Short-wave asymptotics of the information entropy of a circular membrane

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Abstract

The spreading of the position and momentum probability distributions for the stable free oscillations of a circular membrane of radius $l$ is analyzed by means of the associated Boltzmann-Shannon information entropies in the correspondence principle limit ($n \to \infty$, $m$ fixed), where the numbers $(n, m)$, $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, uniquely characterize an oscillation of this two-dimensional system. This is done by solving the short-wave asymptotics of the physical entropies in the two complementary spaces, which boils down to the calculation of the asymptotic behaviour of certain entropic integrals of Bessel functions. It is rigorously shown that the position and momentum entropies behave as $2 \ln(l) + \ln(4\pi) - 2$ and $\ln(n) - 2 \ln(l) + \ln(2\pi^2)$ when $n \to \infty$, respectively. So the total entropy sum has a logarithmic dependence on $n$ and it does not depend on the membrane radius. The former indicates that the ordering of short-wavelength oscillations is exactly identical for the entropic sum and the single-particle energy. The latter holds for all oscillations of the membrane because of the uniform scaling invariance of the entropy sum.

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1 Introduction

The study of the spreading of classical and quantum probability distributions for arbitrary states of physical systems in $D$ dimensions ($D \geq 1$) by means of the Boltzmann-Shannon information entropy is a topic of increasing interest from both fundamental and practical points of view. Indeed, for example, this topic for the ground states is strongly connected with the entropic uncertainty relationship [Bialynicki-Birula et al., 1975, Beckner, 1975], a stronger version of the Heisenberg uncertainty principle, which enables us to get good estimates of physical quantities of the system. Also, for highly-excited states this question is a natural way to gain insight into the domain where the transition from quantum mechanics to classical physics is produced, the correspondence principle operates and the chaotic dynamics takes place.

However the analytical determination of the information entropies is a non-trivial problem in both position and momentum spaces, even for simple quantum systems such as the single-particle ones in central potentials [Yáñez et al., 1994, Majernik et al., 1996, Dehesa et al., 2001 (c)]; see [Dehesa et al., 2001 (a)] for a recent survey. Indeed, the wavefunctions of the harmonic oscillator and of hydrogenic systems are controlled by the Hermite, Laguerre and Gegenbauer polynomials; so, the position and momentum entropies of these systems are expressed in terms of the so-called entropy-like integrals of these families of classical orthogonal polynomials [Aptekarev et al., 1995, Sánchez-Ruiz, 1997 (b), Dehesa et al., 1998, Buyarov et al., 2000]. The explicit expression of these mathematical objects has been computed only recently for certain classical orthogonal polynomials although in an involved form. This is mainly due to the fact that the $L^p$-norms of these polynomials cannot be easily computed in general, except in the asymptotic (i.e. when the degree of the polynomial tends to infinity) case [Aptekarev et al., 1995].

In this paper we shall study the spreading of the probability distribution of stable oscillations of a circular membrane by means of the information entropy of their corresponding wavefunctions in both position and momentum spaces. This problem is equivalent to that of a quantum-mechanical single-particle system with the $z$ component fixed. Now, the associated wavefunctions do not involve any longer a classical orthogonal polynomial but a Bessel function of first kind. We shall show that the physical position and momentum entropies of the membrane can be reduced to certain entropy-like integrals of Bessel functions. These novel mathematical objects cannot be explicitly computed in general. Here, we calculate them in the asymptotic case, which allows us to estimate both the position and momentum entropies in the domain of short-wavelength oscillations of the membrane on a
well-founded mathematical basis; see Sections 2 and 3, respectively. It is interesting to point out that entropy-like integrals of non-polynomial functions are very scarce in the literature; the only known cases apart from the Bessel one here considered, are those which involve the Airy functions and the Mac-Donald functions which have been recently encountered in the calculation of the information entropies of single-particle systems submitted to a linear [Sánchez-Ruiz, 1997 (a)] and a Toda-like potential [Dehesa et al., 2001 (b)], respectively.

2 Asymptotics in the position space

The Schrodinger equation for the stable free oscillation of a circular membrane of a radius $l$ is described by the two-dimensional homogeneous Helmholtz equation

$$\nabla^2 \Psi(x, y) + k^2 \Psi(x, y) = 0,$$

with the Dirichlet boundary condition

$$\Psi(x, y)|_{x^2 + y^2 = l^2} = 0.$$ 

This equation has a discrete set of eigenvalues $k_{mn}^2$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$, with the corresponding eigenfunctions $\Psi_{mn}$. Due to the radial symmetry of the problem it can be solved by separation of variables in polar coordinates $(r, \varphi)$, yielding

$$\Psi_{mn}(x, y) = \frac{1}{\sqrt{2\pi}} R_{mn}(r) e^{im\varphi}, \quad m \in \mathbb{Z}, \ n \in \mathbb{N}, \quad (1)$$

where

$$R_{mn}(r) = C_{mn} J_m(k_{mn} r). \quad \quad (2)$$

Here $J_m$ is the Bessel function of the first kind and order $m$, and

$$lk_{m1} < lk_{m2} < \ldots$$

are its positive zeros. The constant $C_{mn}$ is obtained from the normalization condition

$$\int_0^l R_{mn}^2(r) r \, dr = 1. \quad \quad (3)$$

The information entropy of the wave function $\Psi_{mn}$ is defined as

$$S(\Psi_{mn}) = -\int\int_{x^2 + y^2 \leq l^2} |\Psi_{mn}(x, y)|^2 \ln |\Psi_{mn}(x, y)|^2 \, dx \, dy. \quad \quad (4)$$

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With account of (1) we have
\[ S(\Psi_{mn}) = S_{mn} + \ln(2\pi), \quad (5) \]
where
\[ S_{mn} = -\int_0^l |R_{mn}(r)|^2 \ln |R_{mn}(r)|^2 r \, dr. \]

Our main goal is to investigate the behavior of the entropy component \( S_{mn} \) when \( n \to \infty \) and \( m \in \mathbb{Z} \) is fixed. In fact, we will establish that for every \( m \in \mathbb{Z} \), there exists the limit
\[ \lim_{n \to \infty} S_{mn} = S = \ln \left( 2l^2 \right) - 2, \quad (6) \]
which together with (5) allows us to find the following asymptotic value for the position entropy of the membrane
\[ \lim_{n \to \infty} S(\Psi_{mn}) = 2\ln l + \ln (4\pi) - 2, \quad m \text{ fixed.} \quad (7) \]

Two observations about the semiclassical limit of the position entropy follow. First, it does not depend on the quantum number \( m \), what is consistent with the fact that the set of eigenfunctions (1) behaves asymptotically as the trigonometric system, for which the entropy is constant. Then, the position entropy of a specific membrane has for all oscillations an absolute limit, given by Eq. (7). Second, it does depend on the radius of the membrane so that when \( l \) increases, the entropy logarithmically increases; this indicates that the oscillation of the membrane gets more spread (i.e., its wavelength is larger). It is important to realize that this logarithmic dependence on \( l \) holds not only for short-wavelength oscillations \( (n \to \infty) \) but for any oscillation of the membrane because of the property of invariance of the wave equation under uniform scaling of the coordinates.

The proof of (6) goes according to the following scheme. First, we will find the \( L^q \) norm (Renyi entropy) of the radial component \( R_{mn} \) of the wave function \( \Psi_{mn} \). Then a limit transition allows us to conclude (6).

For \( 0 < q < 2 \), consider the value
\[ N_{mn}(q) = \int_0^l |R_{mn}(r)|^{2q} r \, dr. \]

Using the asymptotics of the Bessel function (see e.g. [Jeffrey, 1995, Formula 17.4.1.1]),
\[ J_m(z) = \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{\pi m}{2} - \frac{\pi}{4} \right) \left( 1 + O(z^{-1/2}) \right), \quad z \to \infty, \quad (8) \]
we conclude that for \( n \to \infty \),

\[
N_{mn}(q) = \int_0^l \left| C_{mn} \sqrt{\frac{2}{\pi k_{mn}}} \cos \left( k_{mn} r - \frac{m \pi}{2} - \frac{\pi}{4} \right) \right|^{2q} r \, dr + o(1)
\]

\[
= C_{mn}^{2q} \left( \frac{2}{\pi k_{mn}} \right)^q \int_0^l \left| \cos \left( k_{mn} r - \frac{m \pi}{2} - \frac{\pi}{4} \right) \right|^{2q} r^{1-q} \, dr + o(1)
\]

Since

\[
k_{mn} \sim \frac{\pi n}{l}, \quad n \to \infty,
\]

we just have to consider the integrals

\[
\left( \frac{1}{\pi} \right)^{2-q} \int_0^\pi \left| \cos(n \theta) \right|^{2q} \theta^{1-q} \, d\theta.
\]

Using Lemma 2.1 from [Aptekarev et al., 1995], we obtain that the integral above converges to

\[
\int_0^\pi \left| \cos(\theta) \right|^{2q} \theta^{1-q} \, d\theta = B(q + 1/2, 1/2) \frac{1}{\pi} \frac{\pi^{2-q}}{2-q},
\]

where \( B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y) \) is the beta function. Thus,

\[
N_{mn}(q) = C_{mn}^{2q} \left( \frac{2}{\pi k_{mn}} \right)^q l^{2-q} B(q + 1/2, 1/2) \frac{1}{\pi} \frac{\pi^{2-q}}{2-q} + o(1), \quad n \to \infty.
\]

Since by (3), \( N_{mn}(1) = 1 \), we obtain that

\[
C_{mn}^2 \sim \frac{\pi k_{mn}}{l}, \quad n \to \infty.
\]

Using it in (10) again we see that

\[
\lim_{n \to \infty} N_{mn}(q) = N(q) = 2^q l^{2-2q} B(q + 1/2, 1/2) \frac{1}{\pi(2-q)}.
\]

In order to get (6) we need to make the last step: it is well known and easy to verify that

\[
S_{mn} = -\frac{\partial}{\partial q} \ln N_{mn}(q) \bigg|_{q=1},
\]

so that

\[
S = \lim_{n \to \infty} S_{mn} = -\frac{\partial}{\partial q} \ln N(q) \bigg|_{q=1}.
\]
Straightforward computation shows that
\[
\frac{\partial}{\partial q} \ln N(q) = \ln 2 - 2 \ln l + \psi(q + 1/2) - \psi(q + 1) + \frac{1}{2 - q},
\]
where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) is the digamma function. Since \( \psi(3/2) - \psi(2) = 1 - \ln 4 \), (6) follows.

3 Asymptotics in the momentum space

Here we shall calculate the asymptotics of the momentum information entropy of the membrane. The wave function now is \( \Xi_{mn} \), the two-dimensional Fourier transform of \( \Psi_{mn} \):
\[
\Xi_{mn}(p_x, p_y) = \frac{1}{2\pi} \iint_{x^2+y^2 \leq l^2} \Psi_{mn}(x, y) e^{i(xp_x + yp_y)} dx dy.
\]
We shall show that the momentum entropy defined by
\[
S(\Xi_{mn}) = -\iint_{\mathbb{R}^2} |\Xi_{mn}(p_x, p_y)|^2 \ln |\Xi_{mn}(p_x, p_y)|^2 dp_x dp_y,
\]
has the following asymptotic behavior
\[
S(\Xi_{mn}) \sim \ln n - 2 \ln l + \ln(2\pi^3), \quad n \to \infty, \quad m \text{ fixed}. \tag{13}
\]

Notice that the semiclassical momentum entropy does not depend on the azimuthal quantum number \( m \) as for the corresponding position quantity. More relevant are its \( n \)- and \( l \)-dependences. Indeed, it logarithmically depends on the principal quantum number \( n \) what means that the momentum entropy of a specific membrane increases without limit as the quantum number \( n \) characterizing its oscillations tends to infinity, although its rate of increase becomes ever more leisurely. This indicates that the Fourier-transformed or momentum-space oscillations of the membrane spreads more and more when \( n \) grows. In addition, Eq.(13) tells us that the semiclassical quantum entropy depends on the radius of the membrane also in a logarithmic form, indicating that the spreading of the momentum-space oscillations of the membrane logarithmically decreases when its radius increases. Here again this logarithmic behavior with \( l \) also holds for arbitrary oscillations, as it happened for the position case but with opposite sign.
In polar coordinates \((x, y) = (r \cos \varphi, r \sin \varphi)\) and \((p_x, p_y) = (p \cos \theta, p \sin \theta)\) we have

\[
\Xi_{mn}(p, \theta) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^l r \, dr \, \frac{1}{\sqrt{2\pi}} R_{mn}(r) e^{im\varphi} e^{i rp \cos(\varphi - \theta)}
\]

\[
= \frac{1}{\sqrt{2\pi}} e^{im\theta} \int_0^l R_{mn}(r) \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{im\varphi} e^{i rp \cos \varphi} \, d\varphi \right\} r \, dr.
\]

But

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{im\varphi} e^{i rp \cos \varphi} \, d\varphi = i^m J_m(rp),
\]

where \(J_m\) is the Bessel function of the first kind and order \(m\). Hence,

\[
\Xi_{mn}(p, \theta) = \frac{i^m}{\sqrt{2\pi}} e^{im\theta} Q_{mn}(p),
\]

where

\[
Q_{mn}(p) = \int_0^l R_{mn}(r) J_m(rp) \, r \, dr = C_{mn} \int_0^l J_m(k_{mn} r) J_m(rp) \, r \, dr.
\]

Using that (cf. [Jeffrey, 1995, Formulas 17.12.1.1])

\[
\int J_m(ar) J_m(br) r \, dr = \begin{cases} 
\frac{r [aJ_m(br)J'_m(ar) - bJ_m(ar)J'_m(br)]}{b^2 - a^2} & \text{if } a \neq b, \\
\frac{r^2}{2} \left\{ [J'_m(ar)]^2 + \left( 1 - \frac{m^2}{a^2 r^2} \right) [J_m(ar)]^2 \right\} & \text{if } a = b,
\end{cases}
\]

and the boundary conditions

\[
J_m(k_{mn} l) = 0,
\]

we obtain for \(p \neq k_{mn}\),

\[
\int_0^l J_m(pr) J_m(k_{mn} r) \, r \, dr = \frac{l k_{mn} J_m(pl) J'_m(k_{mn} l)}{p^2 - k_{mn}^2},
\]

and

\[
\int_0^l [J_m(k_{mn} r)]^2 \, r \, dr = C_{mn}^{-2} = \frac{l^2}{2} [J'_m(k_{mn} l)]^2,
\]

where we have used the normalization condition (3). Finally,

\[
Q_{mn}(p) = \sqrt{2} k_{mn} \frac{J_m(pl)}{p^2 - k_{mn}^2},
\]

\[\text{7}\]
With account of (14), as above, 
\[
S(\Xi_{mn}) = S_{mn}^* + \ln(2\pi) ,
\]
where
\[
S_{mn}^* = - \int_0^\infty |Q_{mn}(p)|^2 \ln |Q_{mn}(p)|^2 \, dp
\]
(17)
\[
= -2k_{mn}^2 \int_0^\infty \frac{|J_m(p l)|^2}{(p^2 - k_{mn}^2)^2} \ln \left( \frac{2k_{mn}^2}{(p^2 - k_{mn}^2)} \right) \, dp
\]
\[
= -2 \int_0^\infty \frac{|J_m(k_{mn}x)|^2}{(x^2 - 1)^2} \ln \left( \frac{2k_{mn}^2}{(x^2 - 1)} \right) x \, dx .
\]

As \( n \to \infty \), by (8) and (9),
\[
S_{mn}^* \sim - \frac{4}{\pi^2 n} \int_0^{+\infty} f_n(x) \ln \left( \frac{4l^2}{\pi^4 n^3} f_n(x) \right) \, dx ,
\]
where
\[
f_n(x) = \frac{\sin^2(\pi n x)}{(x^2 - 1)^2} .
\]
First,
\[
\int_0^\infty f_n(x) \, dx = \frac{\pi^2 n}{4} .
\]

On the other hand, \( f_n \) is a Fejer-type kernel giving in the limit, up to normalization (18), the Dirac delta at \( x = 1 \). Since \( \ln(f_n(x)/x) \) has only integrable singularities on \( \mathbb{R}_+ \), we have that
\[
S_{mn}^* \sim - \frac{4}{\pi^2 n} \ln \left( \frac{4l^2}{\pi^4 n^3} f_n(1) \right) = \ln n - 2 \ln(l/\pi) .
\]
(19)
This expression together with Eq. (16) straightforwardly leads to the asymptotic behavior of the momentum entropy given by Eq. (13).

Finally, let us derive the asymptotic behavior the sum of the entropies \( S(\Psi_{mn}) + S(\Xi_{mn}) \), which represents the joint entropic measure of the position-momentum uncertainty of the membrane. By (7) and (13) we have that as \( n \to \infty \), the entropic sum or combined information entropy behaves like
\[
S(\Psi_{mn}) + S(\Xi_{mn}) \sim \ln n + \ln(8\pi^4) - 2 .
\]
(20)
This result deserves same comments. First, the entropy sum for all oscillations of the membrane does not depend on its radius \( l \) because the
individual position and momentum entropies have the same logarithmic dependence on $l$ but with opposite signs. This is so because the entropy sum is invariant to uniform scaling of the membrane coordinates. Second, the entropic sum logarithmically grows with $n$. This behavior with the principal quantum number shows that the ordering of state oscillations is exactly identical with respect to the entropy sum and the single-particle energy at least for short-wave oscillations. This fact was also numerically observed for neutral atoms [Gadre et al., 1985 (b), Gadre et al., 1987] and, more recently and rigorously, for other simpler quantum-mechanical systems [Dehesa et al., 2001 (c), Dehesa et al., 1998, Dehesa et al., 2001 (b)]. Third, it naturally follows that Eq. (20) is in accordance with the entropic uncertainty principle [Bialynicki-Birula et al., 1975] which for all two-dimensional single-particle systems says that the entropic sum is bounded from below by the value $2(1 + \ln \pi)$.

4 Conclusions

The position and momentum information entropies, as well as their sum, have been rigorously investigated for the short-wavelength oscillations (i.e. in the correspondence principle limit $n \to \infty, m$ fixed) of a circular membrane of radius $l$. It was found that the position entropy depends logarithmically only on the radius of the membrane, and the momentum entropy depends, also in a logarithmic form, on the radius and the natural number $n$ characterizing the state. Finally, two most interesting observations follow on the entropy sum or joint measure of the total position-momentum uncertainty for highly-excited states of a circular membrane of radius $l$. One, it does not depend on the radius because of its uniform scaling invariance. Two, it has a logarithmically increasing behavior with respect to the main quantum number. This behavior is shared by a large set of physical systems having very different spectra of level energies, particularly neutral atoms [Gadre et al., 1985 (b), Gadre et al., 1987] and the quantum systems characterized by certain power-type potentials with bound states such as, for example, the one-dimensional hydrogen atom and the two-dimensional harmonic oscillator [Dehesa et al., 1998], as well as the single-particle systems submitted to the potential $V(x) = Ex^{2k}$, $k \in \mathbb{N}$, $E > 0$ and $x \in \mathbb{R}_0$ [Dehesa et al., 2001 (c)]. All this supports the (largely unexplored) idea to extend the Jaynes’ maximum entropy procedure to the case involving constraints in complementary spaces. The new maximum entropy procedure requires the maximization of the entropy sum subject to the known con-
straints in the position and momentum spaces [Gadre et al., 1985 (a)].

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