On Application of
Gorini-Kossakowski-Sudarshan-Lindblad Equation
in Cognitive Psychology

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Abstract. We proceed towards an application of the mathematical formalism of quantum mechanics to cognitive psychology — the problem of decision-making in games of the Prisoners Dilemma type. These games were used as tests of rationality of players. Experiments performed in cognitive psychology by Shafir and Tversky [1, 2], Croson [3], Hofstader [4, 5] demonstrated that in general real players do not use “rational strategy” provided by classical game theory; this psychological phenomenon was called the disjunction effect. We elaborate a model of quantum-like decision making which can explain this effect (“irrationality” of plays). Our model is based on quantum information theory. The main result of this paper is the derivation of Gorini-Kossakowski-Sudarshan-Lindblad equation whose equilibrium solution gives the quantum state used for decision making. It is the first application of this equation in cognitive psychology.

1. Introduction

Recently several authors discussed the possibility of applying the mathematical formalism of quantum mechanics to cognitive psychology, in particular, to the games of Prisoners Dilemma (PD) type, see [6–18]. It was found that statistical data obtained in some experiments of cognitive psychology, see [1–5], cannot be described by classical probability model (Kolmogorov’s model). These experiments play an important role in behavioural economics; they provide tests of rationality of behaviour of agents acting on the market (including the financial market). The impossibility to use the Kolmogorov model stimulated research in applications of non-classical probabilistic mod-

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Quantum probability is one of the most advanced mathematical models for non-classical probability. Therefore it was natural to try to apply quantum probability to, e.g., PD-type games. However, the experimental data demonstrated that the situation is really nontrivial: known data cannot be described by the conventional Dirac-von Neumann model of quantum mechanics, see [7, 16] for a discussion. In [13] statistical data from one of the experiments (Tversky-Shafir [2]) was described by a quantum Markov chain.

Recently the authors proposed a quantum-like model of decision making by using the generalised quantum formalism based on lifting of density operators [18, 19]. The crucial point of this model is the possibility to entangle possible strategies of two players in the PD-type game through lifting.

In [18] we presented a toy model of quantum-like decision making based on a simple system of differential equations for the equilibrium quantum state. This system was a quantum version of the standard system of equation for chemical equilibrium. It was invented ad hoc; its connection with the standard equations of quantum mechanics was not clear.

In this paper we embed the equilibrium equation of [18] in the standard quantum formalism. We derived the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation whose equilibrium provides the quantum state which is used for decision making. This equation is quite interesting from a purely mathematical viewpoint, but we are merely interested in its cognitive interpretation. It is a nontrivial problem.

It seems that our model presents the first application of GKSL-equation in cognitive science and psychology. We hope that new applications will be found soon.

We emphasise that our model has no direct relation to models of “quantum physical brain”, e.g., [20 – 23]. We consider the brain as a processor of information using the rules described by quantum information theory; we do not try to reduce such information processing to quantum physical processes in the brain.

2. Quantum-like Model for Decision-making in Two-player Game

In [18], we designed a quantum-like model for decision-making process in two-player games. This section explains briefly how a player in our model decides on his own actions.

*Let us mention recent experiments on recognition of ambiguous figures [14] which, similarly, produced non-classical statistical data.
Table 1: Pay-off table of the two-player game. 0 and 1 denote the strategies chosen by players A and B. The symbols \((x/y)\) with \(x, y = a, b, c, d\) represent the pay-offs which are assigned to A and B in four possible situations.

2.1. Pay-off table of two-player game

Let us consider a two-player game with two strategies. We name two players “A” and “B”. Two strategies which A and B can choose are denoted by “0” and “1”. Table 1 shows pay-offs assigned to possible four consequences of “0_A0_B”, “0_A1_B”, “1_A0_B” and “1_A1_B”. Here, \(a, b, c\) and \(d\) denote the values of pay-offs.

For example, the game of PD-type is characterised by the relation of \(c > a > d > b\). For player A, his pay-off will be \(a\) or \(c\) if player B chooses “0” and \(b\) or \(d\) if player B chooses “1”. In both cases, from the relations of \(c > a\) and \(d > b\), A can obtain larger pay-offs if he chooses 1. The situation is the same for player B. Conventional game theory concludes that in PD game a rational player, who wants to maximise his own payoff, always chooses “1”.

However, the above discussion does not explain completely the process of decision-making in real player’s mind. Actually, as seen in statistical data in some experiments on so-called disjunction effect, real players frequently behave “irrationally”. Our model is an attempt to describe such real player’s behaviours by some mathematical formalism. Our model is a “quantum-like model” which is derived from basic concepts of quantum mechanics, but not an expansion of conventional game theory.

2.2. Decision-making process in player’s mind

We present our model for the decision-making process in two-player games. We focus on player A mind. In principle, player A is not informed of which action B chooses. Player A will be aware of two potential actions of B simultaneously, and then he cannot reject any of them. In our model, this indeterminacy of player A is described by the following quantum superposition:

\[
|\phi_B\rangle = \alpha |0_B\rangle + \beta |1_B\rangle \in \mathbb{C}^2. \tag{1}
\]
The values of $\alpha$ and $\beta$ are related to the degree of awareness of B’s actions.\(^b\) We call this complex vector $|\phi_B\rangle$ a prediction state vector. In accordance with the formalism of quantum mechanics, we assume $|\alpha|^2 + |\beta|^2 = 1$.

Player A who is inclined to choose the action “0” will be aware of two consequences of “0\(_A\)0\(_B\)” and “0\(_A\)1\(_B\)” with probability amplitudes of $\alpha$ and $\beta$. This situation is described by a vector from $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$|\Phi\rangle = \alpha |0\(_A\)0\(_B\rangle + \beta |0\(_A\)1\(_B\rangle = |0\(_A\rangle \otimes |\phi_B\rangle.$$  \(^{(2)}\)

Similarly,

$$|\Phi\rangle = |1\(_A\rangle \otimes |\phi_B\rangle,$$  \(^{(3)}\)

describes the situation such that A prefers to choose “1”. By using these state vectors $|\Phi\rangle$ and $|\Phi\rangle$, we define the following vector:

$$|\Psi\rangle = x |\Phi\rangle + y |\Phi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2,$$  \(^{(4)}\)

($|x|^2 + |y|^2 = 1$). This state vector describes A’s intension to act. We call it a mental state vector of A.

Player A’s brain in this mental state chooses his own action probabilistically. His decisions are described as “quantum measurements” of projectors corresponding to the vectors $|\Phi\rangle$ or $|\Phi\rangle$ on the state $|\Psi\rangle$. (Probabilities of “0” and “1” are given by $P(0\(_A\)) = P_{0\(_A\)} = |x|^2$ and $P(1\(_A\)) = P_{1\(_A\)} = |y|^2$.)

In our model, the decision-making process is described as a dynamics changing $|x|^2$ and $|y|^2$, and this dynamics has an equilibrium solution.\(^c\) Such stabilisation of the mental state explains the following psychological activity in the player’s mind: The player has two psychological tendencies, one to choose 0 and the other to choose 1. Degrees of these two opposite tendencies change in his mind, and they become stable with balancing. Fluctuations die and the definite probabilistic picture of the situation is created in A’s mind.

In our previous paper [18], we introduced the equations of chemical equilibration type as the most simple dynamics of the stabilisation of probabilities.\(^d\)

$$\frac{d}{dt} P_{0\(_A\)} = -kP_{0\(_A\)} + \tilde{k}P_{1\(_A\)},$$

\(^{b}\)So to say, these are complex probabilistic amplitudes of A’s intension that B can make decisions 0 or 1, respectively.

\(^{c}\)The existence of unique equilibrium state is one of fundamental assumptions of our model. Thus we speculate that in the process of evolution the brain selected only dynamical systems (in fact, quantum-like dynamics) with this special property. The absence of equilibrium state would imply unnecessary reflections. Such organisms would not survive. Of course, some psychical deviations may induce brain dynamics without equilibrium solutions. But we consider the case of “normal bahaviour” in the process of decision making.

\(^{d}\)Thus in [18] we described only dynamics of probabilities, not of the mental state of A. In fact, only these probabilities are used by A to make the final decision. However, it is interesting to describe also the basic process occurring in the brain, namely, the dynamics of the mental state, its stabilisation to an equilibrium solution. We shall do this in Sect. 3.
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\[ \frac{d}{dt}P_1 = kP_0 - \tilde{k}P_1, \]  
\( (5) \)

The parameter \( k (\tilde{k}) \) corresponds to the velocity of the reaction from 0\(_A\) to 1\(_A\) (from 1\(_A\) to 0\(_A\)), and in the equilibrium state, the probabilities \( P_0 \) and \( P_1 \) are given as:

\[ P_{0A}^E = \frac{\tilde{k}}{k + \tilde{k}}, \quad P_{1A}^E = \frac{k}{k + \tilde{k}}. \]  
\( (6) \)

2.3. Definition of velocities of \( k \) and \( \tilde{k} \)

As we have seen from (5), the player’s tendency to choose 1 or 0 is proportional to the velocity \( k \) or \( \tilde{k} \), and these parameters determine the stability solution (6). The choice of \( k \) and \( \tilde{k} \) is a very important issue in our model. We assume that their values are determined through a comparison of possible consequences, 0\(_A0\(_B\), 0\(_A1\(_B\), 1\(_A0\(_B\), and 1\(_A1\(_B\). The player in our model will consider the following four kinds of comparisons:

\[ 0_A0_B \overset{k_1}{\underset{k_1}{\equiv}} 1_A0_B, \quad 0_A1_B \overset{k_2}{\underset{k_2}{\equiv}} 1_A1_B, \]
\[ 0_A1_B \overset{k_3}{\underset{k_3}{\equiv}} 1_A0_B, \quad 0_A0_B \overset{k_4}{\underset{k_4}{\equiv}} 1_A1_B. \]  
\( (7) \)

These comparisons are represented like the conditions of chemical equilibrium, each of which is specified by the reaction velocities, \( k_i \) and \( \tilde{k}_i \). Here, we introduce four maps from (the space of linear operators) \( \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2) \) to \( \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2) \) of the form

\[ V_i(\cdot) \equiv V_i \cdot V_i^*, \quad \tilde{V}_i(\cdot) \equiv V_i^* \cdot V_i, \quad i = 1, 2, 3, 4, \]  
\( (8) \)

where \( \{V_i\} \) are transition operators defined by

\[ V_1 = |1_A0_B\rangle \langle 0_A0_B|, \quad V_2 = |1_A1_B\rangle \langle 0_A1_B|, \]
\[ V_3 = |1_A0_B\rangle \langle 0_A1_B|, \quad V_4 = |1_A1_B\rangle \langle 0_A0_B|. \]

Each of \( V_i \) gives state transitions in the four comparisons of (7), respectively. Furthermore, we introduce the map providing state transitions between \( |\Phi_{0A}\rangle \langle \Phi_{0A}| \) and \( |\Phi_{1A}\rangle \langle \Phi_{1A}| \):

\[ \mathcal{V}(\cdot) \equiv V \cdot V^*, \quad \tilde{\mathcal{V}}(\cdot) \equiv V^* \cdot V, \]  
\( (9) \)

\*The differential equations (5) are not described as a quantum dynamics. In fact, they give only a part of the complete system of quantum dynamical equations, the dynamics of the diagonal terms of the density matrix. In Sect. 3, we complete the system (5) and obtain a quantum dynamics; a differential equation for density matrix of the mental state.

\( ^f \)In the next subsection, we mention the relation between \( k_i (\tilde{k}_i) \) and the pay-off table of the game.
where

\[ V = |\Phi_{1A}\rangle \langle \Phi_{0A}|. \]  

(10)

By using \( V \) and \( \tilde{V} \), the differential equations (5) are rewritten as

\[
\frac{d}{dt} \text{tr}(V(\Theta)) = -k \text{tr}(V(\Theta)) + \tilde{k} \text{tr}(\tilde{V}(\Theta)),
\]

\[
\frac{d}{dt} \text{tr}(\tilde{V}(\Theta)) = k \text{tr}(V(\Theta)) - \tilde{k} \text{tr}(\tilde{V}(\Theta)),
\]

(11)

\( \Theta \) is the mental state \(|\Psi_A\rangle \langle \Psi_A|\) of (4). The maps \( V \) and \( \tilde{V} \) specify “comparison” in player’s mind. Thus, the values of the parameters \( k \) and \( \tilde{k} \) are determined depending on the forms of the operators \( (V, \tilde{V}) \). First, let us consider the simple forms such as \( (V, \tilde{V}) = (V_i, \tilde{V}_i) \). In such case, \( (k, \tilde{k}) = (k_i, \tilde{k}_i) \) is assumed.

However, by our model the prediction state \(|\phi_B\rangle\) (describing A’s image of possible actions of B) determines the dynamics of A’s mental state (describing A’s image of possible own actions). Thus the operators \( V \) and \( \tilde{V} \) should have forms depending on the amplitudes \( \alpha \) and \( \beta \), which are the coefficients of the prediction state vector \(|\phi_B\rangle\), see (1). This is one of the basic assumptions of our model! In general, \( (V, \tilde{V}) \neq (V_i, \tilde{V}_i) \), if \( \alpha \) or \( \beta \neq 0 \) or 1.

Actually, from the definition of \(|\Phi_{0A,1A}\rangle\), see (2) and (3), one can easily check that

\[ V = |\alpha|^2 |1_{A0B}\rangle \langle 0_{A0B}| + |\beta|^2 |1_{A1B}\rangle \langle 0_{A1B}| + \alpha\beta^* |1_{A0B}\rangle \langle 0_{A1B}| + \alpha^*\beta |1_{A1B}\rangle \langle 0_{A0B}| \]

\[ = \sum_{i=1}^4 c_i V_i, \]  

(12)

\( \{c_i\} = \{|\alpha|^2, |\beta|^2, \alpha\beta^*, \alpha^*\beta\} \), and (9) is rewritten as

\[ V(\cdot) = \sum_{i=1}^4 |c_i|^2 V_i(\cdot) + \sum_{i\neq j} c_i c_j^* V_i \cdot V_j^*, \]

\[ \tilde{V}(\cdot) = \sum_{i=1}^4 |c_i|^2 \tilde{V}_i(\cdot) + \sum_{i\neq j} c_i^* c_j V_i^* \cdot V_j. \]

The term \( \sum_{i} |c_i|^2 V_i(\cdot) \) (or \( \sum_{i} |c_i|^2 \tilde{V}_i(\cdot) \)) indicates that the four kinds of comparisons affect the player’s tendency to choose 0 (or 1) simultaneously. The terms \( \sum_{i\neq j} c_i c_j^* V_i \cdot V_j^* \) and \( \sum_{i\neq j} c_i^* c_j V_i^* \cdot V_j \) indicate that in player A’s mind the four comparisons are not performed independently; they interfere. Player A in our model holds indeterminacy on B’s action, so his concerns
about the consequence of “0A0B” and “0A1B” (or “1A0B” and “1A1B”) always fluctuate. In such a situation, player A cannot perform four comparisons independently. The interference terms \( \sum_{i \neq j} c_i^* V_i \cdot V_j \) and \( \sum_{i \neq j} c_j^* V_i^* \cdot V_j \) represent psychological influences of the situation in which player A is not informed of B’s action. The velocities \( k \) and \( \tilde{k} \) should have the forms reflecting effects of the four comparisons and interferences between them. In order to define these velocities in the appropriative forms, we introduce complex numbers \( \mu \) and \( \tilde{\mu} \), which determine \( k \) and \( \tilde{k} \) by

\[
k = |\mu|^2, \quad \tilde{k} = |\tilde{\mu}|^2,
\]

and define these \( \mu \) and \( \tilde{\mu} \) as

\[
\begin{align*}
\mu &= |\alpha|^2 \mu_1 + |\beta|^2 \mu_2 + \alpha^* \beta^* \mu_3 + \alpha \beta^* \mu_4 = \sum_i c_i \mu_i, \\
\tilde{\mu} &= |\alpha|^2 \tilde{\mu}_1 + |\beta|^2 \tilde{\mu}_2 + \alpha^* \beta^* \tilde{\mu}_3 + \alpha \beta^* \tilde{\mu}_4 = \sum_i c_i^* \tilde{\mu}_i.
\end{align*}
\]

Here \( \mu_i = 1,2,3,4 \) and \( \tilde{\mu}_i = 1,2,3,4 \) are complex numbers satisfying \( |\mu_i|^2 = k_i \) \( |\tilde{\mu}_i|^2 = \tilde{k}_i \) for given \( k_i \) and \( \tilde{k}_i \). Hence, \( k \) and \( \tilde{k} \) are defined as

\[
\begin{align*}
k &= \sum_{i=1,2,3,4} |c_i|^2 k_i + \sum_{i \neq j} c_i^* c_j^* \mu_i \mu_j^*, \\
\tilde{k} &= \sum_{i=1,2,3,4} |c_i|^2 \tilde{k}_i + \sum_{i \neq j} c_i^* c_j \tilde{\mu}_i \tilde{\mu}_j^*.
\end{align*}
\]

These forms are similar to the structures of the operators \( \mathcal{V} \) and \( \tilde{\mathcal{V}} \) represented in (13). In this definition, the most important point is the selection of complex numbers \( \mu_i \) and \( \tilde{\mu}_i \). These parameters will have no meaning as physical quantities, rather, they have meaning similar to amplitudes introduced in the quantum mechanical sense. The terms \( \sum_{i \neq j} c_i^* c_j^* \mu_i \mu_j^* \) will provide effects similar to quantum interference to the value of \( k \), see [6] on quantum-like interference in cognitive and social sciences.\(^8\)

We can simplify the representation of \( k \) and \( \tilde{k} \) by introducing the so-called comparison operator

\[
\mathcal{T}_\sigma \equiv \Phi_1 A \langle \Phi_1 A | T | \Phi_0 A \rangle \langle \Phi_0 A | + | \Phi_0 A \rangle \langle \Phi_0 A | T | \Phi_1 A \rangle \langle \Phi_1 A |,
\]

where \( T \) is a matrix given in the form

\[
T = \begin{pmatrix}
0 & 0 & \tilde{\mu}_1 & \tilde{\mu}_3 \\
0 & 0 & \tilde{\mu}_4 & \tilde{\mu}_2 \\
\mu_1 & \mu_4 & 0 & 0 \\
\mu_3 & \mu_2 & 0 & 0
\end{pmatrix}.
\]

\(^8\)Such effects in the process of decision making will be discussed in more detail in further papers.
The index $\sigma$ of $T_\sigma$ denotes the prediction state $\sigma \equiv |\phi_B\rangle \langle \phi_B|$. With using the term of $T_\sigma$, $k$ and $\tilde{k}$ are represented as

$$k = \langle \Phi_0A | T_\sigma^* T_\sigma | \Phi_0A \rangle \equiv \langle \Phi_0A | K_\sigma | \Phi_0A \rangle,$$

$$\tilde{k} = \langle \Phi_1A | T_\sigma^* T_\sigma | \Phi_1A \rangle \equiv \langle \Phi_1A | K_\sigma | \Phi_1A \rangle. \quad (17)$$

Here, $K_\sigma = T_\sigma^* T_\sigma$ is Hermitian. It can be called an operator of velocity. This operator is different from conventional operators of physical quantities defined in quantum mechanics: its form depends on the prediction state $\sigma$. This property indicates that the dynamics in our model has state adaptivity which is an important concept in the adaptive dynamics theory proposed in [24].

2.4. DECISION-MAKING IN PD-TYPE GAME AND IRRATIONAL CHOICE

The parameters $(k_i, \tilde{k}_i)$ introduced in the previous subsection specify the player’s four kinds of comparisons, see (7). It is natural that these comparisons depend on a given game, namely its pay-off table like that of Table 1. The simplest relation between pay-offs and parameters $(k_i, \tilde{k}_i)$ can be obtained by expressing the order of pay-off values with parameters $k_i$. For example, in the case of PD-type game, if the relation of pay-offs is $c > a > d > b$, then $k_i$ and $\tilde{k}_i$ are given as

$$k_1 = 1, \quad k_2 = 1, \quad k_3 = 1, \quad k_4 = 0,$$

$$\tilde{k}_1 = 0, \quad \tilde{k}_2 = 0, \quad \tilde{k}_3 = 0, \quad \tilde{k}_4 = 1. \quad (18)$$

Such setting is simple, but not realistic. The real player’s decision-making will depend on differences between pay-offs, not only on magnitude relations. That is, the following setting will be more realistic:

$$k_1 = f_1(|a - c|), \quad k_2 = f_2(|b - d|), \quad k_3 = f_3(|b - c|), \quad k_4 = 0,$$

$$\tilde{k}_1 = 0, \quad \tilde{k}_2 = 0, \quad \tilde{k}_3 = 0, \quad \tilde{k}_4 = \tilde{f}_4(|a - d|). \quad (19)$$

The functions $f_i(x)$ are assumed to be monotone increasing.

As a result under the settings of $k_i$ and $\tilde{k}_i$ of (19) or (20), the probability $P_{0A}^E$ of (6) is non-zero. Thus, our model explains that player A generally can potentially make the “irrational” choice of 0 in the PD-game. The reason for this result is that the parameter $\tilde{k}_4$ is non-zero. $\tilde{k}_4$ represents the degree of tendency to choose 0 which arises from the comparison between the consequences of $0_A0_B$ and $1_A1_B$. It should be noted that such comparison is not present in classical game theory.
3. Quantum Completion of Differential Equation (5)

In our model, the dynamics of probabilities corresponding to the mental state is specified by (5)

\[
\frac{d}{dt} P_{0A} = -kP_{0A} + \tilde{k}P_{1A},
\]

\[
\frac{d}{dt} P_{1A} = kP_{0A} - \tilde{k}P_{1A}.
\]

Here, \(P_{0A} = |x|^2\) and \(P_{1A} = |y|^2\), \(x\) and \(y\) are coefficients of the mental state \(|\Psi\rangle = x |0_A\rangle \otimes |\phi_B\rangle + y |1_A\rangle \otimes |\phi_B\rangle = x |\Phi_0\rangle + y |\Phi_1\rangle\), where \(|\phi_B\rangle = \alpha |0_B\rangle + \beta |1_B\rangle \alpha, \beta \in \mathbb{C}\) is the prediction vector defined in (1). The parameters \(k\) and \(\tilde{k}\) are defined in (18).

In this section, we complete the dynamics (5) to a quantum dynamics of the density matrix \(\rho = \langle \psi | \psi \rangle\) corresponding to the mental state \(|\psi\rangle = x |0_A\rangle + y |1_A\rangle\) describing the superposition of A’s decisions. Finally, we shall represent this quantum dynamics in the Gorini-Kossakowski-Sudarshan-Lindblad form. However, we should be very careful with an interpretation of the appearance of such “master equation” in our cognitive model of decision making. In general, it is not reasonable to compare A’s mental image of possible B’s actions with a kind of mental bath for A’s image of possible own actions. However, we consider a simple model in which the prediction state does not change in the process of decision making. This is the case of Born’s approximation. In the model under consideration A created a mental image of possible B’s actions once and up to finalising the process of decision making. The interaction with this image changes permanently the state \(\rho\) of A’s own possible actions, but this change of A’s self-image of her actions has a very small influence on the prediction state — it can be considered as constant. In spite of the mathematical simplicity of such a model, this sort of decision making is quite reasonable.

First, we consider a discrete dynamics of \(\rho\) specified by the following quantum channel \(\Lambda^*\), a map from an initial state \(\rho(0)\) to a state at time \(\tau\)

\[
\Lambda^*(\rho(0)) = \Gamma \rho(0) \Gamma^* + \Gamma' \rho(0) \Gamma'^* = \rho(\tau),
\]

(20)

where

\[
\Gamma = \begin{pmatrix} \sqrt{1 - \Delta} & 0 \\ 0 & \sqrt{1 - \Delta} \end{pmatrix},
\]

\[
\Gamma' = \begin{pmatrix} 0 & \sqrt{\Delta e^{i\phi}} \\ \sqrt{\Delta e^{i\phi}} & 0 \end{pmatrix}.
\]

(21)
Here, $0 \leq \Delta, \tilde{\Delta} \leq 1$, and $-\pi \leq \theta, \tilde{\theta} \leq \pi$. With using components of $\rho(0)$, components of $\rho(\tau)$ are represented as

\begin{align*}
\rho_{00}(\tau) &= (1 - \Delta)\rho_{00}(0) + \tilde{\Delta}\rho_{11}(0), \\
\rho_{01}(\tau) &= \sqrt{(1 - \Delta)(1 - \tilde{\Delta})}\rho_{01}(0) + \sqrt{\Delta \tilde{\Delta}} e^{-i(\theta - \tilde{\theta})}\rho_{10}(0), \\
\rho_{10}(\tau) &= \sqrt{(1 - \Delta)(1 - \tilde{\Delta})}\rho_{10}(0) + \sqrt{\Delta \tilde{\Delta}} e^{i(\theta - \tilde{\theta})}\rho_{01}(0), \\
\rho_{11}(\tau) &= (1 - \Delta)\rho_{11}(0) + \Delta\rho_{00}(0). \tag{22}
\end{align*}

We assume the dynamics by $\Lambda^*\tau$ in the time interval $\tau$ occurs from some interaction with the system of $\sigma$, that is, prediction state. Note that $\Gamma^\ast \Gamma + \Gamma^\ast \Gamma' = I$ is satisfied. It means that this channel has Kraus representation that is linear, trace preserving and complete positive. In this discrete dynamics, $\rho(n\tau)$ is given by $n$ times maps from $\rho(t)$

$$
\rho(n\tau) = \Lambda^\ast_n (\Lambda^\ast_n (\cdots (\Lambda^\ast_n (\rho(0)) \cdots)).
$$

Here, we redefine $\Delta$, $\tilde{\Delta}$ and $\theta, \tilde{\theta}$ in the following form relating with $\mu$ and $\tilde{\mu}$ in (14)

\begin{align*}
\Delta(\tau) &= \frac{|\mu|^2}{|\mu|^2 + |\tilde{\mu}|^2} \tau \equiv |\mu'|^2 \tau, \\
\tilde{\Delta}(\tau) &= \frac{|\tilde{\mu}|^2}{|\mu|^2 + |\tilde{\mu}|^2} \tau \equiv |\tilde{\mu}'|^2 \tau, \\
\mu &= |\mu| e^{i\theta}, \quad \tilde{\mu} = |\tilde{\mu}| e^{i\tilde{\theta}}. \tag{23}
\end{align*}

Then, one can describe diagonal parts of $\rho_{00}(n\tau)$ and $\rho_{11}(n\tau)$ as

\begin{align*}
\rho_{00}(n\tau) &= (1 - \Delta - \tilde{\Delta})^n\rho_{00}(0) + (1 - (1 - \Delta - \tilde{\Delta})^n) \frac{\Delta}{\Delta + \tilde{\Delta}} \\
&= (1 - \tau)^n\rho_{00}(0) + (1 - (1 - \tau)^n) |\tilde{\mu}'|^2, \\
\rho_{11}(n\tau) &= (1 - \Delta - \tilde{\Delta})^n\rho_{11}(0) + (1 - (1 - \Delta - \tilde{\Delta})^n) \frac{\Delta}{\Delta + \tilde{\Delta}} \\
&= (1 - \tau)^n\rho_{11}(0) + (1 - (1 - \tau)^n) |\mu'|^2. \tag{24}
\end{align*}

These parts approach to the values of $|\tilde{\mu}'|^2$ and $|\mu'|^2$ respectively, if $\tau < 1$. Especially, if $\tau \ll 1$, the non-diagonal parts of (22) are approximated as

\begin{align*}
\rho_{01}(\tau) &\approx \left[1 - \frac{1}{2}(\Delta + \tilde{\Delta})\right] \rho_{01}(0) + \sqrt{\Delta \tilde{\Delta}} e^{-i(\theta - \tilde{\theta})}\rho_{10}(0), \\
\rho_{10}(\tau) &\approx \left[1 - \frac{1}{2}(\Delta + \tilde{\Delta})\right] \rho_{10}(0) + \sqrt{\Delta \tilde{\Delta}} e^{i(\theta - \tilde{\theta})}\rho_{01}(0).
\end{align*}
One can easily check that the absolute values of non-diagonal parts decrease for any values of $\theta(\tilde{\theta})$. It should be noted here, under the condition of $\tau \ll 1$, this dynamics has a property of Markovian process

$$
\rho(n\tau) = \Lambda_*^n(\Lambda_*^* (\Lambda_*^* (\rho(0)) \cdots )) \approx \Lambda_*^n \rho(0).
$$

Next, we consider a van Hove-like limit such that $n\tau = t$ fixed with $\tau \to 0$ and $n \to \infty$. From (22), the following differential equations are obtained

$$
\begin{align*}
\frac{d}{dt}\rho_{00}(t) &= \lim_{\tau \to 0} \rho_{00}(t + \tau) - \rho_{00}(t) = -|\mu'|^2 \rho_{00} + |\tilde{\mu}'|^2 \rho_{11}, \\
\frac{d}{dt}\rho_{01}(t) &= \lim_{\tau \to 0} \rho_{01}(t + \tau) - \rho_{01}(t) = -\frac{1}{2} \rho_{01} + \mu'^* \tilde{\mu}' \rho_{10}, \\
\frac{d}{dt}\rho_{10}(t) &= \lim_{\tau \to 0} \rho_{10}(t + \tau) - \rho_{10}(t) = -\frac{1}{2} \rho_{10} + \mu' \tilde{\mu}'^* \rho_{01}, \\
\frac{d}{dt}\rho_{11}(t) &= \lim_{\tau \to 0} \rho_{11}(t + \tau) - \rho_{11}(t) = -|\tilde{\mu}'|^2 \rho_{11} + |\mu'|^2 \rho_{00}.
\end{align*}
$$

It is clear that the equations for diagonal parts correspond to the equations of (1) essentially.

In the form of polar coordinate the non-diagonal part $\rho_{01}(t) = r(t) e^{i\phi(t)}$ has the absolute value $r(t)$ of the form

$$
r(t) = e^{-t/2} \sqrt{A_0^2 e^{-2|\mu'|t} + B_0^2 e^{-2|\tilde{\mu}'|t}},
$$

where $\rho_{01}(0) = A_0 + iB_0, A_0, B_0 \in \mathbb{R}$. One can check that $r(t)$ goes to zero for $t \to \infty$, see also numerical modeling Fig. 1.

Thus, in this dynamics, any initial state of $\rho(0)$ approaches the unique equilibrium state

$$
\rho_E = \left( \begin{array}{cc} |\mu'|^2 & 0 \\ 0 & |\tilde{\mu}'|^2 \end{array} \right) = \left( \begin{array}{cc} |\mu|^2 & 0 \\ 0 & |\tilde{\mu}|^2 \end{array} \right).
$$

The differential equations (25) can be rewritten as

$$
\frac{d}{dt}\rho(t) = Uh\rho h^* U - \frac{1}{2}(h\rho h^* + Jh\rho h^* J) - \frac{1}{4}(\rho - J\rho J),
$$

where

$$
\begin{align*}
h &= \mu'|0_B\rangle \langle 0_B| + \tilde{\mu}'|1_B\rangle \langle 1_B| = \left( \begin{array}{cc} \mu' & 0 \\ 0 & \tilde{\mu}' \end{array} \right), \\
U &= \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \\
J &= \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).
\end{align*}
$$

(28)
Fig. 1: The numerical solution for $\rho_{01}(t)$ in case of $\rho_{01}(0) = \frac{1}{2} \exp(i\pi/5)$ and $\mu' \tilde{\mu}' = \frac{1}{3} \exp(i\pi/2)$.

4. Gorini-Kossakowski-Sudarshan-Lindblad Form

Denote the space of $k \times k$ matrices by the symbol $M(k)$. The equation of (27) is defined by a linear map $M(2) \mapsto M(2)$

$$L(\rho) = U h \rho h^* U - \frac{1}{2} (h \rho h^* + J h \rho h^* J) - \frac{1}{4} (\rho - J \rho J) \equiv \mathcal{L} \rho. \quad (29)$$

We rewrite this linear map into the form of Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation

$$\mathcal{L} = \mathcal{H} + \mathcal{D},$$

$\mathcal{H}$ is a Hamiltonian part giving a unitary evolution, and $\mathcal{D}$ is a dissipative part. GKSL theory explains that if a linear operator $L: M(N) \mapsto M(N)$ satisfies the conditions $\langle L(A) \rangle^* = L(A^*)$ and $\text{tr}(LA) = 0$ for all $A \in M(N)$, such $L$ can be uniquely written in the form

$$L(A) = \sum_{i,j=0}^{N^2-1} c_{ij} F_i A F_j^*$$

$$= -i[H, A] + \frac{1}{2} \sum_{i,j=1}^{N^2-1} c_{ij} \left( [F_i, A F_j^*] + [F_i A, F_j^*] \right). \quad (30)$$

The set of $\{F_i\}_{i=0,1,2,...,N^2-1}$ is a complete orthonormal set in $M(N)$ such that $F_0 = (1/N)^{\frac{1}{2}} I$. $\{c_{ij}\}$ is uniquely determined by the choice of the $F_i$'s.
The operator $H$ is defined by $(1/2i)(F^* - F)$ with $F = (1/N)^{\frac{1}{2}} \sum_{i=1}^{N^2-1} c_{i0}F_i$. We call this $H$ “Hamiltonian” satisfying $H = H^*$ and $\text{tr}(H) = 0$.

In our case of (35), we give a set of \{\text{\textit{F}}_{\text{\textit{i}}}\} as

\[
\text{\textit{F}}_{\text{\textit{0}}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{\textit{F}}_{\text{\textit{1}}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{\textit{F}}_{\text{\textit{2}}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{\textit{F}}_{\text{\textit{3}}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{31}
\]

The operators $h$, $Uh$ and $Jh$ in (27) is represented as

\[
h = \frac{1}{\sqrt{2}}(\mu' + \tilde{\mu}')\text{\textit{F}}_{\text{\textit{0}}} + \frac{1}{\sqrt{2}}(\mu' - \tilde{\mu}')\text{\textit{F}}_{\text{\textit{1}}},
\]
\[
Uh = \frac{1}{\sqrt{2}}(\mu' + \tilde{\mu}')\text{\textit{F}}_{\text{\textit{2}}} - \frac{1}{\sqrt{2}}(\mu' - \tilde{\mu}')\text{\textit{F}}_{\text{\textit{3}}},
\]
\[
Jh = \frac{1}{\sqrt{2}}(\mu' - \tilde{\mu}')\text{\textit{F}}_{\text{\textit{0}}} + \frac{1}{\sqrt{2}}(\mu' + \tilde{\mu}')\text{\textit{F}}_{\text{\textit{1}}}. \tag{32}
\]

Then, the \{\text{\textit{c}}_{\text{\textit{ij}}}\}, $i, j = 1, 2, 3$, is represented by the $3 \times 3$ matrix of

\[
\frac{1}{2} \begin{pmatrix} 1 - |\mu'|^2 - |\tilde{\mu}'|^2 & 0 & 0 \\ 0 & |\mu' + \tilde{\mu}'|^2 & -(\mu' + \tilde{\mu}')(\mu'^* + \tilde{\mu}'^*) \\ 0 & -(\mu' - \tilde{\mu}')(\mu'^* + \tilde{\mu}'^*) & |\mu' - \tilde{\mu}'|^2 \end{pmatrix}, \tag{33}
\]

and $H = 0$. One can easily check that the above matrix of \{\text{\textit{c}}_{\text{\textit{ij}}}\}, $i, j = 1, 2, 3$, is a positive matrix. It means the equation $\frac{d}{dt}\rho = L\rho$ in our model is a master equation of complete positive dynamical semigroup. Namely, the map defined as $\Lambda_t^\tau \equiv e^{L\tau}$ is complete positive, linear, and Markovian process. The result of $H = 0$ is interpreted as follows: our model assumes that the player’s decision making strongly depends on his prediction of another player’s action and a comparison between consequences of game. These effects are described in the dissipative term in open system dynamics, because the prediction state $\sigma$ is treated as an environment and the comparison occurs from some interaction with this environment. In a sense, our model is simple and ideal. We can expect other psychological effects, which is not related with his prediction (environment), will be described in Hamiltonian part.

5. Conclusion

In the framework of quantum information theory we constructed a model of decision making in games of the Prisoners Dilemma type which are important in cognitive psychology, economics and finances. The process of decision making

\[^b\text{As mentioned in previous section, the channel } \Lambda_t^\tau (\tau \ll 1) \text{ has the property } \Lambda_t^\tau \Lambda_s^\tau \approx \Lambda_{t+s}^\tau.\]
making is based on the GKSL-equation, the basic equation of quantum information processing for open systems. This is the first application of this equation in cognitive psychology. In physics GKSL describes a system in interaction with environment. The separation between the system and environment is done by an observer. A tricky point in applications of the GKSL-equation to cognitive science is that the brain is a self-observer (!). The role of a system under observation is played by Self. And the brain is also able to create the mental bath representing Self’s picture of some mental phenomenon. In the particular model under consideration Self is reduced to A’s representation of his possible decisions and the bath is reduced to again A’s representation of possible actions of another player, B. The internal mental dynamics of Self is described by a unitary evolution, so in our model self-reflections are unitary. The mental interaction with (its own) picture of B’s possible actions is described by Lindblad operators.

We remark that GKSL-dynamics is Markovian, cf. [13]. This assumption is only approximately valid for the process of brain’s functioning. Therefore it is natural to proceed towards cognitive models based on non-Markovian quantum dynamics.

We remark that the GKSL dynamics in our model strongly depends on the environment of prediction state $|\phi_B\rangle$, which is also a state specifying the player’s state of mind. Such property is called state-adaptivity in the concept of adaptive dynamics theory, see [24]. Our model is a specific example of adaptive dynamical system.

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Bibliography


