) is stabilizable and detectable,
- \( D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \).
Fig. 1. The basic feedback loop.

\[ \begin{bmatrix} B^T_1 & D^T_{21} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \]

For \( N \geq 1 \), define the space of strictly proper finite impulse response (FIR) transfer matrices by 
\[ \mathcal{X} = \bigoplus_{i=1}^{N} \frac{1}{z} \mathbb{C}^{p_2 \times q_2}, \]
and denote the real FIR transfer matrices by 
\[ \mathcal{R}\mathcal{X} = \bigoplus_{i=1}^{N} \frac{1}{z} \mathbb{R}^{p_2 \times q_2}. \]
Note that \( \frac{1}{z} \mathcal{H}_2 \) can be decomposed into orthogonal subspaces as

\[ \frac{1}{z} \mathcal{H}_2 = \mathcal{X} \oplus \frac{1}{z^{N+1}} \mathcal{H}_2. \]

Let \( S \subset \frac{1}{z} \mathbb{R}^{p_2} \) be a subspace of the form

\[ S = \mathcal{Y} \oplus \frac{1}{z^{N+1}} \mathbb{R}^{p_2}, \]

where

\[ \mathcal{Y} = \bigoplus_{i=1}^{\infty} \frac{1}{z^i} \mathcal{Y}_i, \]

and \( \mathcal{Y}_i \subset \mathbb{R}^{p_2 \times q_2} \) defines a sparsity pattern over matrices. Delay patterns satisfying the decomposition in (1) will be called strongly connected, since delay patterns arising from strongly-connected communication networks always have this form. (See subsection II-C.)

The set \( S \) is assumed to be quadratically invariant with respect to \( G_{22} \), which means that for all \( \mathcal{K} \in S \), \( \mathcal{K}G_{22}\mathcal{K} \in S \). The key property of quadratic invariance is that \( \mathcal{K} \in S \) if and only if \( \mathcal{K}(I - G_{22}\mathcal{K})^{-1} \in S \) [2].

The decentralized \( \mathcal{H}_2 \) problem studied in this paper is given by

\[ \min_{\mathcal{K}} \| G_{11} + G_{12}\mathcal{K}(I - G_{22}\mathcal{K})^{-1}G_{21} \|_{\mathcal{H}_2}^2 \]

s.t. \( \mathcal{K} \in S \).

The quadratic invariance assumption guarantees that the corresponding model matching problem is convex [2]. Reduction to model matching is discussed in Section III-B.

The decomposition of \( S \) in (1) is crucial for the results of this paper. The property that \( \frac{1}{z} \mathbb{R}^{p_2} \subset S \) implies that every measurement is available to all controller subsystems within \( N \) time steps. Concrete examples of delay patterns of this form are described in the next subsection.

For technical simplicity, controllers in this paper are assumed to be strictly proper (that is, in \( \frac{1}{z} \mathbb{R}^{p_2} \)). The results in this paper can be extended to non-strictly proper controllers but more complicated formulas would result.
C. Communication Delay Patterns

Equation (1) can be used to model many delayed information sharing patterns. Indeed, any delay constraint of the form
\[
\begin{bmatrix}
K_{11} & \cdots & K_{1q_2} \\
\vdots & \ddots & \vdots \\
K_{p_11} & \cdots & K_{p_1q_2}
\end{bmatrix} \in \begin{bmatrix}
\frac{1}{z^{t_{11}}} R_{p_1} & \cdots & \frac{1}{z^{t_{q_2}}} R_{p_1} \\
\vdots & \ddots & \vdots \\
\frac{1}{z^{t_{p_1}}} R_{p_1} & \cdots & \frac{1}{z^{t_{p_2}}} R_{p_1}
\end{bmatrix},
\]
for positive integers \( t_{ij} \), is strongly connected. This subsection will discuss a class of strongly connected delay patterns that arise from graphs.

As an example, consider an infinite-horizon, strictly proper version of the 1-step delayed information sharing pattern studied in [5], [6], [7]. This delay constraint may be represented using (1) with \( N = 1 \) and \( \mathcal{Y} \) corresponding to block diagonal FIR matrices
\[
\mathcal{Y} = \frac{1}{z} \begin{bmatrix}
\mathbb{R}^{P_{21} \times q_{21}} & 0 \\
0 & \mathbb{R}^{P_{22} \times q_{22}}
\end{bmatrix}.
\]
Similarly, for \( N > 1 \), a strictly proper version of the \( N \)-step delay information sharing pattern studied in [11], [12], [13], [14] can be characterized by \( \mathcal{Y} \) of the form
\[
\mathcal{Y} = \bigoplus_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix}
\mathbb{R}^{P_{21} \times q_{21}} & 0 \\
0 & \mathbb{R}^{P_{22} \times q_{22}}
\end{bmatrix}.
\]
The strictly proper \( N \)-step delay pattern can be described by the graph in Figure 2.

More generally, assume that communication between the controller subsystems is specified by a strongly-connected graph \( (V, E) \) with self-loops at each node. Computational delays are specified by positive integers on the self-loops, while communication delays are represented by non-negative integers on the edges between distinct nodes. Requiring positive computational delays ensures that the controller is strictly proper.

A constraint space of the form (1) can be constructed as follows. For nodes \( i \) and \( j \) let \( c_i \) be the computational delay at node \( i \) and let \( \tilde{d}_{ij} \) be the sum of communication delays along the directed path with shortest aggregate delay. Define the delay matrix, \( d \), such that
\[
d_{ij} = c_i + \tilde{d}_{ij}.
\]
Let \( N = \max\{d_{ij} : i, j \in V\} - 1 \). The corresponding constraint space is defined by

\[
S = \begin{bmatrix}
\frac{1}{z_{|V|}} R_p & \cdots & \frac{1}{z_{|V|}} R_p \\
\vdots & \ddots & \vdots \\
\frac{1}{z_{|V|}} R_p & \cdots & \frac{1}{z_{|V|}} R_p
\end{bmatrix}.
\]

Thus, the \( S \) can be decomposed as in (1) by defining

\[
Y = \bigoplus_{k=1}^N \frac{1}{z^k} \begin{bmatrix}
Y_{kk}^{11} & \cdots & Y_{kk}^{1|V|} \\
\vdots & \ddots & \vdots \\
Y_{kk}^{|V|1} & \cdots & Y_{kk}^{|V||V|}
\end{bmatrix}
\]

where

\[
Y_{ij}^k = \begin{cases} 
R_{2i \times 2j} & \text{if } d_{ij} \leq k \\
0 & \text{if } d_{ij} > k.
\end{cases}
\]

Necessary and sufficient conditions for such constraints to be quadratically invariant are given in [20].

In the \( N \)-step delay example, the delay matrix is given by

\[
d = \begin{bmatrix} 1 & N + 1 \\ N + 1 & 1 \end{bmatrix}.
\]

As another example, consider the strictly proper version of the three-player chain problem discussed in [9], [16].

The graph describing the delays is given in Figure 3, leading to a delay matrix

\[
d = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}.
\]

Thus, the constraint space is defined by

\[
Y = \frac{1}{z} \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \oplus \frac{1}{z^2} \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix},
\]

where, for compactness, * is used to denote a space of appropriately sized real matrices.

Using this convention, all measurements, \( y_j(t) \), are available to all controllers by time \( t + N + 1 \).
III. DECENTRALIZED STABILIZATION

This section parametrizes the set of controllers \( K \in S \) which internally stabilize the plant \( G \). The parametrization naturally leads to a convex model matching formulation of \( H_2 \) problem. In analogy with results on sparse transfer matrices [1], the parametrization is based on quadratic invariance and the classical Youla parametrization.

A. All Stabilizing Decentralized Controllers

A collection of stable transfer matrices, \( \hat{M}, \hat{N}, \hat{X}, \hat{Y}, \tilde{M}, \tilde{N}, \tilde{X}, \) and \( \tilde{Y} \in \mathcal{RH}_\infty \), defines a doubly-coprime factorization of \( G_{22} \) if

\[
\begin{bmatrix}
\hat{N} & \hat{M} \\
\tilde{M} & \tilde{N}
\end{bmatrix} = I, \tag{5}
\]

As long as \((A, B_2, C_2)\) is stabilizable and detectable, there are numerous ways to construct a doubly coprime factorization of \( G_{22} \).

The following theorem is well known [21].

**Theorem 1:** Assume that \( G_{22} \) has a double doubly-coprime factorization of the form in (5). A controller \( K \in \mathcal{R}_p \) internally stabilizes \( G \) if and only if there is a transfer matrix \( Q \in \mathcal{R}_\mathcal{H}_\infty \) such that

\[
K = (\hat{Y} - \hat{M}Q)(\hat{X} - \tilde{N}Q)^{-1} = (\hat{X} - Q\tilde{N})^{-1}(\hat{Y} - Q\hat{M}). \tag{6}
\]

From [2], if \( G_{22} \) is quadratically invariant under \( S \), then \( K \in S \) if and only if \( K(I - G_{22}K)^{-1} \in \mathcal{S} \). As in [1], a straightforward calculation shows that

\[
K(I - G_{22}K)^{-1} = (\hat{Y} - \hat{M}Q)\hat{M}, \tag{7}
\]

and thus \( K \in S \) if and only if \((\hat{Y} - \hat{M}Q)\hat{M} \in S\). Recalling the strongly-connected structure of \( S \) gives the following theorem.

**Theorem 2:** A controller \( K \in S \) internally stabilizes \( G_{22} \) if and only if there are transfer matrices \( U \in \frac{1}{z\mathcal{H}_\infty} \mathcal{R}_\mathcal{H}_\infty \) and \( V \in \mathcal{R}_X \) such that

\[
K = (\hat{Y} - \hat{M}(U + V))(\hat{X} - \tilde{N}(U + V))^{-1}
\]

and

\[
P_X \left((\hat{Y} - \hat{M}V)\hat{M}\right) \in \mathcal{Y}. \tag{8}
\]

**Proof:** Say that \( K \) internally stabilizes \( G \). Then, by Theorem 1, there exists \( Q \in \mathcal{R}_\mathcal{H}_\infty \) such that (6) holds. Given \( Q \in \mathcal{R}_\mathcal{H}_\infty \), there are unique \( U \in \frac{1}{z\mathcal{H}_\infty} \mathcal{R}_\mathcal{H}_\infty \) and \( V \in \mathcal{R}_X \) such that \( Q = U + V \). Now, (7) implies that \( K \in S \) if and only if

\[
(\hat{Y} - \hat{M}(U + V))\hat{M} \in S.
\]
Furthermore, note that \((\hat{Y} - \hat{M}(U + V))\tilde{M} \in \mathcal{RH}_\infty = \mathcal{RH}_2\), and thus
\[
(\hat{Y} - \hat{M}(U + V))\tilde{M} = \mathbb{P}_X \left((\hat{Y} - \hat{M}(U + V))\tilde{M}\right)
+ \mathbb{P}_{\frac{1}{z^{N+1}}} h_2 \left((\hat{Y} - \hat{M}(U + V))\tilde{M}\right)
= \mathbb{P}_X \left((\hat{Y} - \hat{M}V)\tilde{M}\right)
+ \mathbb{P}_{\frac{1}{z^{N+1}}} h_2 \left((\hat{Y} - \hat{M}(U + V))\tilde{M}\right)
\]
(9)

The second equality follows because \(U \in \frac{1}{z^{N+1}}\mathcal{RH}_2\), and thus \(\hat{M}U\tilde{M} \in \frac{1}{z^{N+1}}\mathcal{RH}_2\). Since \(\mathcal{S} = \mathcal{Y} \oplus \frac{1}{z^{N+1}}\mathcal{R}_p\), the right of (9) is in \(\mathcal{S}\) if and only if (8) holds.

The converse is proved by reversing the steps.

Note that (8) reduces to a finite-dimensional linear constraint on the FIR term, \(V \in \mathcal{R}\mathcal{X}\). The other term, \(U\), is delayed, but otherwise unconstrained.

**B. Model Matching**

Given a doubly-coprime factorization, (7) implies that the closed-loop transfer matrix is given by
\[
G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} = P_{11} + P_{12}Q P_{21},
\]
where
\[
P_{11} = G_{11} + G_{12}\hat{Y}\hat{M}G_{21}
\]
\[
P_{12} = -G_{12}\hat{M}
\]
\[
P_{21} = \hat{M}G_{21}.
\]

Using the decomposition \(Q = U + V\), with \(V \in \mathcal{R}\mathcal{X}\) and \(U \in \frac{1}{z^{N+1}}\mathcal{RH}_2\), the decentralized \(\mathcal{H}_2\) problem, (3), is equivalent to the following model matching problem:
\[
\begin{align*}
\min_{U,V} & \quad \|P_{11} + P_{12}(U + V)P_{21}\|^2_{\mathcal{H}_2} \\
\text{s.t.} & \quad U \in \frac{1}{z^{N+1}}\mathcal{RH}_2, \quad V \in \mathcal{R}\mathcal{X} \\
& \quad \mathbb{P}_X \left((\hat{Y} - \hat{M}V)\tilde{M}\right) \in \mathcal{Y}.
\end{align*}
\]
(11)

IV. RESULTS

This section first gives the main result of the paper, a quadratic programming solution of the decentralized control problem, (3). Next, the quadratic program is reduced to a finite-horizon decentralized LQG problem. A vectorization method for computing the optimal solution is also given.
A. Quadratic Programming Formulation

In the previous section, it was shown that the decentralized feedback problem is equivalent to a model matching problem, (11). It will be shown that for an appropriately chosen doubly-coprime factorization, the model matching problem reduces to a quadratic program. The reduction is due to the special nature of strongly-connected delay patterns.

Let \( X \) and \( Y \) be the stabilizing solutions of the Riccati equations associated with the linear quadratic regulator and Kalman filter, respectively:

\[
X = C_1^T C_1 + A^T X A - A^T X B_2 (I + B_2^T X B_2)^{-1} B_2^T X A \tag{12}
\]

\[
Y = B_1 B_1^T + A Y A^T - A Y C_2^T (I + C_2 Y C_2^T)^{-1} C_2 Y A^T. \tag{13}
\]

Define \( \Omega = I + B_2^T X B_2 \) and \( \Psi = I + C_2 Y C_2^T \), the corresponding gains are given by

\[
K = -\Omega^{-1} B_2^T X A \tag{14}
\]

\[
L = -A Y C_2^T \Psi^{-1}. \tag{15}
\]

Furthermore, \( A + B_2 K \) and \( A + L C_2 \) are stable.

It is well known (e.g. [21]) that a doubly-coprime factorization of \( G_{22} \) is given by

\[
\begin{bmatrix}
\hat{M} & \hat{Y} \\
\hat{N} & \hat{X}
\end{bmatrix} =
\begin{bmatrix}
A + B_2 K & B_2 & -L \\
K & I & 0 \\
C_2 & 0 & I
\end{bmatrix},
\]

\[
\begin{bmatrix}
\hat{X} & -\hat{Y} \\
-\hat{N} & \hat{M}
\end{bmatrix} =
\begin{bmatrix}
A + L C_2 & B_2 & -L \\
-K & I & 0 \\
-C_2 & 0 & I
\end{bmatrix}. \tag{16}
\]

The following theorem is the main result of the paper.

**Theorem 3:** Consider the doubly-coprime factorization of \( G_{22} \) defined by (16). The optimal solution to the decentralized \( H_2 \) problem defined by (3) is given by

\[
\mathcal{K}^* = (\hat{Y} - \hat{M} V^*)(\hat{X} - \hat{N} V^*)^{-1},
\]

where \( V^* \) is the unique optimal solution to the quadratic program

\[
\min_{V \in \mathbb{R}^X} \quad ||\Omega^{1/2} V \Psi^{1/2}||^2_{\mathcal{H}_2} \tag{17}
\]

s.t. \( \mathbb{P}_X \left( (\hat{Y} - \hat{M} V)\hat{M} \right) \in \mathcal{Y} \).

Furthermore, the optimal cost is given by

\[
||P_{11}||^2_{\mathcal{H}_2} + ||\Omega^{1/2} V^* \Psi^{1/2}||^2_{\mathcal{H}_2}.
\]
Proof: For the doubly-coprime factorization given by (16) the model matching matrices, (10), have state space realizations given by

\[
P_{11} = \begin{bmatrix}
A + B_2K & -B_2K & B_1 \\
0 & A + LC_2 & B_1 + LD_{21} \\
C_1 + D_{12}K & -D_{12}K & 0
\end{bmatrix}
\]

\[
P_{12} = -\begin{bmatrix}
A + B_2K & B_2 \\
C_1 + D_{12}K & D_{12}
\end{bmatrix}
\]

\[
P_{21} = \begin{bmatrix}
A + LC_2 & B_1 + LD_{21} \\
C_2 & D_{21}
\end{bmatrix}
\]  

(18)

For a fixed \( V \in \mathcal{H}_2 \), the optimal \( U \in \mathcal{H}_2 \) is found by solving

\[
\min_{U \in \mathcal{H}_2} \| P_{11} + P_{12}VP_{21} + P_{12}UP_{21} \|_{\mathcal{H}_2}^2.
\]

A necessary condition for \( U \) to be optimal, given \( V \), is

\[
P_{12}^\sim P_{11} P_{21}^\sim + P_{12}^\sim P_{12}V P_{21} P_{21}^\sim + P_{12}^\sim P_{12}U P_{21} P_{21}^\sim \in \left( \frac{1}{N+1} \mathcal{H}_2 \right) \perp.
\]

Lemma A.2 implies that

\[
P_{12}^\sim P_{12} = \Omega, \quad \text{and} \quad P_{21} P_{21}^\sim = \Psi.
\]

Thus, the optimality condition becomes

\[
P_{12}^\sim P_{11} P_{21}^\sim + \Omega V \Psi + \Omega U \Psi \in \left( \frac{1}{N+1} \mathcal{H}_2 \right) \perp.
\]

Lemma A.2 also implies that

\[
P_{12}^\sim P_{12} = \Omega, \quad \text{and} \quad P_{21} P_{21}^\sim = \Psi.
\]

Thus, the optimality condition becomes

\[
P_{12}^\sim P_{11} P_{21}^\sim + \Omega V \Psi + \Omega U \Psi \in \left( \frac{1}{N+1} \mathcal{H}_2 \right) \perp.
\]

It follows that

\[
U = -\mathbb{P}_{\mathcal{H}_2} \left( \Omega^{-1} P_{12}^\sim P_{11} P_{21}^\sim \Psi^{-1} + V \right) = 0.
\]

Thus, the optimal \( U \) is 0 for any \( V \in \mathcal{H}_2 \).

Plugging \( V \) into the cost of (11) gives

\[
\| P_{11} + P_{12}VP_{21} \|_{\mathcal{H}_2}^2 = \| P_{11} \|_{\mathcal{H}_2}^2 + \| P_{12}VP_{21} \|_{\mathcal{H}_2}^2 + 2\langle P_{11}, P_{12}VP_{21} \rangle
\]

The second term on the right of (19) simplifies as

\[
\| P_{12}VP_{21} \|_{\mathcal{H}_2}^2 = \langle P_{12}^\sim P_{12}VP_{21} P_{21}^\sim, V \rangle
\]

\[
= \langle \Omega V \Psi, V \rangle
\]

\[
= \| \Omega^{1/2} V \Psi^{1/2} \|_{\mathcal{H}_2}^2.
\]
For the third term on the right of (19), $V \in \mathcal{H}_2$ implies that
\[
\langle P_{11}, P_{12} V P_{21} \rangle = \langle P_{12}^* P_{11} V, V \rangle = 0.
\]
It follows that for a fixed $V$, the cost is given by
\[
\|P_{11}\|_{\mathcal{H}_2}^2 + \|\Omega^{1/2} V \Psi^{1/2}\|_{\mathcal{H}_2}^2.
\]
Thus, Theorem 2 and the model matching formulation, (11), imply that the optimal $V$ must solve (17). Note that
\[
\|\Omega^{1/2} V \Psi^{1/2}\| = \sum_{i=1}^{N} \text{Tr}(\Omega_i V_i \Psi_i V_i^T)
\]
is a positive definite quadratic function of $V$, while the constraint is linear. Thus (17) is a quadratic program and it must have a unique optimal solution.

For completeness, a state-space realization will be given for $\mathcal{K}$ of the form
\[
\mathcal{K} = (\hat{Y} - \hat{M} V)(\hat{X} - \hat{N} V)^{-1}
\]
with $V \in \mathcal{R} \mathcal{X}$. Note that $V$ has a realization
\[
V = \begin{bmatrix}
I_{q_2} & 0_{q_2 \times q_2} & \cdots & 0_{q_2 \times q_2} \\
0_{q_2 \times q_2} & I_{q_2} & \cdots & 0_{q_2 \times q_2} \\
\vdots & \vdots & \ddots & \vdots \\
0_{q_2 \times q_2} & 0_{q_2 \times q_2} & \cdots & I_{q_2}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_N
\end{bmatrix}
=: \begin{bmatrix}
A_V & B_V \\
C_V & 0
\end{bmatrix}.
\]
Standard state-space manipulations show that
\[
\mathcal{K} = \begin{bmatrix}
A + B_2 K + LC_2 & B_2 C_V & -L \\
B_V C_2 & A_V & -B_V \\
K & C_V & 0
\end{bmatrix}.
\]
Thus, $\mathcal{K}$ has a state-space realization of order $n + q_2 N$. If $N$ is the smallest integer such that a decomposition of the form (1) holds, then $\mathcal{K}$ must have entries in $\frac{1}{2^{n+q_2}} \mathcal{R}_p$. In this case, any minimal realization must have order at least $N$. Thus, the order of the realization in (21) is within a constant factor of the minimal realization order.

**B. Finite-Horizon Decentralized LQG**

In this subsection, the quadratic program from Theorem 3 is cast as a finite-horizon decentralized LQG problem. It is hoped that special cases of the resulting finite-horizon problem could be solved more explicitly in the future, perhaps using techniques similar to those in [22].
Theorem 4: The optimal feedback controller, $K^*$, is uniquely determined by the first $N$ terms of its power series expansion, $\mathbb{P}_X(K^*)$. Furthermore, $\mathbb{P}_X(K^*)$ is the optimal solution to the following decentralized LQG problem:

$$\min_H \mathbb{E} \left[ (Kx_N - u_N)^T \Omega (Kx_N - u_N) \right]$$

s.t. $x_{i+1} = Ax_i + B_2 u_i - Lw_i, \quad x_0 = 0$

$$y_i = C_2 x_i + w_i$$

$$u_i = \sum_{j=1}^{i} H_j y_{i-j}$$

$$H_i \in \mathcal{Y}_i,$$

where the noise terms are independent Gaussian random variables with mean 0 and covariance $\Psi$.

Proof: First assume that a feasible controller has the form in (20). In particular, $K^*$ has this form. Solving for $V$ shows that

$$V = (K^* N - \hat{M})^{-1}(K^* \hat{X} - \hat{Y}).$$

Furthermore, since all the transfer functions are proper and $V \in \mathcal{X}$,

$$V = \mathbb{P}_X \left( (\mathbb{P}_X(K) \hat{N} - \hat{M})^{-1}(\mathbb{P}_X(K) \hat{X} - \hat{Y}) \right).$$

Plugging the right of this equation into (20) shows that $K$ can be computed in terms of $\mathbb{P}_X(K)$.

Now the LQG problem for $\mathbb{P}_X(K)$ will be derived. For simpler notation, let $H = \mathbb{P}_X(K)$. A bit of algebra using (5) shows that

$$(\hat{M} - H \hat{N})^{-1}(\hat{Y} - H \hat{X})$$

$$= \hat{M}^{-1}(I - HG_{22})^{-1}(I - H G_{22}) \hat{Y} + H \hat{M}^{-1}(\hat{N} \hat{Y} - \hat{M} \hat{X}))$$

$$= \hat{M}^{-1} \hat{Y} - \hat{M}^{-1}(I - HG_{22})^{-1} H \hat{M}^{-1}. \quad (24)$$

The matrices on the right have realizations given by

$$\hat{M}^{-1} = \begin{bmatrix} A & -B_2 \\ K & I \end{bmatrix}, \quad \hat{M}^{-1} = \begin{bmatrix} A & -L \\ C_2 & I \end{bmatrix},$$

$$\hat{M}^{-1} \hat{Y} = \begin{bmatrix} A & -L \\ K & 0 \end{bmatrix}. \quad (25)$$

Combining (23) and (24) with the state-space realizations shows that

$$V = \mathbb{P}_X \left( \begin{bmatrix} A & -L \\ K & 0 \end{bmatrix} \right) \quad (25)$$

$$+ \mathbb{P}_X \left( \begin{bmatrix} A & B_2 \\ K & -I \end{bmatrix} H(I - G_{22} H)^{-1} \begin{bmatrix} A & -L \\ C_2 & I \end{bmatrix} \right).$$
Thus, the convolution $z_i = \sum_{j=1}^{i} V_j w_{i-j}$ has a time-domain representation given by

$$x_{i+1} = Ax_i + B_2 u_i - L w_i, \quad x_0 = 0$$

$$y_i = C_2 x_i + w_i$$

$$u_i = \sum_{j=1}^{i} H_j y_{i-j}$$

$$z_i = K x_i - u_i.$$ 

Furthermore, recall that $K \in S$ if and only if $H = \mathbb{P}_X(K) \in \mathcal{Y}$. 

Finally, say that $w_i$ are independent Gaussian random variables with zero mean and covariance $\Psi$. The proof is completed by noting that

$$\|\Omega^{1/2} V_1^{1/2}\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{N} \text{Tr}((\Omega^{1/2} V_i^{1/2} \Psi^{1/2} V_i^T \Omega^{1/2}))$$

$$= \sum_{i=1}^{N} \mathbb{E}[\text{Tr}(\Omega^{1/2} V_i^T w_{N-i} w_{N-i}^T V_i^T \Omega^{1/2})]$$

$$= \mathbb{E}[\text{Tr}(\Omega^{1/2} z_N^T z_N^T \Omega^{1/2})]$$

$$= \mathbb{E}[(K x_N - u_N)^T (K x_N - u_N)].$$ 

The finite-horizon LQG problem, (22), is somewhat non-standard, since the policy, $H$, is time-invariant. Typically, finite-horizon control problems allow for time-varying policies, i.e.

$$u_i = \sum_{j=1}^{i} H_{i,j} y_{i-j}.$$ 

Thus, more work is needed to see if the method from [22], which assumes that the controller is time-varying, can be modified to handle problems of the form (22).

While specialized methods for solving (22) do not appear to exist, it can be solved using results from team decision theory. Consider the model matching parameter

$$J = -\mathbb{P}_X(\langle \tilde{Y} - \tilde{M} V \rangle \tilde{M}) = -\mathbb{P}_X((I - HG_{22})^{-1} H).$$  

(26)

Using (25), the quadratic program of Theorem 3 reduces to a finite-dimensional decentralized model matching problem:

$$\min_{J \in \mathcal{Y}^N} \|\Omega^{1/2} \mathbb{P}_X(\tilde{M}^{-1} \tilde{Y} + \tilde{M}^{-1} J \tilde{M}^{-1}) \Psi^{1/2}\|_{\mathcal{H}_2}^2.$$  

(27)

This problem is a special case of the static team decision problem [23], [24], which in turn reduces to a system of linear equations. The system of equations will be large, however, since the recursive structure of the model matching problem is not exploited.
C. Vectorization

In this subsection, the quadratic program of Theorem 3 will be cast as a finite-horizon state-feedback problem using vectorization techniques. The finite-dimensional model matching problem from (27) can also be transformed into a state-feedback problem by a similar vectorization method.

First, by defining \( R = \Psi \otimes \Omega \), the vectorized form of the cost function becomes

\[
\|\Omega^{1/2}V^{1/2}\|^2_{H_2} = \sum_{i=1}^{N} \text{Tr}(\Omega_i \Psi_i V_i^T) = \sum_{i=1}^{N} \text{vec}(V_i)^T R \text{vec}(V_i).
\]  

(28)

Recall the definition of the model matching parameter, \( J \), in (26), and let its series expansion be given by

\[ J = \sum_{i=1}^{N} \frac{1}{z_i^2} J_i. \]

If \( \mathcal{Y} \) is defined by (2), then the model matching constraint of (11) is equivalent to \( J_i \in \mathcal{Y}_i \). The vectorized form of \( J \) is computed from

\[ \text{vec}(-\hat{Y} \hat{M} + \hat{M}V \hat{M}) = -\text{vec}(\hat{Y} \hat{M}) + (\hat{M}^T \otimes \hat{M}) \text{vec}(V). \]

By Lemma A.3 in the appendix, the terms of \( J_i \) can be computed by the recursion

\[
x_1 = \begin{bmatrix} \text{vec}(L) \\ 0_{nq_2 \times 1} \end{bmatrix}
\]

\[
x_{i+1} = A_v x_i + B_v \text{vec}(V_i)
\]

\[
\text{vec}(J_i) = C_v x_i + \text{vec}(V_i),
\]

(29)

where

\[
\begin{bmatrix}
A_v \\
C_v
\end{bmatrix}
\begin{bmatrix}
B_v \\
D_v
\end{bmatrix}
= \begin{bmatrix}
I_{q_2} \otimes (A + B_2 K) & 0_{nq_2 \times nq_2} & I_{q_2} \otimes B_2 \\
C_v^T \otimes K & (A + L C_2)^T \otimes I_{p_2} & C_v^T \otimes I_{p_2} \\
I_{q_2} \otimes K & L^T \otimes I_{p_2} & I_{p_2 \times q_2}
\end{bmatrix}.
\]

Note that \( A_v \) is stable since \( A + B_2 K \) and \( A + L C_2 \) are.

Now let \( E_i \) and \( F_i \) be matrices with columns that form orthonormal bases of \( \text{vec}(\mathcal{Y}_i) \) and \( \text{vec}(\mathcal{Y}_i^\perp) \), respectively. The term \( \text{vec}(V_i) \) can then be decomposed as

\[ \text{vec}(V_i) = E_i u_i + F_i u_i^\perp, \]

for some vectors \( u_i \) and \( u_i^\perp \).

Using (29), the constraint that \( J_i \in \mathcal{Y}_i \) can be equivalently cast as

\[
F_i^T (C_v x_i + \text{vec}(V_i)) = F_i^T C_v x_i + u_i^\perp = 0.
\]

(30)
Plugging (30) into the cost (28) and the recursion (29) leads to the following optimal control problem:

\[
\min_u \sum_{i=1}^{N} \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}
\]

\[\text{s.t.} \quad x_1 = \begin{bmatrix} \text{vec}(L) \\ 0_{np_i \times 1} \end{bmatrix}, \quad x_{i+1} = A_i x_i + B_i u_i,\]

where the time-varying matrices are given by:

\[
\begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} = \begin{bmatrix} C_v^T F_i^T R F_i C_v & -C_v^T F_i^T R E_i \\ -E_i^T R F_i F_i^T C_v & E_i^T R E_i \end{bmatrix}, \quad \begin{bmatrix} A_i & B_i \end{bmatrix} = \begin{bmatrix} A_v - B_v F_i F_i^T C_v & B_v E_i \end{bmatrix}.
\]

As is standard, [25], the optimal controller can be computed as \( u_i = K_i x_i \), where

\[
K_i = -(R_i + B_i^T X_{i+1} B_i)^{-1}(B_i^T X_{i+1} A_i + S_i^T)
\]

and \( X_i \) is computed from the recursion

\[
X_{N+1} = 0, \quad X_i = Q_i + A_i^T X_{i+1} A_i - (A_i^T X_{i+1} A_i + S_i)(R_i + B_i^T X_{i+1} B_i)^{-1}(B_i^T X_{i+1} A_i + S_i^T).
\]

Furthermore, the optimal cost is given by \( x_1^T X_1 x_1 \). The next theorem follows immediately from Theorem 3 and the preceding discussion.

**Theorem 5:** The optimal \( V \) is computed as

\[
x_1 = \begin{bmatrix} \text{vec}(L) \\ 0_{np_i \times 1} \end{bmatrix}, \quad x_{i+1} = (A_i + B_i K_i)x_i, \quad \text{vec}(V_i) = (E_i K_i - F_i F_i^T C_v)x_i.
\]

Furthermore, the decentralized \( \mathcal{H}_2 \) problem of (3) has optimal value

\[
\|P_{11}\|^2 + x_1^T X_1 x_1.
\]

**V. Numerical Examples**

This section gives some numerical examples of optimal controllers computed using the vectorization method of the previous section.

\[\text{This definition is a slight abuse of notation, since } B_i \text{ here are distinct from the original input matrices } B_1 \text{ and } B_2, \text{ etc.}\]
A. The Chain Problem

Recall the three-player chain structure from Figure 3 with constraint specified by (4). Consider the plant given by

\[
A = \begin{bmatrix}
1.5 & 1 & 0 \\
1 & 1.5 & 1 \\
0 & 1 & 1.5
\end{bmatrix}, \quad B = \begin{bmatrix}
I_{3 \times 3} & 0_{3 \times 3} & I_{3 \times 3}
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
I_{3 \times 3} \\
0_{3 \times 3} \\
I_{3 \times 3}
\end{bmatrix}, \quad D = \begin{bmatrix}
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & I_{3 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3}
\end{bmatrix}.
\]

For comparison purposes, the optimal $\mathcal{H}_2$ norm was computed using model matching from this paper and the LMI method of [3], [4]. In all both cases the norm was found to be 34.9304. In contrast, the centralized controller gives a norm of 24.236.

B. Increasing Delays

Consider the plant defined by

\[
G = \begin{bmatrix}
0.9 & 0 & 0.1 & 0 & 10 & 0 \\
0 & -2 & 20 & 0 & 0 & 0 & 0.1 \\
2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 10 & 0 & 0 & 1 & 0 \\
0 & 0.1 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

For $N \geq 1$, let $K_{Tri}^N$, $K_{Di}^N$, and $K_{Low}^N$ solve the decentralized $\mathcal{H}_2$ problem, (3), with constraints of the form

\[
K_{Tri}^N \in \frac{1}{z^{N+1}} \mathbb{R}_p \bigoplus \bigoplus_{i=1}^N \frac{1}{z^i} \begin{bmatrix}
\mathbb{R} & 0 \\
0 & \mathbb{R}
\end{bmatrix},
\]

\[
K_{Di}^N \in \frac{1}{z^{N+1}} \mathbb{R}_p \bigoplus \bigoplus_{i=1}^N \frac{1}{z^i} \begin{bmatrix}
\mathbb{R} & 0 \\
0 & \mathbb{R}
\end{bmatrix},
\]

\[
K_{Low}^N \in \frac{1}{z^{N+1}} \mathbb{R}_p \bigoplus \bigoplus_{i=1}^N \frac{1}{z^i} \begin{bmatrix}
0 & 0 \\
0 & \mathbb{R}
\end{bmatrix}.
\]

The resulting norms are plotted in Figure 4.
Fig. 4. This plot shows the closed-loop norm for $K_{Tri}^N$, $K_{Di}^N$, and $K_{Low}^N$. For a given $N$, the controllers with fewer sparsity constraints give rise to lower norms. As $N$ increases, all of the norms increase monotonically since the controllers have access to less information. The dotted lines correspond to the optimal norms for sparsity structures given in (32).

As $N \to \infty$, the resulting controllers appear to approach optimal sparse controllers

$$
K_{Tri}^\infty \in \begin{bmatrix} \frac{1}{2}R_p & 0 \\ \frac{1}{2}R_p & \frac{1}{2}R_p \end{bmatrix}, \\
K_{Di}^\infty \in \begin{bmatrix} \frac{1}{2}R_p & 0 \\ 0 & \frac{1}{2}R_p \end{bmatrix}, \\
K_{Low}^\infty \in \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}R_p \end{bmatrix}, 
$$

(32)

which can be computed by the vectorization technique from [2]. Evidence for the convergence is shown by the fact that the norms limit to the values computed for the sparse controllers (Figure 4).
VI. Conclusion

This paper derives a novel solution for a class of output feedback $H_2$ control problems with strongly connected communication delay patterns. First, all stabilizing decentralized controllers are characterized via doubly coprime factorization. Then, by a standard change of variables, the $H_2$ problem is cast as a convex model matching problem. The main theorem shows that for a doubly coprime factorization based on the LQR and Kalman filter gains, the model matching problem reduces to a finite-dimensional quadratic program. The quadratic program can be transformed further into a finite-horizon decentralized LQG problem. A solution to the quadratic program based on vectorization is also presented.

Many open problems remain for optimal control with strongly-connected delay constraints. For specialized delay patterns, it may be possible to find more efficient and explicit solutions to the finite-horizon LQG problem of the paper. It is also likely that optimization methods for analyzing [26] and designing [27] delay patterns can be modified to apply to the quadratic program in this paper. Furthermore, work is needed to understand how the controllers in this paper could be realized [28], [29], [30] and computed [31], [32] in a distributed fashion. With a distributed implementation in place, model reduction techniques might also be useful for decreasing the amount of local computation required.

VII. Acknowledgements

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References

This appendix collects state-space formulas that are useful for deriving the results in the paper.

**Lemma A.1:** Let $G$, $H$, and $J$ be real rational transfer matrices given by

$$G = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix}, \quad H = \begin{bmatrix} A_H & B_H \\ C_H & D_H \end{bmatrix}, \quad J = \begin{bmatrix} A_J & B_J \\ C_J & D_J \end{bmatrix},$$

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such that \( A_G, A_H, \) and \( A_J \) are stable matrices and \( G^\sim H J^\sim \) is well defined. Let \( \Theta \) and \( \Gamma \) satisfy the following Lyapunov equations:

\[
\Gamma = A_G^T \Gamma A_H + C_H^T C_H \\
\Theta = A_H \Theta A_J^T + B_H B_J^T.
\]

Then the following equations hold:

\[
G^\sim H = D_G^T D_H + B_G^T \Gamma B_H \\
H J^\sim = D_H D_J^T + C_H \Theta C_J^T
\]

\[
\mathbb{P}_{\frac{1}{z}}(G^\sim H J^\sim) = \begin{bmatrix}
A_H & A_H \Theta C_J^T + B_H D_J^T \\
B_G^T \Gamma A_H + D_G^T C_H & 0
\end{bmatrix}
\]

**Proof:** First (33) will be proved.

\[
G^\sim H = \left( B_G^T \left( \frac{1}{z} I - A_G^T \right)^{-1} C_H^T + D_G^T \right) \\
\cdot (C_H(zI - A_H)^{-1}B_H + D_H)
= D_G^T D_H \\
+ D_G^T C_H(zI - A_H)^{-1}B_H + D_G^T \left( \frac{1}{z} I - A_G^T \right)^{-1} C_H^T D_H \\
+ B_G^T \left( \frac{1}{z} I - A_G^T \right)^{-1} C_H^T C_H(zI - A_H)^{-1}B_H.
\]

The Lyapunov equation for \( \Gamma \) and some simple algebra shows that

\[
C_G^T C_H = \Gamma - A_G^T \Gamma A_H
= \left( \frac{1}{z} I - A_G^T \right) \Gamma(zI - A_H) \\
+ \left( \frac{1}{z} I - A_G^T \right) \Gamma A_H + A_G^T \Gamma(zI - A_H).
\]
Combining (37) with the last term of (36) gives
\[
\left(\frac{1}{z} I - A_G^T\right)^{-1} C_G^T C_H (z I - A_H)^{-1} = \Gamma + \Gamma A_H (z I - A_H)^{-1} + \left(\frac{1}{z} I - A_G^T\right)^{-1} A_G^T \Gamma,
\]
and so
\[
G^{-1} H = D_G^T D_H + B_G^T \Gamma B_H + (B_G^T \Gamma A_H + D_G^T C_H) (z I - A_H)^{-1} B_H
+ B_G^T \left(\frac{1}{z} I - A_G^T\right) (A_G^T \Gamma B_H + C_G^T D_H).
\]
Thus (33) has been proved. The proof of (34) is analogous.

Now (35) will be proved. Since the product of anticausal transfer matrices is in \((\frac{1}{z} \mathcal{H}_2)^\perp\), (33) implies that
\[
\mathbb{P}_{\frac{1}{z} \mathcal{H}_2}(G^{-1} H J^{-1}) = \mathbb{P}_{\frac{1}{z} \mathcal{H}_2} \left[ \begin{bmatrix} A_H & B_H \\ B_G^T \Gamma A_H + D_G^T C_H & 0 \end{bmatrix} \begin{bmatrix} A_J & B_J \\ C_J & D_J \end{bmatrix} \right].
\]
Now applying (34) to the product on the right proves (35).

Using Lemma A.1, the transfer matrix products used in Theorem 3 have simple formulas given by the following lemma.

**Lemma A.2:** Let \( P_{11}, P_{12}, \) and \( P_{21} \) be defined as in (18). The following equations hold:
\[
P_{12}^* P_{12} = \Omega, \quad P_{21}^* P_{21} = \Psi, \quad \mathbb{P}_{\frac{1}{z} \mathcal{H}_2}(P_{12}^* P_{11} P_{21}^*) = 0.
\]

**Proof:** First \( P_{12}^* P_{12} = \Omega \) will be proved using (33). Note that \( X \) defined in (12) satisfies the Lyapunov equation
\[
X = (A + B_2 K)^T X (A + B_2 K) + (C_1 + D_{12} K)^T (C_1 + D_{12} K).
\]
Furthermore, the output matrix in the last term of (33) specializes to
\[
B_G^T X (A + B_2 K) + D_G^T C_H = B_G^T X A + (I + B_2^T X B_2) K = 0.
\]
So the final term of (33) must be 0. Similarly, the middle term is 0. Thus the only remaining term is the constant matrix
\[
D_G^T D_H + B_G^T X B_2 = \Omega.
\]
Proving that \( P_{21}^* P_{21} = \Psi \) is similar.
Now it will be shown that $\mathbb{E}_{\frac{1}{2}H_{2}}(P_{12}^{*}P_{11}P_{21}^{*}) = 0$. First note that there are matrices $\Gamma_{2}$ and $\Theta_{1}$ such that the following Lyapunov equations hold:

$$\begin{bmatrix} X & \Gamma_{2} \end{bmatrix} = (A + B_{2}K)^{T}\begin{bmatrix} X & \Gamma_{2} \end{bmatrix}\begin{bmatrix} A + B_{2}K & -B_{2}K \\ 0 & A + LC_{2} \end{bmatrix}$$

$$+ (C_{1} + D_{12}K)^{T}\begin{bmatrix} C_{1} + D_{12}K & -D_{12}K \end{bmatrix}$$

$$\begin{bmatrix} \Theta_{1} \\ Y \end{bmatrix} = \begin{bmatrix} A + B_{2}K & -B_{2}K \\ 0 & A + LC_{2} \end{bmatrix}\begin{bmatrix} \Theta_{1} \\ Y \end{bmatrix}(A + LC_{2})^{T}$$

$$+ \begin{bmatrix} B_{1} \\ B_{1} + LD_{21} \end{bmatrix}(B_{1} + LD_{21})^{T}$$

By (35), the output matrix of $\mathbb{E}_{\frac{1}{2}H_{2}}(P_{12}^{*}P_{11}P_{21}^{*})$ becomes

$$B_{2}^{T}\begin{bmatrix} X & \Gamma_{2} \end{bmatrix}\begin{bmatrix} A + B_{2}K & -B_{2}K \\ 0 & A + LC_{2} \end{bmatrix}$$

$$+ D_{12}^{T}\begin{bmatrix} C_{1} + D_{12}K & -D_{12}K \end{bmatrix} = \begin{bmatrix} 0 & * \end{bmatrix}.$$  

where $*$ denotes an irrelevant entry. Similarly, the input matrix is given by

$$\begin{bmatrix} A + B_{2}K & -B_{2}K \\ 0 & A + LC_{2} \end{bmatrix}\begin{bmatrix} \Theta_{1} \\ Y \end{bmatrix}C_{2}^{T} + \begin{bmatrix} B_{1} \\ B_{1} + LD_{21} \end{bmatrix}D_{21}^{T} = \begin{bmatrix} * \\ 0 \end{bmatrix}.$$  

Thus, the transfer matrix must have the form

$$\begin{bmatrix} 0 & * \\ * & * \end{bmatrix}\begin{bmatrix} * \\ 0 \end{bmatrix} = 0.$$

\[\textbf{Lemma A.3:} \text{ For } \hat{Y}, \hat{M}, \text{ and } \tilde{M} \text{ defined as in (16), the following equation holds.} \]

$$\begin{bmatrix} -\text{vec}(\hat{Y} \tilde{M}) \\ \hat{M}^{T} \otimes \tilde{M} \end{bmatrix} = \begin{bmatrix} I_{q_{2}} \otimes (A + B_{2}K) & 0_{n_{q_{2}} \times n_{p_{2}}} \\ C_{2}^{T} \otimes K & 0_{n_{p_{2}} \times 1} \end{bmatrix} \begin{bmatrix} \text{vec}(L) \end{bmatrix} I_{q_{2}} \otimes B_{2}$$

$$\begin{bmatrix} I_{q_{2}} \otimes K \\ L^{T} \otimes I_{p_{2}} \end{bmatrix} \begin{bmatrix} I_{q_{2}} \otimes (A + LC_{2})^{T} \otimes I_{p_{2}} \\ C_{2}^{T} \otimes I_{p_{2}} \end{bmatrix}.$$  

\[\textbf{Proof:} \text{ For more compact notation, let } A_{K} = A + B_{2}K \text{ and } A_{L} = A + LC_{2}. \text{ The Kronecker product } \hat{M}^{T} \otimes \tilde{M} \text{ is computed as:} \]

$$\hat{M}^{T} \otimes \tilde{M} = (\hat{M}^{T} \otimes I_{p_{2}})(I_{q_{2}} \otimes \tilde{M})$$

$$= \begin{bmatrix} A_{L}^{T} \otimes I_{p_{2}} \\ L^{T} \otimes I_{p_{2}} \end{bmatrix} \begin{bmatrix} C_{2}^{T} \otimes I_{p_{2}} \\ I_{q_{2}} \otimes I_{p_{2}} \end{bmatrix} \begin{bmatrix} I_{q_{2}} \otimes A_{K} \\ I_{q_{2}} \otimes B_{2} \end{bmatrix}.$$  

$$\begin{bmatrix} I_{q_{2}} \otimes K \\ I_{q_{2}} \otimes I_{p_{2}} \end{bmatrix}.$$
Computing $-\vec{Y \tilde{M}}$ is similar after noting that

$$-\vec{Y \tilde{M}} = \begin{pmatrix} 
M^T \otimes \left[ \begin{array}{c|c} 
A & I_n \\
K & 0_{p^2 \times n} 
\end{array} \right] 
\end{pmatrix} \vec{L}.$$
The $\mathcal{H}_2$ Control Problem for Decentralized Systems with Delays

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Abstract

This paper gives a new solution to the output feedback $\mathcal{H}_2$ problem for communication delay patterns specified by a strongly connected graph. A characterization of all stabilizing controllers satisfying the delay constraints is given and the decentralized $\mathcal{H}_2$ problem is cast as a convex model matching problem. The main result shows that the model matching problem can be reduced to a finite-dimensional quadratic program. This quadratic program can be reformulated as a finite-horizon decentralized LQG problem. A recursive state-space method for computing the optimal controller based on vectorization is given.

I. INTRODUCTION

Decentralized control problems arise when inputs to a dynamic system are chosen by multiple controllers with access to different information. In decentralized control with delays, local measurements are passed to the various controllers over a communication network with delays. As a result of the delays, some controllers will have access to measurements before others. This paper provides a new solution to the $\mathcal{H}_2$ optimal control problem, subject to communication delays, based on the Youla parametrization and vectorization.

A. Contributions

This paper solves the decentralized $\mathcal{H}_2$ problem for a class of delay patterns arising from strongly-connected communication networks. The main insight in the paper is that for such delay patterns, the $\mathcal{H}_2$ problem can be reformulated as a finite-horizon decentralized control problem. Thus, finite-dimensional optimization methods can be used to solve the original infinite-horizon optimal control problem.

To derive the finite-horizon optimal control problem, a Youla parametrization framework developed for sparsity problems, [1], is adapted to communication delay patterns. The parametrization is then used to characterize all stabilizing controllers that satisfy a given delay pattern. The corresponding decentralized model matching problem is also derived. As is typical, the model matching problem depends on the choice of a doubly-coprime factorization. For a doubly-coprime factorization based on the centralized LQG controller, it is shown that the model matching problem is, in fact, finite-dimensional.
B. Related Work

Output feedback $\mathcal{H}_2$ problems with various communication delay patterns have been previously solved using approaches based on vectorization [2] and linear matrix inequalities (LMIs) [3], [4]. The primary difference between those methods and the method in this paper, is that the infinite-horizon decentralized problems are reduced to infinite-horizon centralized problems with state dimensions that grow with the size of the delay. This paper, on the other hand, reduces the infinite-horizon problem to a decentralized finite-horizon problem of the same state-dimension, with horizon growing with the size of the delay.

Most work on decentralized optimal control with delays is based on dynamic programming. For the special case known as the one-step delay information sharing pattern, the output feedback $\mathcal{H}_2$ problem was solved in the 1970s by dynamic programming [5], [6], [7]. For more complex delay patterns, dynamic programming has extensions to decentralized state feedback [8], [9], [10], but output feedback is difficult because appropriate sufficient statistics for dynamic programming are not obvious [11], [12], [13]. Recently, methods based on POMDPs have been developed for output feedback control of nonlinear systems with general delay patterns [14], [15]. It would be interesting to see how these recent dynamic programming methods can be adapted to the $\mathcal{H}_2$ problem studied in this paper.

In contrast to dynamic programming approaches, this paper uses operator theoretic calculations to reduce the decentralized model matching problem to a quadratic program. It is an extension of [16], which uses spectral factorization to derive a similar quadratic program. This paper, on the other hand, relies on a well-chosen doubly-coprime factorization to simplify the model matching problem and remove the need for spectral factorization. Furthermore, the method in this paper applies to unstable systems, while the work in [16] is restricted to stable systems.

More broadly, this paper is influenced by recent developments in decentralized optimal control with sparsity constraints. As mentioned above, the Youla parametrization method in this paper is based on the parametrization of sparse controllers from [1]. Many of the operator theoretic calculations are modified from spectral factorization methods for sparsity constraints such as [17], [18], [19].

C. Overview

The paper is structured as follows. Section II defines the general problem studied in this paper, the decentralized $\mathcal{H}_2$ problem with a strongly-connected delay pattern. Section III gives a parametrization of all stabilizing controllers that satisfy a given delay pattern, and presents the corresponding model matching problem. In Section IV, the decentralized $\mathcal{H}_2$ problem is reduced to a quadratic program, and this program is solved by vectorization. Numerical results are given in Section V and finally conclusions are given in VI.

II. Problem

This section introduces the basic notation and the model matching problem of interest. Subsection II-C describes how common delayed information sharing patterns can be cast in the framework of this paper.
A. Preliminaries on $H_2$

Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disc of complex numbers and let $\overline{\mathbb{D}}$ be its closure. A function $G : (\mathbb{C} \cup \{ \infty \}) \setminus \overline{\mathbb{D}} \to \mathbb{C}^{p \times q}$ is in $H_2$ if it can be expanded as

$$G(z) = \sum_{i=0}^{\infty} \frac{1}{z^i} G_i,$$

where $G_i \in \mathbb{C}^{p \times q}$ and $\sum_{i=0}^{\infty} \text{Tr}(G_i G_i^*) < \infty$. Define the conjugate of $G$ by

$$G(z) = \sum_{i=0}^{\infty} z^i G_i^*.$$

For a real rational transfer matrix, $G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the conjugate is given by

$$(C(zI - A)^{-1} B + D)^\sim = B^T \left( \frac{1}{z} I - A^T \right)^{-1} C^T + D^T.$$

The space $H_2$ is a Hilbert space with inner product defined by

$$\langle G, H \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Tr} \left( G(e^{j\theta}) H(e^{j\theta})^\sim \right) d\theta = \sum_{i=0}^{\infty} \text{Tr} \left( G_i H_i^* \right),$$

where the second equality follows from Parseval’s identity.

If $\mathcal{M}$ is a subspace of $H_2$, denote the orthogonal projection onto $\mathcal{M}$ by $P_{\mathcal{M}}$.

A function $G : (\mathbb{C} \cup \{ \infty \}) \setminus \overline{\mathbb{D}} \to \mathbb{C}^{p \times q}$ is in $H_\infty$ if it is analytic, bounded, and has a well-defined limit $G(e^{j\theta}) \in \mathbb{C}^{p \times q}$ almost everywhere on the unit circle.

Let $\mathcal{R}_p$ denote the space of proper real rational transfer matrices. Furthermore, denote $\mathcal{R}_p \cap H_2$ and $\mathcal{R}_p \cap H_\infty$ by $\mathcal{R}H_2$ and $\mathcal{R}H_\infty$, respectively. Note that $\mathcal{R}H_2 = \mathcal{R}H_\infty$, since both correspond to transfer matrices with no poles outside of $\mathbb{D}$.

B. Formulation

This subsection introduces the generic problem of interest. Let $G$ be a discrete-time plant given by

$$G = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} = \begin{bmatrix} G_11 & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

with inputs of dimension $p_1, p_2$ and outputs of dimension $q_1, q_2$.

For the existence of solutions of the appropriate Riccati equations, as well as simplicity of formulas, assume that

- $(A, B_1, C_1)$ is stabilizable and detectable,
- $(A, B_2, C_2)$ is stabilizable and detectable,
- $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$. 

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\[ D_{21} \begin{bmatrix} B_1^T & D_{21}^T \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}. \]

For \( N \geq 1 \), define the space of strictly proper finite impulse response (FIR) transfer matrices by

\[ X = \mathcal{O}_{i=1}^N \frac{1}{z^i} \mathbb{C}^{p_2 \times q_2}, \]

and denote the real FIR transfer matrices by \( \mathcal{R}X = \mathcal{O}_{i=1}^N \frac{1}{z^i} \mathbb{R}^{p_2 \times q_2} \). Note that \( \frac{1}{z} L_2 \) can be decomposed into orthogonal subspaces as

\[ \frac{1}{z} L_2 = X \oplus \frac{1}{z^{N+1}} H_2. \]

Let \( S \subset \frac{1}{z} \mathbb{R}_p \) be a subspace of the form

\[ S = \mathcal{Y} \oplus \frac{1}{z^{N+1}} \mathbb{R}_p, \]

where

\[ \mathcal{Y} = \mathcal{O}_{i} \frac{1}{z^i} \mathcal{Y}_i, \]

and \( \mathcal{Y}_i \subset \mathbb{R}^{p_2 \times q_2} \) defines a sparsity pattern over matrices. Delay patterns satisfying the decomposition in (1) will be called strongly connected, since delay patterns arising from strongly-connected communication networks always have this form. (See subsection II-C.)

The set \( S \) is assumed to be quadratically invariant with respect to \( G_{22} \), which means that for all \( K \in S \), \( K G_{22} K \in S \). The key property of quadratic invariance is that \( K \in S \) if and only if \( K(I - G_{22} K)^{-1} \in S \) [2].

The decentralized \( H_2 \) problem studied in this paper is given by

\[ \min_K \| G_{11} + G_{12} K(I - G_{22} K)^{-1} G_{21} \|_{H_2}^2 \]

s.t. \( K \in S \).

The quadratic invariance assumption guarantees that the corresponding model matching problem is convex [2]. Reduction to model matching is discussed in Section III-B.

The decomposition of \( S \) in (1) is crucial for the results of this paper. The property that \( \frac{1}{z^{N+1}} \mathbb{R}_p \subset S \) implies that every measurement is available to all controller subsystems within \( N \) time steps. Concrete examples of delay patterns of this form are described in the next subsection.

For technical simplicity, controllers in this paper are assumed to be strictly proper (that is, in \( \frac{1}{z^i} \mathbb{R}_p \)). The results in this paper can be extended to non-strictly proper controllers but more complicated formulas would result.
Fig. 2. The strictly proper $N$-step delay information pattern can be visualized as a two-node graph. The delay-1 self-loops specify computational delays of 1 at each node, while the delay-$N$ edges specify communication delays. Self-loops are drawn as dashed arrows to distinguish them as denoting computational delays.

C. Communication Delay Patterns

Equation (1) can be used to model many delayed information sharing patterns. Indeed, any delay constraint of the form
\[
\begin{bmatrix}
K_{11} & \cdots & K_{1q_2} \\
\\
\vdots & & \vdots \\
K_{p1} & \cdots & K_{p2q_2}
\end{bmatrix}
\in
\begin{bmatrix}
\frac{1}{z^{t_{11}}}R_p & \cdots & \frac{1}{z^{t_{1q_2}}}R_p \\
\\
& \vdots & \\
& \frac{1}{z^{t_{p1}}}R_p & \cdots & \frac{1}{z^{t_{p2q_2}}}R_p
\end{bmatrix},
\]
for positive integers $t_{ij}$, is strongly connected. This subsection will discuss a class of strongly connected delay patterns that arise from graphs.

As an example, consider an infinite-horizon, strictly proper version of the 1-step delayed information sharing pattern studied in [5], [6], [7]. This delay constraint may be represented using (1) with $N = 1$ and $Y$ corresponding to block diagonal FIR matrices
\[
Y = \frac{1}{z}
\begin{bmatrix}
\mathbb{R}^{p_2 \times q_2} & 0 \\
0 & \mathbb{R}^{p_2 \times q_2}
\end{bmatrix}.
\]
Similarly, for $N > 1$, a strictly proper version of the $N$-step delay information sharing pattern studied in [11], [12], [13], [14] can be characterized by $Y$ of the form
\[
Y = \bigoplus_{i=1}^{N} \frac{1}{z^i}
\begin{bmatrix}
\mathbb{R}^{p_2 \times q_2} & 0 \\
0 & \mathbb{R}^{p_2 \times q_2}
\end{bmatrix}.
\]
The strictly proper $N$-step delay pattern can be described by the graph in Figure 2.

More generally, assume that communication between the controller subsystems is specified by a strongly-connected graph $(V,E)$ with self-loops at each node. Computational delays are specified by positive integers on the self-loops, while communication delays are represented by non-negative integers on the edges between distinct nodes. Requiring positive computational delays ensures that the controller is strictly proper.

A constraint space of the form (1) can be constructed as follows. For nodes $i$ and $j$ let $c_i$ be the computational delay at node $i$ and let $\tilde{d}_{ij}$ be the sum of communication delays along the directed path with shortest aggregate delay. Define the delay matrix, $d$, such that
\[
d_{ij} = c_i + \tilde{d}_{ij}.
\]
Let $N = \max\{d_{ij} : i, j \in V\} - 1.$ The corresponding constraint space is defined by

$$S = \begin{bmatrix} \frac{1}{z_{i1}}R_p & \cdots & \frac{1}{z_{i|V|}}R_p \\ \vdots & \ddots & \vdots \\ \frac{1}{z_{|V||V|}}R_p & \cdots & \frac{1}{z_{|V||V|}}R_p \end{bmatrix}.$$  

Thus, the $S$ can be decomposed as in (1) by defining

$$Y = \bigoplus_{k=1}^{N} \frac{1}{z_k} Y_k \begin{bmatrix} Y_{11}^{11} & \cdots & Y_{1|V|}^{1|V|} \\ \vdots & \ddots & \vdots \\ Y_{|V|1}^{1V} & \cdots & Y_{|V||V|}^{1V} \end{bmatrix},$$  

where

$$Y_{ij}^{k} = \begin{cases} R_{p_{1 \times q_2}} & \text{if } d_{ij} \leq k \\ 0 & \text{if } d_{ij} > k. \end{cases}$$

Necessary and sufficient conditions for such constraints to be quadratically invariant are given in [20].

In the $N$-step delay example, the delay matrix is given by

$$d = \begin{bmatrix} 1 & N + 1 \\ N + 1 & 1 \end{bmatrix}.$$  

As another example, consider the strictly proper version of the three-player chain problem discussed in [9], [16]. The graph describing the delays is given in Figure 3, leading to a delay matrix

$$d = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$  

Thus, the constraint space is defined by

$$Y = \frac{1}{z} \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & \end{bmatrix} \bigoplus \frac{1}{z^2} \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix},$$  

where, for compactness, * is used to denote a space of appropriately sized real matrices.

\(^1\)Using this convention, all measurements, $y_j(t)$, are available to all controllers by time $t + N + 1.$
III. DECENTRALIZED STABILIZATION

This section parametrizes the set of controllers $K \in \mathcal{S}$ which internally stabilize the plant $G$. The parametrization naturally leads to a convex model matching formulation of $H_2$ problem. In analogy with results on sparse transfer matrices [1], the parametrization is based on quadratic invariance and the classical Youla parametrization.

A. All Stabilizing Decentralized Controllers

A collection of stable transfer matrices, $\hat{M}$, $\hat{N}$, $\hat{X}$, $\hat{Y}$, $\tilde{M}$, $\tilde{N}$, and $\tilde{Y}$, and $\hat{Y} \in \mathcal{RH}_\infty$, defines a doubly-coprime factorization of $G_{22}$ if

$$G_{22} = \hat{N} \hat{M}^{-1} = \tilde{M}^{-1} \tilde{N}$$

and

$$\begin{bmatrix}
\hat{X} & -\hat{Y} \\
-\hat{N} & \hat{M}
\end{bmatrix}
\begin{bmatrix}
\hat{M} & \hat{Y} \\
\tilde{N} & \tilde{X}
\end{bmatrix}
= I. \quad (5)$$

As long as $(A, B_2, C_2)$ is stabilizable and detectable, there are numerous ways to construct a doubly coprime factorization of $G_{22}$.

The following theorem is well known [21].

**Theorem 1:** Assume that $G_{22}$ has a double doubly-coprime factorization of the form in (5). A controller $K \in \mathcal{R}_p$ internally stabilizes $G$ if and only if there is a transfer matrix $Q \in \mathcal{RH}_\infty$ such that

$$K = (\hat{Y} - \hat{M}Q)(\hat{X} - \tilde{N}Q)^{-1} = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}). \quad (6)$$

From [2], if $G_{22}$ is quadratically invariant under $\mathcal{S}$, then $K \in \mathcal{S}$ if and only if $K(I - G_{22}K)^{-1} \in \mathcal{S}$. As in [1], a straightforward calculation shows that

$$K(I - G_{22}K)^{-1} = (\hat{Y} - \hat{M}Q)\tilde{M}, \quad (7)$$

and thus $K \in \mathcal{S}$ if and only if $(\hat{Y} - \hat{M}Q)\tilde{M} \in \mathcal{S}$. Recalling the strongly-connected structure of $\mathcal{S}$ gives the following theorem.

**Theorem 2:** A controller $K \in \mathcal{S}$ internally stabilizes $G_{22}$ if and only if there are transfer matrices $U \in \frac{1}{z^{1+n}}\mathcal{RH}_\infty$ and $V \in \mathcal{RX}$ such that

$$K = (\hat{Y} - \hat{M}(U + V))(\hat{X} - \tilde{N}(U + V))^{-1}$$

and

$$\mathfrak{P}_X \left( (\hat{Y} - \hat{M}V)\tilde{M} \right) \in \mathcal{Y}. \quad (8)$$

**Proof:** Say that $K$ internally stabilizes $G$. Then, by Theorem 1, there exists $Q \in \mathcal{RH}_\infty$ such that (6) holds. Given $Q \in \mathcal{RH}_\infty$, there are unique $U \in \frac{1}{z^{1+n}}\mathcal{RH}_\infty$ and $V \in \mathcal{RX}$ such that $Q = U + V$. Now, (7) implies that $K \in \mathcal{S}$ if and only if

$$(\hat{Y} - \hat{M}(U + V))\tilde{M} \in \mathcal{S}.$$
Furthermore, note that \((\hat{Y} - \hat{M}(U + V))\hat{M} \in \mathcal{RH}_\infty = \mathcal{RH}_2\), and thus
\[
(\hat{Y} - \hat{M}(U + V))\hat{M} = P_\mathcal{X} \left( (\hat{Y} - \hat{M}(U + V))\hat{M} \right) + \mathbb{P}_{\frac{1}{z^n+1}} \mathcal{N}_2 \left( (\hat{Y} - \hat{M}(U + V))\hat{M} \right)
\]
\[
= P_\mathcal{X} \left( (\hat{Y} - \hat{M}(U + V))\hat{M} \right) + \mathbb{P}_{\frac{1}{z^n+1}} \mathcal{N}_2 \left( (\hat{Y} - \hat{M}(U + V))\hat{M} \right)
\]  
(9)

The second equality follows because \(U \in \frac{1}{z^n+1}\mathcal{RH}_2\), and thus \(\hat{M}U\hat{M} \in \frac{1}{z^n+1}\mathcal{RH}_2\). Since \(\mathcal{S} = \mathcal{Y} \oplus \frac{1}{z^n+1}\mathcal{R}_p\), the right of (9) is in \(\mathcal{S}\) if and only if (8) holds.

The converse is proved by reversing the steps. 

Note that (8) reduces to a finite-dimensional linear constraint on the FIR term, \(V \in \mathcal{RH}_\mathcal{X}\). The other term, \(U\), is delayed, but otherwise unconstrained.

**B. Model Matching**

Given a doubly-coprime factorization, (7) implies that the closed-loop transfer matrix is given by
\[
G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} = P_{11} + P_{12}QP_{21},
\]
where
\[
P_{11} = G_{11} + G_{12}\hat{Y}\hat{M}G_{21}
\]
\[
P_{12} = -G_{12}\hat{M}
\]
\[
P_{21} = \hat{M}G_{21}.
\]  
(10)

Using the decomposition \(Q = U + V\), with \(V \in \mathcal{RH}_\mathcal{X}\) and \(U \in \frac{1}{z^n+1}\mathcal{RH}_2\), the decentralized \(\mathcal{H}_2\) problem, (3), is equivalent to the following *model matching problem*:
\[
\min_{U,V} \quad \|P_{11} + P_{12}(U + V)P_{21}\|_{\mathcal{H}_2}^2
\]
\[
\text{s.t.} \quad U \in \frac{1}{z^n+1}\mathcal{RH}_2, \quad V \in \mathcal{RH}_\mathcal{X}
\]
\[
P_\mathcal{X} \left( (\hat{Y} - \hat{M}V)\hat{M} \right) \in \mathcal{Y}.
\]  
(11)

**IV. RESULTS**

This section first gives the main result of the paper, a quadratic programming solution of the decentralized control problem, (3). Next, the quadratic program is reduced to a finite-horizon decentralized LQG problem. A vectorization method for computing the optimal solution is also given.
A. Quadratic Programming Formulation

In the previous section, it was shown that the decentralized feedback problem is equivalent to a model matching problem, (11). It will be shown that for an appropriately chosen doubly-coprime factorization, the model matching problem reduces to a quadratic program. The reduction is due to the special nature of strongly-connected delay patterns.

Let $X$ and $Y$ be the stabilizing solutions of the Riccati equations associated with the linear quadratic regulator and Kalman filter, respectively:

$$X = C_1^T C_1 + A^T X A - A^T X B_2 (I + B_2^T X B_2)^{-1} B_2^T X A,$$

$$Y = B_1 B_1^T + A Y A^T - A Y C_2^T (I + C_2 Y C_2^T)^{-1} C_2 Y A^T.$$  

Define $\Omega = I + B_2^T X B_2$ and $\Psi = I + C_2 Y C_2^T$, the corresponding gains are given by

$$K = -\Omega^{-1} B_2^T X A,$$

$$L = -AYC_2^T \Psi^{-1}.$$  

Furthermore, $A + B_2 K$ and $A + L C_2$ are stable.

It is well known (e.g. [21]) that a doubly-coprime factorization of $G_{22}$ is given by

$$\begin{bmatrix} \hat{M} & \hat{Y} \\ \hat{N} & \hat{X} \end{bmatrix} = \begin{bmatrix} A + B_2 K & B_2 & -L \\ K & I & 0 \\ C_2 & 0 & I \end{bmatrix},$$

$$\begin{bmatrix} \hat{X} & -\hat{Y} \\ -\hat{N} & \hat{M} \end{bmatrix} = \begin{bmatrix} A + L C_2 & B_2 & -L \\ -K & I & 0 \\ -C_2 & 0 & I \end{bmatrix}.  \quad (16)$$

The following theorem is the main result of the paper.

**Theorem 3:** Consider the doubly-coprime factorization of $G_{22}$ defined by (16). The optimal solution to the decentralized $H_2$ problem defined by (3) is given by

$$K^* = (\hat{Y} - \hat{M} V^*) (\hat{X} - \hat{N} V^*)^{-1},$$

where $V^*$ is the unique optimal solution to the quadratic program

$$\begin{align*}
\min_{V \in \mathbb{R}^X} & \quad \|\Omega^{1/2} V \Psi^{1/2}\|_{H_2}^2 \\
\text{s.t.} & \quad P_X \left( (\hat{Y} - \hat{M} V) \hat{M} \right) \in \mathcal{Y}.
\end{align*}  \quad (17)$$

Furthermore, the optimal cost is given by

$$\|P_{11}\|_{H_2}^2 + \|\Omega^{1/2} V^* \Psi^{1/2}\|_{H_2}^2.$$
Proof: For the doubly-coprime factorization given by (16) the model matching matrices, (10), have state space realizations given by

\[
P_{11} = \begin{bmatrix} A + B_2 K & -B_2 K & B_1 \\ 0 & A + L C_2 & B_1 + L D_{21} \\ C_1 + D_{12} K & -D_{12} K & 0 \end{bmatrix}
\]

\[
P_{12} = -\begin{bmatrix} A + B_2 K & B_2 \\ C_1 + D_{12} K & D_{12} \end{bmatrix}
\]

\[
P_{21} = \begin{bmatrix} A + L C_2 & B_1 + L D_{21} \\ C_2 & D_{21} \end{bmatrix}
\]

(18)

For a fixed \( V \in \mathcal{R} \mathcal{X} \), the optimal \( U \in \mathcal{R} \mathcal{H}^2 \) is found by solving

\[
\min_{U \in \mathcal{R} \mathcal{H}^2} \| P_{11} + P_{12} V P_{21} + P_{12} U P_{21} \|^2_{\mathcal{H}^2}.
\]

A necessary condition for \( U \) to be optimal, given \( V \), is

\[
(P_{12} P_{11} P_{21})^\dagger + P_{12} V P_{21} P_{21}^\dagger + P_{12} U P_{21} P_{21}^\dagger \in \left( \frac{1}{\sqrt{\mathcal{N}+1}} \mathcal{H}^2 \right)^\perp.
\]

Lemma A.2 implies that

\[
P_{12} P_{12} = \Omega, \quad \text{and} \quad P_{21} P_{21}^\dagger = \Psi.
\]

Thus, the optimality condition becomes

\[
(P_{12} P_{11} P_{21})^\dagger + \Omega V \Psi + \Omega U \Psi \in \left( \frac{1}{\sqrt{\mathcal{N}+1}} \mathcal{H}^2 \right)^\perp.
\]

Lemma A.2 also implies that

\[
P_{12} P_{11} P_{21} \in \mathcal{F}_{\frac{1}{\sqrt{\mathcal{N}+1}} \mathcal{H}^2} (P_{12}^\dagger P_{11} P_{21}^\dagger) = 0.
\]

It follows that

\[
U = -\mathcal{F}_{\frac{1}{\sqrt{\mathcal{N}+1}} \mathcal{H}^2} (\Omega^{-1} P_{12}^\dagger P_{11} P_{21}^\dagger \Psi^{-1} + V) = 0.
\]

Thus, the optimal \( U \) is 0 for any \( V \in \mathcal{R} \mathcal{X} \).

Plugging \( V \) into the cost of (11) gives

\[
\| P_{11} + P_{12} V P_{21} \|^2_{\mathcal{H}^2} = \| P_{11} \|^2_{\mathcal{H}^2} + \| P_{12} V P_{21} \|^2_{\mathcal{H}^2} + 2\langle P_{11}, P_{12} V P_{21} \rangle
\]

(19)

The second term on the right of (19) simplifies as

\[
\| P_{12} V P_{21} \|^2_{\mathcal{H}^2} = \langle P_{12}^\dagger P_{12} V P_{21} P_{21}^\dagger, V \rangle = \langle \Omega V \Psi, V \rangle = \| \Omega^{1/2} V \Psi^{1/2} \|^2_{\mathcal{H}^2}.
\]
For the third term on the right of (19), \( V \in \mathcal{H}_2 \) implies that
\[
\langle P_{11}, P_{12}VP_{21} \rangle = \langle P_{12}^*P_{11}^*V, V \rangle = 0.
\]
It follows that for a fixed \( V \), the cost is given by
\[
\|P_{11}\|_{\mathcal{H}_2}^2 + \|\Omega^{1/2}V\Psi^{1/2}\|_{\mathcal{H}_2}^2.
\]
Thus, Theorem 2 and the model matching formulation, (11), imply that the optimal \( V \) must solve (17). Note that
\[
\|\Omega^{1/2}V\Psi^{1/2}\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{N} \text{Tr}(\Omega_i^{1/2}\Psi_i^{1/2} \Omega_i^{1/2}\Psi_i^{1/2})
\]
is a positive definite quadratic function of \( V \), while the constraint is linear. Thus (17) is a quadratic program and it must have a unique optimal solution.

For completeness, a state-space realization will be given for \( K \) of the form
\[
K = (\hat{Y} - \hat{M}V)(\hat{X} - \hat{N}V)^{-1}
\] (20)
with \( V \in \mathcal{R}\mathcal{X} \). Note that \( V \) has a realization
\[
V = \begin{bmatrix}
I_{q_2} & 0_{q_2 \times q_2} & \cdots & 0_{q_2 \times q_2} \\
I_{q_2} & \cdots & \cdots & \cdots \\
0_{q_2 \times q_2} & \cdots & \cdots & \cdots \\
0_{q_2 \times q_2} & 0_{q_2 \times q_2} & \cdots & 0_{q_2 \times q_2} \\
V_1 & V_2 & \cdots & V_N \\
\end{bmatrix}
\]
\[
= \begin{bmatrix}
A_V & B_V \\
C_V & 0 \\
\end{bmatrix}
\]
Standard state-space manipulations show that
\[
K = \begin{bmatrix}
A + B_2K + LC_2 & B_2C_V & -L \\
B_VC_2 & A_V & -B_V \\
K & C_V & 0 \\
\end{bmatrix}
\] (21)
Thus, \( K \) has a state-space realization of order \( n + q_2N \). If \( N \) is the smallest integer such that a decomposition of the form (1) holds, then \( K \) must have entries in \( \frac{1}{n+q_2} \mathcal{R}_p \). In this case, any minimal realization must have order at least \( N \). Thus, the order of the realization in (21) is within a constant factor of the minimal realization order.

### B. Finite-Horizon Decentralized LQG

In this subsection, the quadratic program from Theorem 3 is cast as a finite-horizon decentralized LQG problem. It is hoped that special cases of the resulting finite-horizon problem could be solved more explicitly in the future, perhaps using techniques similar to those in [22].
Theorem 4: The optimal feedback controller, $K^*$, is uniquely determined by the first $N$ terms of its power series expansion, $\mathbb{P}_X(K^*)$. Furthermore, $\mathbb{P}_X(K^*)$ is the optimal solution to the following decentralized LQG problem:

$$
\min_H \mathbb{E} \left[ (Kx_N - u_N)^T \Omega (Kx_N - u_N) \right]
$$

s.t. $x_{i+1} = Ax_i + B_2u_i - Lw_i, \quad x_0 = 0$

$$
y_i = C_2x_i + w_i
$$

$$
u_i = \sum_{j=1}^i H_j y_{i-j}
$$

$$
H_i \in \mathcal{Y}_i,
$$

where the noise terms are independent Gaussian random variables with mean 0 and covariance $\Psi$.

Proof: First assume that a feasible controller has the form in (20). In particular, $K^*$ has this form. Solving for $V$ shows that

$$
V = (K\tilde{N} - \tilde{M})^{-1}(K\tilde{X} - \tilde{Y}).
$$

Furthermore, since all the transfer functions are proper and $V \in X$,

$$
V = \mathbb{P}_X \left( (\mathbb{P}_X(K)\tilde{N} - \tilde{M})^{-1}(\mathbb{P}_X(K)\tilde{X} - \tilde{Y}) \right).
$$

Plugging the right of this equation into (20) shows that $K$ can be computed in terms of $\mathbb{P}_X(K)$.

Now the LQG problem for $\mathbb{P}_X(K)$ will be derived. For simpler notation, let $H = \mathbb{P}_X(K)$. A bit of algebra using (5) shows that

$$
(\tilde{M} - H\tilde{N})^{-1}(\tilde{Y} - H\tilde{X})
$$

$$
= \tilde{M}^{-1}(I - HG_{22})^{-1}((I - HG_{22})\tilde{Y} + H\tilde{M}^{-1}(\tilde{N}\tilde{Y} - \tilde{M}\tilde{X}))
$$

$$
= \tilde{M}^{-1}\tilde{Y} - \tilde{M}^{-1}(I - HG_{22})^{-1}H\tilde{M}^{-1}.
$$

The matrices on the right have realizations given by

$$
\tilde{M}^{-1} = \begin{bmatrix} A & -B_2 \\ K & I \end{bmatrix}, \quad \tilde{M}^{-1} = \begin{bmatrix} A & -L \\ C_2 & I \end{bmatrix},
$$

$$
\tilde{M}^{-1}\tilde{Y} = \begin{bmatrix} A \\ K \end{bmatrix} - \begin{bmatrix} -L \\ 0 \end{bmatrix}.
$$

Combining (23) and (24) with the state-space realizations shows that

$$
V = \mathbb{P}_X \left( \begin{bmatrix} A & -L \\ K & 0 \end{bmatrix} \right)
$$

$$
+ \mathbb{P}_X \left( \begin{bmatrix} A & B_2 \\ K & -I \end{bmatrix} H(I - G_{22}H)^{-1} \begin{bmatrix} A & -L \\ C_2 & I \end{bmatrix} \right).
$$
Thus, the convolution $z_i = \sum_{j=1}^{i} V_j w_{i-j}$ has a time-domain representation given by
\[
x_{i+1} = A x_i + B_2 u_i - L w_i, \quad x_0 = 0
\]
\[
y_i = C_2 x_i + w_i
\]
\[
u_i = \sum_{j=1}^{i} H_j y_{i-j}
\]
\[
z_i = K x_i - u_i.
\]
Furthermore, recall that $K \in S$ if and only if $H = P_X(K) \in \mathcal{Y}$.

Finally, say that $w_i$ are independent Gaussian random variables with zero mean and covariance $\Psi$. The proof is completed by noting that
\[
\|\Omega^{1/2} V \Psi^{1/2}\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{N} \text{Tr}(\Omega^{1/2} V_i^{1/2} \Psi^{1/2} V_i^{1/2} \Omega^{1/2})
\]
\[
= \sum_{i=1}^{N} \mathbb{E}[\text{Tr}(\Omega^{1/2} V_i w_{N-i} w_{N-i}^T V_i^{1/2})]
\]
\[
= \mathbb{E}[\text{Tr}(\Omega^{1/2} z_N z_N^T \Omega^{1/2})]
\]
\[
= \mathbb{E}[(K x_N - u_N)^T \Omega (K x_N - u_N)].
\]

The finite-horizon LQG problem, (22), is somewhat non-standard, since the policy, $H$, is time-invariant. Typically, finite-horizon control problems allow for time-varying policies, i.e.
\[
u_i = \sum_{j=1}^{i} H_{i,j} y_{i-j}.
\]
Thus, more work is needed to see if the method from [22], which assumes that the controller is time-varying, can be modified to handle problems of the form (22).

While specialized methods for solving (22) do not appear to exist, it can be solved using results from team decision theory. Consider the model matching parameter
\[
J = -\mathbb{P}_X((\hat{Y} - \hat{M} V) \hat{M}) = -\mathbb{P}_X((I - H G_{22})^{-1} H).
\]
Using (25), the quadratic program of Theorem 3 reduces to a finite-dimensional decentralized model matching problem:
\[
\min_{J \in \mathcal{Y}} \|\Omega^{1/2} \mathbb{P}_X(\hat{M}^{-1} \hat{Y} + \hat{M}^{-1} J \hat{M}^{-1}) \Psi^{1/2}\|_{\mathcal{H}_2}^2.
\]
This problem is a special case of the static team decision problem [23], [24], which in turn reduces to a system of linear equations. The system of equations will be large, however, since the recursive structure of the model matching problem is not exploited.
Chapter 1: Introduction

In this subsection, the quadratic program of Theorem 3 will be cast as a finite-horizon state-feedback problem using vectorization techniques. The finite-dimensional model matching problem from (27) can also be transformed into a state-feedback problem by a similar vectorization method.

First, by defining $R = \Psi \otimes \Omega$, the vectorized form of the cost function becomes

$$\|\Omega^{1/2} V \Psi^{1/2}\|_H^2 = \sum_{i=1}^N \text{Tr}(\Omega_i V_i \Psi_i^T) = \sum_{i=1}^N \text{vec}(V_i)^T R \text{vec}(V_i).$$

(28)

Recall the definition of the model matching parameter, $J$, in (26), and let its series expansion be given by

$$J = \sum_{i=1}^N \frac{1}{z_i} J_i.$$

If $\mathcal{Y}$ is defined by (2), then the model matching constraint of (11) is equivalent to $J_i \in \mathcal{Y}_i$. The vectorized form of $J$ is computed from

$$\text{vec}(-\hat{Y} \hat{M} + \hat{M} V \hat{M}) = -\text{vec}(\hat{Y} \hat{M}) + (\hat{M}^T \otimes \hat{M}) \text{vec}(V).$$

By Lemma A.3 in the appendix, the terms of $J_i$ can be computed by the recursion

$$\begin{align*}
x_1 &= \begin{bmatrix} \text{vec}(L) \\ 0_{np_2 \times 1} \end{bmatrix} \\
x_{i+1} = A_v x_i + B_v \text{vec}(V_i) \\
\text{vec}(J_i) = C_v x_i + \text{vec}(V_i),
\end{align*}$$

(29)

where

$$\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} = \begin{bmatrix} I_{q_2} \otimes (A + B_2 K) & 0_{nq_2 \times np_2} & I_{q_2} \otimes B_2 \\ C_2^T \otimes K & (A + L C_2)^T \otimes I_{p_2} & C_2^T \otimes I_{p_2} \\ I_{q_2} \otimes K & L^T \otimes I_{p_2} & I_{p_2 q_2} \end{bmatrix}.$$  

Note that $A_v$ is stable since $A + B_2 K$ and $A + L C_2$ are.

Now let $E_i$ and $F_i$ be matrices with columns that form orthonormal bases of $\text{vec}(\mathcal{Y}_i)$ and $\text{vec}(\mathcal{Y}_i^\perp)$, respectively. The term $\text{vec}(V_i)$ can then be decomposed as

$$\text{vec}(V_i) = E_i u_i + F_i u_i^\perp,$$

for some vectors $u_i$ and $u_i^\perp$.

Using (29), the constraint that $J_i \in \mathcal{Y}_i$ can be equivalently cast as

$$F_i^T (C_v x_i + \text{vec}(V_i)) = F_i^T C_v x_i + u_i^\perp = 0.$$

(30)
Plugging (30) into the cost (28) and the recursion (29) leads to the following optimal control problem:

$$\min_{u} \sum_{i=1}^{N} \left[ x_i^T \quad u_i^T \right] \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

(31)

s.t.

$$x_1 = \begin{bmatrix} \text{vec}(L) \\ 0_{np_1 \times 1} \end{bmatrix}$$

$$x_{i+1} = A_i x_i + B_i u_i,$$

where the time-varying matrices are given by

$$\begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} = \begin{bmatrix} C_v F_i F_i^T R_i F_i C_v & -C_v F_i F_i^T R E_i \\
-C_v F_i F_i^T R E_i & E_i^T R E_i \end{bmatrix}$$

$$\begin{bmatrix} A_i & B_i \end{bmatrix} = \begin{bmatrix} A_v - B_v F_i F_i^T C_v & B_v E_i \end{bmatrix}.$$
A. The Chain Problem

Recall the three-player chain structure from Figure 3 with constraint specified by (4). Consider the plant given by

\[
A = \begin{bmatrix}
1.5 & 1 & 0 \\
1 & 1.5 & 1 \\
0 & 1 & 1.5
\end{bmatrix}, \quad B = \begin{bmatrix}
I_{3 \times 3} & 0_{3 \times 3} & I_{3 \times 3}
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
I_{3 \times 3} \\
0_{3 \times 3} \\
I_{3 \times 3}
\end{bmatrix}, \quad D = \begin{bmatrix}
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\
0_{3 \times 3} & I_{3 \times 3} & I_{3 \times 3} \\
0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3}
\end{bmatrix}.
\]

For comparison purposes, the optimal $\mathcal{H}_2$ norm was computed using model matching from this paper and the LMI method of [3], [4]. In all both cases the norm was found to be 34.9304. In contrast, the centralized controller gives a norm of 24.236.

B. Increasing Delays

Consider the plant defined by

\[
G = \begin{bmatrix}
0.9 & 0 & 0.1 & 0 & 0 & \vdots & 10 & 0 \\
0 & -2 & 20 & 0 & 0 & \vdots & 0 & 1 \\
2 & -2 & 0 & 0 & 0 & \vdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \vdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \vdots & 0 & 1 \\
10 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\
0 & 0.1 & 0 & 0 & 0 & \vdots & 0 & 0
\end{bmatrix}.
\]

For $N \geq 1$, let $\mathcal{K}_{\text{Tri}}^N$, $\mathcal{K}_{\text{Di}}^N$, and $\mathcal{K}_{\text{Low}}^N$ solve the decentralized $\mathcal{H}_2$ problem, (3), with constraints of the form

\[
\mathcal{K}_{\text{Tri}}^N \in \frac{1}{z^{N+1}} \mathcal{R}_p \oplus \bigoplus_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix}
\mathbb{R} & 0 \\
\mathbb{R} & \mathbb{R}
\end{bmatrix}
\]

\[
\mathcal{K}_{\text{Di}}^N \in \frac{1}{z^{N+1}} \mathcal{R}_p \oplus \bigoplus_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix}
\mathbb{R} & 0 \\
0 & \mathbb{R}
\end{bmatrix}
\]

\[
\mathcal{K}_{\text{Low}}^N \in \frac{1}{z^{N+1}} \mathcal{R}_p \oplus \bigoplus_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix}
0 & 0 \\
0 & \mathbb{R}
\end{bmatrix}
\]

The resulting norms are plotted in Figure 4.
Fig. 4. This plot shows the closed-loop norm for $K_{\text{Tri}}^N$, $K_{\text{Di}}^N$, and $K_{\text{Low}}^N$. For a given $N$, the controllers with fewer sparsity constraints give rise to lower norms. As $N$ increases, all of the norms increase monotonically since the controllers have access to less information. The dotted lines correspond to the optimal norms for sparsity structures given in (32).

As $N \to \infty$, the resulting controllers appear to approach optimal sparse controllers

$$
K_{\text{Tri}}^\infty \in \begin{bmatrix}
\frac{1}{2}R_p & 0 \\
\frac{1}{2}R_p & \frac{1}{2}R_p
\end{bmatrix},
K_{\text{Di}}^\infty \in \begin{bmatrix}
\frac{1}{2}R_p & 0 \\
0 & \frac{1}{2}R_p
\end{bmatrix},
K_{\text{Low}}^\infty \in \begin{bmatrix}
0 & 0 \\
0 & \frac{1}{2}R_p
\end{bmatrix}
$$

(32)

which can be computed by the vectorization technique from [2]. Evidence for the convergence is shown by the fact that the norms limit to the values computed for the sparse controllers (Figure 4).
VI. Conclusion

This paper derives a novel solution for a class of output feedback $\mathcal{H}_2$ control problems with strongly connected communication delay patterns. First, all stabilizing decentralized controllers are characterized via doubly coprime factorization. Then, by a standard change of variables, the $\mathcal{H}_2$ problem is cast as a convex model matching problem. The main theorem shows that for a doubly coprime factorization based on the LQR and Kalman filter gains, the model matching problem reduces to a finite-dimensional quadratic program. The quadratic program can be transformed further into a finite-horizon decentralized LQG problem. A solution to the quadratic program based on vectorization is also presented.

Many open problems remain for optimal control with strongly-connected delay constraints. For specialized delay patterns, it may be possible to find more efficient and explicit solutions to the finite-horizon LQG problem of the paper. It is also likely that optimization methods for analyzing [26] and designing [27] delay patterns can be modified to apply to the quadratic program in this paper. Furthermore, work is needed to understand how the controllers in this paper could be realized [28], [29], [30] and computed [31], [32] in a distributed fashion. With a distributed implementation in place, model reduction techniques might also be useful for decreasing the amount of local computation required.

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References

APPENDIX

This appendix collects state-space formulas that are useful for deriving the results in the paper.

**Lemma A.1:** Let $G$, $H$, and $J$ be real rational transfer matrices given by

$$G = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix}, \quad H = \begin{bmatrix} A_H & B_H \\ C_H & D_H \end{bmatrix},$$

$$J = \begin{bmatrix} A_J & B_J \\ C_J & D_J \end{bmatrix},$$
such that $A_G$, $A_H$, and $A_J$ are stable matrices and $G^\sim \cdot H J^\sim$ is well defined. Let $\Theta$ and $\Gamma$ satisfy the following Lyapunov equations:

\[
\Gamma = A_G^T \Gamma A_H + C_G^T C_H
\]

\[
\Theta = A_H \Theta A_J^T + B_H B_J^T.
\]

Then the following equations hold:

\[
G^\sim \cdot H = D_G^T D_H + B_G^T \Gamma B_H
\]  \hspace{1cm} (33)

\[
H J^\sim = D_H D_J^T + C_H \Theta C_J^T
\]  \hspace{1cm} (34)

\[
\mathbb{P}_{\frac{1}{z} H z} (G^\sim \cdot H J^\sim) = \begin{bmatrix}
A_H & A_H \Theta C_J^T + B_H D_J^T \\
B_G^T \Gamma A_H + D_G^T C_H & 0
\end{bmatrix}^\sim
\]  \hspace{1cm} (35)

**Proof:** First (33) will be proved.

\[
G^\sim \cdot H = \left( B_G^T \left( \frac{1}{z} I - A_G^T \right)^{-1} C_G^T + D_G^T \right)
\]

\[
\cdot (C_H (zI - A_H)^{-1} B_H + D_H)
\]

\[
= D_G^T D_H
\]  \hspace{1cm} (36)

\[
+ D_G^T C_H (zI - A_H)^{-1} B_H + B_G^T \left( \frac{1}{z} I - A_G^T \right)^{-1} C_G^T D_H
\]

\[
+ B_G^T \left( \frac{1}{z} I - A_G^T \right)^{-1} C_G^T C_H (zI - A_H)^{-1} B_H.
\]

The Lyapunov equation for $\Gamma$ and some simple algebra shows that

\[
C_G^T C_H = \Gamma - A_G^T \Gamma A_H
\]

\[
= \left( \frac{1}{z} I - A_G^T \right) \Gamma (zI - A_H)
\]  \hspace{1cm} (37)

\[
+ \left( \frac{1}{z} I - A_G^T \right) \Gamma A_H + A_G^T \Gamma (zI - A_H).
\]
Combining (37) with the last term of (36) gives
\[
\left(\frac{1}{z} I - A_G^T\right)^{-1} C_G^T C_H (z I - A_H)^{-1} = \Gamma + \Gamma A_H (z I - A_H)^{-1} + \left(\frac{1}{z} I - A_G^T\right)^{-1} A_G^T \Gamma,
\]
and so
\[
G^- H = D_G^T D_H + B_G^T \Gamma B_H + (B_G^T \Gamma A_H + D_G^T C_H)(z I - A_H)^{-1} B_H
\]
\[
+ B_G^T \left(\frac{1}{z} I - A_G^T\right) (A_H^T \Gamma B_H + C_H^T D_H).
\]
Thus (33) has been proved. The proof of (34) is analogous.

Now (35) will be proved. Since the product of anticausal transfer matrices is in \( \left(\frac{1}{z} \mathcal{H}_2\right)^\perp\), (33) implies that
\[
\mathbb{P}_{\frac{1}{z} \mathcal{H}_2} (G^- H J^-)
\]
\[
= \mathbb{P}_{\frac{1}{z} \mathcal{H}_2} \left(\begin{bmatrix} A_H & B_H \\ B_G^T \Gamma A_H + D_G^T C_H & 0 \end{bmatrix} \begin{bmatrix} A_J & B_J \\ C_J & D_J \end{bmatrix}^{-1}\right).
\]
Now applying (34) to the product on the right proves (35).

Using Lemma A.1, the transfer matrix products used in Theorem 3 have simple formulas given by the following lemma.

**Lemma A.2:** Let \( P_{11}, P_{12}, \) and \( P_{21} \) be defined as in (18). The following equations hold:
\[
P_{12}^* P_{12} = \Omega, \quad P_{21}^* P_{21} = \Psi, \quad \mathbb{P}_{\frac{1}{z} \mathcal{H}_2}(P_{12}^* P_{11} P_{21}^*) = 0.
\]

**Proof:** First \( P_{12}^* P_{12} = \Omega \) will be proved using (33). Note that \( X \) defined in (12) satisfies the Lyapunov equation
\[
X = (A + B_2 K)^T X (A + B_2 K) + (C_1 + D_{12} K)^T (C_1 + D_{12} K).
\]
Furthermore, the output matrix in the last term of (33) specializes to
\[
B_G^T X (A + B_2 K) + D_{12}^T (C_1 + D_{12} K) = B_G^T X A + (I + B_G^T X B_2) K
\]
\[
= 0.
\]
So the final term of (33) must be 0. Similarly, the middle term is 0. Thus the only remaining term is the constant matrix
\[
D_{12}^T D_{12} + B_2^T X B_2 = \Omega.
\]
Proving that \( P_{21} P_{21}^* = \Psi \) is similar.
Now it will be shown that \( P_{1} P_{12} P_{21} = 0 \). First note that there are matrices \( \Gamma_{2} \) and \( \Theta_{1} \) such that the following Lyapunov equations hold:

\[
\begin{bmatrix}
X & \Gamma_{2}
\end{bmatrix} = (A + B_{2} K)^{T} \begin{bmatrix}
X & \Gamma_{2}
\end{bmatrix} \begin{bmatrix}
A + B_{2} K & -B_{2} K \\
0 & A + L C_{2}
\end{bmatrix}
+ (C_{1} + D_{12} K)^{T} \begin{bmatrix}
C_{1} + D_{12} K & -D_{12} K
\end{bmatrix}
\]

By (35), the output matrix of \( P_{1} P_{12} P_{21} \) becomes

\[
\begin{bmatrix}
A + B_{2} K & -B_{2} K \\
0 & A + L C_{2}
\end{bmatrix}
+ D_{12}^{T} \begin{bmatrix}
C_{1} + D_{12} K & -D_{12} K
\end{bmatrix} \begin{bmatrix}
0 & *
\end{bmatrix},
\]

\[
\text{where} \; * \text{\; denotes an irrelevant entry. Similarly, the input matrix is given by}
\]

\[
\begin{bmatrix}
A + B_{2} K & -B_{2} K \\
0 & A + L C_{2}
\end{bmatrix}
+ \begin{bmatrix}
C_{2}^{T} \Theta_{1} \\
0
\end{bmatrix} \begin{bmatrix}
B_{1} \\
B_{1} + L D_{21}
\end{bmatrix} = \begin{bmatrix}
* \\
0
\end{bmatrix}.
\]

Thus, the transfer matrix must have the form

\[
\begin{bmatrix}
0 & * & * & *
\end{bmatrix} \begin{bmatrix}
0 & *
\end{bmatrix} = 0.
\]

\[\blacksquare\]

**Lemma A.3:** For \( \check{Y}, \check{M}, \text{and} \check{\check{M}} \) defined as in (16), the following equation holds.

\[
\begin{bmatrix}
-\text{vec}(\check{Y} \check{M}) \\
\check{M}^{T} \otimes \check{M}
\end{bmatrix} =
\begin{bmatrix}
I_{q_{2}} \otimes (A + B_{2} K) & 0_{n q_{2} \times n p_{2}} & \text{vec}(L) \; I_{q_{2}} \otimes B_{2} \\
C_{2}^{T} \otimes K & (A + L C_{2})^{T} \otimes I_{p_{2}} & 0_{n p_{2} \times 1} \; C_{2}^{T} \otimes I_{p_{2}} \\
I_{q_{2}} \otimes K & L^{T} \otimes I_{p_{2}} & 0_{p_{2} q_{2} \times 1} \; I_{p_{2} q_{2}}
\end{bmatrix}.
\]

**Proof:** For more compact notation, let \( A_{K} = A + B_{2} K \) and \( A_{L} = A + L C_{2} \). The Kronecker product \( \check{M}^{T} \otimes \check{M} \) is computed as:

\[
\check{M}^{T} \otimes \check{M} = (\check{M}^{T} \otimes I_{p_{2}})(I_{q_{2}} \otimes \check{M})
\]

\[
= \begin{bmatrix}
A_{L}^{T} \otimes I_{p_{2}} & C_{2}^{T} \otimes I_{p_{2}} \\
L^{T} \otimes I_{p_{2}} & I_{q_{2}} \otimes I_{p_{2}}
\end{bmatrix} \begin{bmatrix}
I_{q_{2}} \otimes A_{K} \\
I_{q_{2}} \otimes B_{2}
\end{bmatrix}.
\]
Computing $-\text{vec}(\hat{Y}\tilde{M})$ is similar after noting that

$$-\text{vec}(\hat{Y}\tilde{M}) = \left(\tilde{M}^T \otimes \begin{bmatrix} A_K & I_n \\ K & 0_{p_2 \times n} \end{bmatrix}\right) \text{vec}(L).$$