Extremism, Campaigning and Ambiguity*

Andreas Westermark†

May 19, 2000

Abstract

This paper studies a model of how political parties use resources for campaigning to inform voters. Each party has a predetermined ideology that is drawn from some distribution. To affect the voters, the parties use campaigning to inform the voters about the policy of the party. We study a symmetric model with two parties and find that, the farther away parties are from each other (on average), the less resources is spent on campaigning (on average) in equilibrium. Thus, the main result is that if parties are extreme, less information is supplied, than if parties are moderate.

Keywords: Political Parties, Campaigning.

JEL Classification: C72, D72, D89

---

*I am very grateful to John Duggan for his extensive comments. I am also grateful for helpful discussions with Nils Gottfries, César Martinelli and William Thomson and from participants at seminars/workshops at University of Rochester and Uppsala University. Financial support from the Wallander-Hedelius foundation is gratefully acknowledged.

†Department of Economics, Uppsala University, P.O.Box 513, S-751 20 Uppsala, Sweden. E-mail: Andreas.Westermark@nek.uu.se
1 Introduction

In political science, an important issue is information transmission between political representatives and the electorate. This issue has several aspects. One is the question whether parties send truthful information to voters, or not. This issue has been studied by Banks (1990), Harrington (1992) and others. Banks finds that, if the realized platform of a party is far away from the median of the voter distribution, voters are able to infer the true platform of that party. If the platform is close to the median of the voter distribution, this is not the case. Martinelli (1997) has studied whether voters can learn from parties that have private information during the electoral process. Schultz (1996) studies a situation where parties possess more information about the true state of the world compared with voters. He finds that polarization leads to non-revealing sequential equilibria. However, in none of these papers is it costly to send messages to voters. An important aspect in campaigning is that it is costly to send information to voters. This motivates the study of a model with costly information transmission.

The main aim of this paper is to analyze how parties use resources for campaigning to inform voters. We describe how the resources spent on campaigning depend on how close parties are to each other and how this in turn affects voters.

In the model there is an exogenously given number of parties. Each party has a predetermined ideology that is drawn from some distribution. This ideology could be determined by the history of the group, for example. The parties care about the number of votes as well as the consumption of some private good. Each party has access to resources that can be used for campaigning. Initially, voters do not know the platform of the parties. To affect the voters the parties use campaigning to inform the voters about the policy of the party. The parties are assumed to use only truthful messages. Thus, we disregard strategic issues of platform choice and focus on the strategic issues related to campaign spending. If a voter is informed by some party, it is assumed that he knows the platform of the party with certainty. This, combined with risk aversion, makes informed voters on average more positive to the party. Given the platforms and the strategies of the parties, voters update their beliefs and
then vote sincerely for the parties.

We study a symmetric model with two parties and find that, the farther away parties are from each other (on average), the less information is supplied (on average) in equilibrium. Note that informing a voter eliminates the risk of voting for that party. Then, consider voters that are informed by one party only, say party 1, and assume that the platform of party 1 is close to the median voter. For the indifferent voter, the expected platform of party 2 is closer to the voters’ peak than the actual platform of party 1. Since voters are risk-averse, the voter would otherwise strictly prefer to vote for party 1. If both parties move closer to each other, then the actual and expected platform moves closer to the indifferent voters peak. Since voter preferences are concave, the closer to the peak a platform is, the flatter preferences are. This implies that the increase in payoff of voting for the party that informed is bigger than the increase in payoff of voting for the other party. Thus, the previously indifferent voter now strictly prefers party 1. The effect makes parties gain more votes by informing when parties are moderate. Since spending increases, voters are (on average) more informed when parties are moderates. Thus, extremism leaves more voters uninformed. Also, the uninformed voters are going to be more uncertain if the parties are further away from each other. The reason is that the variability of spending goes down. Then we study a general model and are able to show existence of equilibrium.

One of the influences of this paper is Harrington and Hess (1996). In Harrington and Hess campaigning is explicitly modeled. Parties are assumed to have a fixed ideology. Parties can use resources either to move their platform closer to the opponent (positive campaigning) or to move their opponents platform further away from the party’s own platform (negative campaigning). However, there is no explicit model of why expenditures can affect voter’s perceptions of the parties. Thus, the model of influencing voters is modeled as a black box. As in this paper, concavity of voter preferences is important for the results.

The paper by Chappell (1994) has a more sophisticated model of voter behavior. In the model campaigning is assumed to be truthful. There are two parties that can choose either to spend an endowment on campaigning or not. Thus, only two possible levels of
campaigning are allowed. Existence of equilibrium cannot be proven even in this simple setup. In contrast, in the model presented here, equilibria generally exist. Also, the paper does not address the issue studied in this paper.

In section 2 the model is described and in section 4 existence of equilibrium is analyzed. In section 3 we study how spending depends on how extreme parties are and how this affects voters. Finally, section 5 concludes.

2 The Model

Let $Y \subseteq \mathbb{R}$ denote the policy space. There is a finite set of parties, denoted by $P$, and a continuum of voters. Let $X_k \subseteq Y$ denote the set of possible platforms for party $k$. For all $k \in P$, $X_k$ is compact. Let $X = \times_{k \in P} X_k$ denote the set of platform profiles. Also, for all $S \subseteq P$, let $X_S = \times_{k \in S} X_k$. For each $k \in P$, the platform is drawn from a probability distribution $g_k : X_k \to \mathbb{R}_+$. Let $p_k \in X_k$ denote the platform that is drawn. Let $p = (p_k)_{k \in P}$ and for all $S \subseteq P$, let $p_S = (p_k)_{k \in S}$. The platforms are not necessarily known to voters. Instead, they are perceived as uncertain. However, voters (and parties) know the distribution from which the platforms are drawn. Let $g(p) = \prod_{k \in P} g_k(p_k)$ and for all $S \subseteq P$, let $g_S(p_S) = \prod_{k \in S} g_k(p_k)$. Thus, the probability that a platform $p_k$ is drawn for party $k$ is independent of the platforms drawn for the parties other than $k$.

The parties can use resources to inform voters of their platform by campaigning. The parties truthfully reveal their platforms. Thus, a voter that is informed by party $k$ knows $p_k$ with certainty. A motivation for that assumption is that parties repeatedly take part in elections. By observing the actions of the parties in parliament, voters can infer whether parties have told the truth. The cost in terms of loss of reputation by sending untruthful messages deter parties from using such messages. Empirical evidence seems to justify this assumption. See for example Budge and Hofferbert (1990).

Each party $k$ has access to some resource $\omega_k \geq 0$, which can be used either for informing voters or for consumption. Let $\omega_{\text{max}} = \max_{k \in P} \omega_k$. Let $c_k \in [0, \omega_k]$ denote the resources party
Party $k$ has the following utility function, defined over pairs $(c_k, v_k)$,

$$u_k(c_k, v_k) = \omega_k - c_k + \nu v_k,$$

where $\nu > 0$ is a weight reflecting the importance of power. A motivation for the assumption that parties care about the number of votes is that the power of a party depends positively on the number of seats it controls. Another motivation is that the distribution of the electorate may be uncertain. Then $v_k$ is the probability of winning.

Voters vote sincerely, i.e. vote for the party that gives them the highest expected utility, given their beliefs concerning the platforms. Since there is a continuum of voters, strategic voting is not an issue. Let $\Pr_k(p_k)$ denote the voters (common) posterior belief that the true platform of party $k$ is $p_k$. Posterior beliefs are determined in equilibrium.

The preferences for voters are single peaked. Let $x^i$ denote the ideal point of voter $i$. All voters with the same ideal point have the same preferences. Then the population of voters can be described by the distribution of the voters’ ideal points. Let the density function be denoted by $f : Y \to \mathbb{R}_+$ and the cumulative distribution function by $F : Y \to [0, 1]$. Let $\underline{x}$ denote the lower bound of the voter distribution, if a lower bound exists. Otherwise, let $\underline{x} = -\infty$. Let $\overline{x}$ denote the upper bound of the voter distribution, if an upper bound exists. Otherwise, let $\overline{x} = \infty$. The figure below describes the model if there are two parties.

Let $V : \mathbb{R} \to \mathbb{R}$ be a concave function that is symmetric around zero and continuously differentiable. The function $V : \mathbb{R} \to \mathbb{R}$ is assumed to represent voters’ preferences. We assume that $V$ is concave, symmetric around zero and continuously differentiable.

Consider party $k$ and suppose that voter $i$ is not informed by party $k$. Then, the expected utility for the voter, when voting for party $k$, is given by

$$\int_{z \in X_k} V(z - x^i) \Pr_k(z)dz.$$ 

If party $k$ has informed the voter the payoff when voting for $k$ is

$$V(p_k - x^i).$$
2.1 Campaigning

The timing is the following. First, the platforms are revealed to the parties. Then the parties choose campaign expenditures. Parties have complete information. Campaign expenditures and initially, platforms, are unobservable by voters. Otherwise, voters could infer the true platform of a party by observing the resources spent, since they generally depend on the platform of the party.

To see this, consider an example with two political parties. Suppose the strategy of party 2 assigns a different spending level to each platform profile. Consider a voter informed about the policy of party 1 and knowing the spending level party 2 but not the platform. The voter can, by observing the cost, then infer the exact platform of party 2. The parties can observe the other parties’ expenditure choices, as well as the platforms of the other parties.

Voters are informed about the policy of party $k$ with probability $\rho(c_k)$ where $\rho : \mathbb{R}_+ \to [0, 1]$. We assume $\rho(0) \geq 0$, $\rho(y) < 1$ and $\rho'(y) \geq 0$ for all $y \in [0, \omega_{\max}]$. Also, for all $y > \omega_{\max}$ we have $\rho(y) = \rho(\omega_{\max})$. This assumption is made for technical convenience only. Also, $\rho$ is concave. A motivation for $\rho(0) > 0$ is that a voter could receive information from other
sources such as newspapers and television broadcasts. Since the population is large, \( p(c_k) \) is also the fraction of the population informed by party \( k \).

Consider the incentives for parties to inform voters. First, informing a voter leads to a reduction in risk of voting for the party that informed. Since voters are risk averse, this effect makes the voter like the party that informed the voter more. Second, by revealing the truth, a party also affect the platform the voters believe the party to have. For example, if the expected platform is closer to the voters peak than the actual platform, the utility of a voter decreases.

Consider the updating of voter beliefs. Given the information voters get they use the information to revise their beliefs about the parties. Clearly, if a party informs a voter, then the voter knows the platform of that party with certainty. However, if a voter receives information from party \( k \) but not from party \( j \), then the voter knows that party \( j \) takes the platform \( p_k \) as given when choosing his strategy. For each possible platform \( p_j \), given the strategy profile of party \( j \), there is some probability that the voter is not informed. Given some platform \( p_j \), if the probability that the voter is informed by \( j \) is lower than the average probability of being informed by \( j \), then the voter increases the weight on this platform \( p_j \). This follows, since being uninformed is more likely when \( p_j \) is true. A similar argument shows that, given \( p_j \), if the probability that the voter is informed by \( j \) is higher than the average probability of being informed by \( j \), then the voter decreases the weight on the platform \( p_j \). Thus, the voter can use Bayes rule to update beliefs over the types of party \( j \). This process of revising beliefs would in general be different for different \( p_k \), since the strategy of party \( j \) in general varies with \( p_k \).

Messages cannot be directed to specific groups of voters. Also, the probability that a voter is reached by party \( k \) is independent of the probability that he is reached by \( j \neq k \).
3 Extreme versus Moderate Parties

The main issue in this paper is what happens with campaigning and voter ambiguity if parties become more or less extreme. To study this, let $f$ be symmetric with mean 0. We also assume $f'(x) > 0$ whenever $x < 0$. Furthermore, let $X_k = \{p_kL, p_kH\}$ where $p_kH = p_kL + 2l$, $l > 0$ and $p_{2L} = -p_{1H}$. Each platform is drawn with probability $\frac{1}{2}$. This implies that the prior variance of the platforms for both parties is $l^2$. The parties have access to the same amount of resources. Let $X = X_1 \times X_2$. The strategy space for party $k$ is then $T_k = \{c : X \rightarrow [0, \omega]\}$. Let $T = T_1 \times T_2$. Define $\delta = E\{p_2\} = -E\{p_1\}$ as the mean of the prior distribution for party 2. Thus, a party is either far away from the median voter or close to the median voter, relative to the expected platform.

Also, we assume that $p_{2L} > 0 > p_{1H}$ which implies $\delta > l$. Thus, the platform of party 1 is always smaller that the platform of party 2. A motivation for this assumption is that, for example, voters usually know that republicans are to the right of democrats. If this were not the case the following might occur. Say that $l < p_{2L} < 0 < p_{1H}$ and the strategy profiles are symmetric. Then, if voters are not informed by any of the parties, all voters with ideal points below 0 prefer party 1 and the other voters party 2. Suppose voters are informed by both parties. Then, all voters with ideal points below 0 prefer party 2 and the other voters party 1. Thus, the electorate “switches” completely. The parties have access to the same amount of resources. Then, we let $\omega = \omega_1 = \omega_2$. Let $c_k(p_{1i}, p_{2j})$ be denoted $c_{kij}$. Given some platform $p \in X$, strategy profile $c \in T$ and voter beliefs $d \in T$, the votes for party 1 is

$$v_k(c \mid d, p) = \rho(c_{1ij})\rho(c_{2ij}) \int_{\mathbb{R}} f(y)dy + [1 - \rho(c_{1ij})][1 - \rho(c_{2ij})] \int_{\mathbb{R}} f(y)dy$$

$$+ [1 - \rho(c_{1ij})][1 - \rho(c_{2ij})] \int_{\mathbb{R}} f(y)dy + \rho(c_{1ij})\rho(c_{2ij}) \int_{\mathbb{R}} f(y)dy.$$ (1)

This section assumes that parties maximize votes. However, note that the function $v_k(c \mid d, p)$ can also be interpreted as the probability of winning an election. Furthermore, similar
results to those presented below can be shown if parties care about the outcome instead of maximizing votes or the probability of winning the election.

As in the paper by Banks, we restrict attention to symmetric equilibria. Also, we restrict attention to quadratic preferences.

### 3.1 Quadratic Preferences

Since the model in general is too complicated to solve analytically we restrict the analysis in section 3 to quadratic preferences. Then

\[ V(z - x^i) = -(z - x^i)^2. \]

Given some \( S \subseteq P \), \( c \in T \) and \( p_S \), let \( E[p_k \mid S, c, p_S] \) denote the expected platform of party \( k \) and let \( Var[p_k \mid S, c, p_S] \) denote the variance of the platform of party \( k \). Now consider two parties \( j \) and \( k \). Quadratic preferences implies that

\[ x_{jk}(S, c, p_S) = \]

\[ \frac{E[p_k \mid S, c, p_S] + E[p_j \mid S, c, p_S]}{2} + \frac{Var[p_k \mid S, c, p_S] - Var[p_j \mid S, c, p_S]}{2 (E[p_k \mid S, c, p_S] - E[p_j \mid S, c, p_S])}. \]  

To see this, note that the payoff when voting for party \( j \) is

\[ -E[(x_j - x^i)^2 \mid S, c, p_S] = -(E[x_j \mid S, c, p_S] - x^i)^2 - Var[p_j \mid S, c, p_S]. \]

Similarly, the payoff when voting for party \( k \) is

\[ -E[(x_k - x^i)^2 \mid S, c, p_S] = -(E[x_k \mid S, c, p_S] - x^i)^2 - Var[p_k \mid S, c, p_S]. \]

Setting expression 3 equal to 4 and solving for \( x^i \) gives expression 2.

### 3.2 Equilibrium

An equilibrium is a strategy profile for the parties such that, firstly, each party chooses an optimal strategy, taking the other parties’ strategies and voter beliefs as given and secondly, voters revise their beliefs given the strategy profile of the parties.
Definition 1 An Informational Nash Equilibrium is a $c^* \in T$ such that

i) for $k = 1, 2$, $c^*_k(z) = \arg \max_{e \in [0, \omega]} \omega - e + v_{k}(e, c^*_k \mid c^*, z)$ for all $z \in X$

ii) given $c^*$, voters revise beliefs using Bayes rule whenever possible.

We use this notion since voters - the players without full information - only passively take part in the game. The “real” players here are the parties. The parties have access to all information. Therefore, we do not use the term Bayesian Nash equilibrium.

The following Lemma gives posterior variances and expectations when a voter is reached by one party only. Then we are able to compute indifferent voters when a voter is reached by one party only. Note that, since we study symmetric equilibria, we can deduce that the indifferent voter is at 0, when voters are informed by none of the parties.

Lemma 2 Let $c$ be a strategy profile. Suppose $S = \{2\}$. If $p_2 = p_{2i}$ and $c_{2ji} > 0$ for some $j \in \{L, H\}$ then

$E[p_1 \mid S, c, p_S] = -\delta + l \frac{\rho(c_{2Hi}) [1 - \rho(c_{1Hi})] - \rho(c_{2Li}) [1 - \rho(c_{1Li})]}{\rho(c_{2Li}) [1 - \rho(c_{1Li})] + \rho(c_{2Hi}) [1 - \rho(c_{1Hi})]}$

and

$Var[p_1 \mid S, c, p_S] = 4l^2 \frac{\rho(c_{2Hi}) \rho(c_{2Li}) [1 - \rho(c_{1Li})] [1 - \rho(c_{1Hi})]}{(\rho(c_{2Li}) [1 - \rho(c_{1Li})] + \rho(c_{2Hi}) [1 - \rho(c_{1Hi})])^2}$

for each $i \in \{L, H\}$.

Proof. Updating of voters. Since $c_{2ji} > 0$ for some $j \in \{L, H\}$ Bayes rule applies.

Let $S = 2$. Bayes rule gives us

$Pr_1(p_{1L} \mid 2, c, p_2) = \frac{\rho(c_{2Li}) [1 - \rho(c_{1Li})]}{\rho(c_{2Li}) [1 - \rho(c_{1Li})] + \rho(c_{2Hi}) [1 - \rho(c_{1Hi})]} \frac{1}{\frac{1}{2} + \frac{1}{2}}$

for $i \in \{L, H\}$. Let $A_1 = \rho(c_{2Li}) [1 - \rho(c_{1Li})]$ and $B_1 = \rho(c_{2Li}) [1 - \rho(c_{1Li})] + \rho(c_{2Hi}) [1 - \rho(c_{1Hi})]$ for $i \in \{L, H\}$. Then $Pr_1(p_{1L} \mid 2, c, p_S) = \frac{A_1}{B_1}$ and

$E[p_1 \mid 2, c, p_S] = -\delta + l \frac{B_1 - 2A_1}{B_1}$. 

10
Also,
\[
V ar \left[p_1 \mid 2, c, p_S\right] = 4l^2 \frac{A_1[B_1 - A_1]}{B_l^2}.
\]
Substituting in the expressions for \(A_1\) and \(B_1\) gives the desired result. ■

A similar result holds for party 2.

Now consider the optimal spending choice when a party is far away from the median voter, relative to the voters prior.

**Lemma 3** Suppose \(c^*\) is a symmetric INE. For each \(i \in \{L, H\}\) we have \(c^*_{1Li} = 0\) and \(c^*_{2iH} = 0\).

**Proof.** Consider party 1. By symmetry, a similar argument holds for party 2. Let \(c\) be an arbitrary symmetric strategy profile. By hypothesis, we have \(p_1 = p_{1L}\). Note first that beliefs not necessarily is given by the result in Lemma 2. This follows, since positive spending is not required.

**Step 1:** \(x_{12}(1, c, p_1) \leq x_{12}(\emptyset, c, p_\emptyset)\) and \(x_{12}(12, c, p_{12}) \leq x_{12}(2, c, p_2)\).

**Case 1:** \(x_{12}(1, c, p_1) \leq x_{12}(\emptyset, c, p_\emptyset)\).

We have
\[
x_{12}(1, c, p_1) - x_{12}(\emptyset, c, p_\emptyset) = \frac{[p_{1L}^2 - p_{2L}^2 \Pr[p_{2L} \mid 1, c, p_1] - p_{2H}^2 \Pr[p_{2H} \mid 1, c, p_1]]}{2(p_{1L} - \mathbb{E}[p_2 \mid 1, c, p_1])}.
\]
Since \(p_{1L} < p_{2L} < p_{2H}\) and \(\Pr[p_{2L} \mid 1, c, p_1] + \Pr[p_{2H} \mid 1, c, p_1] = 1\) the numerator is non-negative and since \(p_{1L} < \mathbb{E}[p_2 \mid 1, c, p_1]\) the denominator is negative for all possible beliefs. Then \(x_{12}(1, c, p_1) - x_{12}(\emptyset, c, p_\emptyset) \leq 0\).

**Case 2:** \(x_{12}(12, c, p_{12}) \leq x_{12}(2, c, p_2)\).

We have
\[
x_{12}(12, c, p_{12}) - x_{12}(2, c, p_2) = \frac{p_{1L} - \mathbb{E}[p_1 \mid 2, c, p_2]}{2} - \frac{V[p_1 \mid 2, c, p_2]}{2(\mathbb{E}[p_1 \mid 2, c, p_2] - p_2)}
\]
\[
= \frac{(p_{1L} - p_{1H})p_{1H} \Pr[p_{1H} \mid 1, c, p_2] + p_2(\mathbb{E}[p_1 \mid 2, c, p_2] - p_{1L})}{2(\mathbb{E}[p_1 \mid 2, c, p_2] - p_2)}.
\]
Since $(p_{1L} - p_{1H}) < 0$, $p_2 > 0$, $p_{1H} < 0$ and $E[p_1 | 2, c, p_2] - p_{1L} \geq 0$ the numerator is nonnegative for all possible beliefs. Also, since $E[p_1 | 2, c, p_2] < p_2$, the denominator is negative for all possible beliefs. Then we have $x_{12}(12, c, p_{12}) - x_{12}(2, c, p_2) \leq 0$.

**Step 2:** $c_{1Li}^* = 0$. Since $x_{12}(1, c, p_{12}) - x_{12}(\varnothing, c, p_{\varnothing}) \leq 0$ and $x_{12}(12, c, p_{12}) - x_{12}(2, c, p_2) \leq 0$ by Step 1, party 1 loses by informing. Note that this holds for any symmetric strategy profile. Then $c_{1Li}^* = 0$. A similar argument holds for party 2.

Thus, each party spends nothing if the realized platform is far away from the median voter, compared with the prior. Note first that, to increase the vote share, a party must convince the voters with peaks in the middle of the voter distribution. Also, recall that when informing voters, the voters become aware of the true platform of the party. Since the true platform is farther away from the median voter than expected, voters in the middle then like the party less. There is also a reduction in risk of voting for the party. However, this reduction in risk is not sufficient to counter the negative surprise for voters in the middle. The total effect on voters in the middle is that they like the party that informed them less. This leads to zero spending. This result makes it easy to compute expenditures for the party.

Now consider the expenditures when parties have a realization of the platform that is close to the median voter, compared with the prior. The following Lemma shows that the spending level of a party is independent of the realized platform of the other party.

**Lemma 4** Suppose $c^*$ is a symmetric INE. For each $i, j \in \{L, H\}$ we have $c_{1Hi}^* = c_{2jL}^* = c_m^* \in [0, \omega]$.

**Proof.** Let $c$ be an arbitrary symmetric strategy profile. Consider party 1. By hypothesis we have $p_1 = p_{1H}$.

**Step 1:** The votes for party 1.

Note that, since $x_{12}(\varnothing, c, p_2) = 0$, we have $\int_{\varnothing}^{x_{12}(\varnothing, c, p_2)} f(y)dy = \frac{1}{2}$.

**Case 1:** $p_2 = p_{2L}$.  

---

12
By using expression 1 the votes for party 1 is

$$
\rho(c_{1HL}^*)\rho(c_{2HL}^*) \left[ \frac{1}{2} \right] + \rho(c_{1HL}^*) \left[ 1 - \rho(c_{2HL}^*) \right] \left[ \frac{1}{2} + \int_0^{x_{12}(1,c,p_1)} f(y)dy \right] + \\
\left[ 1 - \rho(c_{1HL}^*) \right] \rho(c_{2HL}^*) \left[ \frac{1}{2} - \int_{x_{12}(2,c,p_2)}^0 f(y)dy \right] + \left[ 1 - \rho(c_{1HL}^*) \right] \left[ 1 - \rho(c_{2HL}^*) \right] \frac{1}{2}.
$$

**Case 2:** \( p_2 = p_{2H} \).

Then \( c_{2HH}^* = 0 \) and \( \rho(c_{2HH}^*) = 0 \). The votes for party 1 is then

$$
\rho(c_{1HH}^*) \left[ \frac{1}{2} + \int_0^{x_{12}(1,c,p_1)} f(y)dy \right] + \left[ 1 - \rho(c_{1HH}^*) \right] \frac{1}{2}.
$$

(6)

**Step 2: Optimal spending choice.**

By symmetry, we have \( x_{12}(2,c,p_2) = -x_{12}(1,c,p_1) \). Then, by symmetry of the voter distribution, we have (in Case 1)

$$
\int_0^{x_{12}(1,c,p_1)} f(y)dy = \int_{x_{12}(2,c,p_2)}^0 f(y)dy.
$$

From Case 1 and 2 in Step 1, the first-order condition is

$$
\nu \rho'(c_{1Hi}^*) \int_0^{x_{12}(1,c,p_1)} f(y)dy - 1 = 0
$$

(7)

for all \( i \in \{L, H\} \). Since expression 7 only depend on \( c_2 \) through beliefs, the optimal choice for party 1 is independent of the choice of \( c_2 \), given voter beliefs. This implies \( c_{1HL}^* = c_{1HH}^* \).

Then, let \( c_m^* = c_{1HL}^* \) denote the solution to 7. By symmetry, we have \( c_m^* = c_{2HL}^* = c_{1LL}^* \).

The reason behind this result is the following. Consider the effects when party 1 informs voters and when both parties have platforms close to the median voter. Note that the change in indifferent voter when informing, \( x_{12}(1,c,p_1) - x_{12}(\emptyset,c,p_\emptyset) \) or \( x_{12}(12,c,p_{12}) - x_{12}(2,c,p_1) \), is the same, since \( x_{12}(\emptyset,c,p_\emptyset) = x_{12}(12,c,p_{12}) = 0 \) and \( x_{12}(1,c,p_1) = -x_{12}(2,c,p_1) \). By
symmetry of the voter distribution, the effect of informing on the vote share is independent
of the strategy of party 2, given voter beliefs.

Let

\[ z(b, \delta) = \frac{l\delta}{(b(2 - \rho(b)) - l(1 - \rho(b)))}. \]  

(8)

Note that, if \( c \) is symmetric and if \( c_{2HL} = c_{2LL} = b \) then \( z(b, \delta) = x_{12}(1, c, p_S) = -x_{12}(2, c, p_S) \).

From the Lemma above, the set of symmetric equilibria can be described by

\[ \nu\int \frac{z(c_m^*, \delta)}{\rho'(c_m^*)} f(y)dy - 1 = 0. \]

Uniqueness cannot be guaranteed. A common method when selecting among equilibria is
to find restrictions on out of equilibrium beliefs. When \( c_m^* > 0 \), and if both parties have
platforms close to the median voter, all information sets are reached. If a party have a
platform far away from the median voter, that party spends nothing, as can be seen from
Lemma 3. The only information set that are not reached in equilibrium are those where
a party with a platform far away from the median informs a voter. However, restricting
beliefs in these information do not help in selecting equilibria. The reason is that, as can be
seen from Lemma 3, zero spending is optimal for \textit{all} possible beliefs at this information set.
Instead, we focus attention on stable equilibria.

3.3 Stability

To analyze stability, we consider an equilibrium and perturb the equilibrium profile. The
players take the new profile as given and reoptimize. This again gives a new profile. Given
this profile, the players reoptimize again and so forth. This gives a system of differential
equations by using the first-order conditions. Let \( c^0 \) denote the starting value of \( c \). Let the
system of differential equations be denoted \( c(t) \). For all \( c, d \in T \), let \( \|d - c\|_e \) denote the
(euclidean) distance between \( c \) and \( d \). Let a solution to the system of differential equations
be denoted \( \phi(t, c^0) \).
Definition 5 An equilibrium $c^*$ is **locally stable** if there exists a $\eta > 0$ such that

$$\|c^0 - c^*\|_e \leq \eta \Rightarrow \lim_{t \to \infty} \phi(t, c^0) = c^*.$$ 

Unfortunately, existence of a locally stable equilibrium cannot be proven. However, if we weaken the notion of stability a little, we can show existence. We require that a strategy profile is stable among the set of symmetric strategy profiles. A motivation for this is the following. Suppose voters believe that parties use symmetric strategy profiles. Suppose voter beliefs are perturbed, **assuming** that beliefs are symmetric. Then the best response for the parties is symmetric. A tanónnement adjustment would then only consist of symmetric strategy profiles.

Definition 6 An equilibrium $c^*$ is **weakly locally stable** if Definition 5 holds among the set of symmetric strategy profiles.

Let $\emptyset \subset \mathbb{R}_{++}$ denote the (open) set of possible parameter values of $\delta$. Also, note that the function $z$, defined in expression 8, is decreasing in $\delta$. Since $\delta > l$ and $\rho(b) < 1$, we have, for all $\delta \in \Delta, \frac{1}{2} < z(b, \delta) < l$. The following Lemma shows existence. The conditions on $\rho$ and the distribution guarantees the existence of an interior solution.

**Lemma 7** Suppose $\nu \rho'(0) \int_0^1 f(y)dy - 1 > 0, \nu \rho'(\omega) \int_0^1 f(y)dy - 1 < 0$. There exists a symmetric, weakly locally stable INE a.e. on $\Delta$.

**Proof.** Symmetric equilibria.

**Step 1:** The fixed point mapping.

Consider

$$\nu \rho'(c_m) \int_0^{z(b, \delta)} f(y)dy - 1 = 0.$$ 

This expression defines a mapping $k : [0, \omega] \to [0, \omega]$, such that $k(b) = c_m$. Given voter beliefs $b$ and $\delta$, $c_m$ is the best response of parties. Also, by construction, $k(b)$ is the optimal
campaign expenditure level for a party that has a platform close to the median voter, for beliefs resulting from \( b \).

Since
\[
\frac{\partial z}{\partial b} = \frac{\rho'(b)l \delta [\delta - l]}{(\delta (2 - \rho(b)) - l (1 - \rho(b)))^2} > 0,
\]
\( \rho'(c_m) > 0 \) and \( \rho''(c_m) < 0 \) for any \( c_m \in (0, \omega) \), it follows that \( k \) is single-valued and nondecreasing. The fixed points of the mapping \( k \) is the set of symmetric equilibria. Since \( z \) is continuously differentiable and \( \rho \) is twice continuously differentiable, if \( b \in (0, \omega) \) and \( k(b) \in (0, \omega) \), then \( k \) is differentiable.

Since \( k \) is a continuous function, there exists at least one symmetric equilibrium.

**Figure 2.** The mapping \( k \) and the set of symmetric equilibria.

**Step 2:** Using \( k \) to find a candidate equilibrium.

Consider any interior fixed point \( c_m^* \) such that
\[
k'(c_m^*) = -\frac{\rho'(c_m^*)f(z(c_m^*, \delta)) \frac{\partial z}{\partial c_m^*}}{\rho''(c_m^*) \int_0^\omega f(y)dy} < 1.
\]

16
Any fixed point satisfying 10 is a candidate equilibrium. The technical conditions in the statement of the Lemma imply that \( k(0) > 0 \) and \( k(\omega) < \omega \). Then, Proposition 8.3.1 in Mas-Colell (1989) implies that there exists such a fixed point for almost all \( \delta \in \Delta \). In Figure 2, the fixed points with the lowest and highest spending levels are candidate equilibria.

**Step 3.** First, we show that we can restrict attention to the stability of parties where the realized platform is close to the median voter.

By using a similar argument as in Lemma 2 it can be shown that

\[
x_{12}(\emptyset, c, p_\emptyset) = \frac{l \delta (A_2^\emptyset - A_1^\emptyset)}{((\delta - l) B^\emptyset + l A_2^\emptyset + l A_1^\emptyset)}
\]

where

\[
A_1^\emptyset = (1 - \rho(c_{1LL})) (1 - \rho(c_{2LL})) + (1 - \rho(c_{1HH})) (1 - \rho(c_{2HH})) ,
\]

\[
A_2^\emptyset = (1 - \rho(c_{1HH})) (1 - \rho(c_{2HH})) + (1 - \rho(c_{1LL})) (1 - \rho(c_{2LL}))
\]

and

\[
B^\emptyset = \sum_{j \in \{L, H\}} \sum_{i \in \{L, H\}} (1 - \rho(c_{1ij})) (1 - \rho(c_{2ij})).
\]

Using that \( c_{2LL} = c_{1HH} = c_f \) and \( c_{1HL} = c_{2HL} = c_c \) by symmetry gives

\[
x_{12}(\emptyset, c, p_\emptyset) = 0.
\]

Note that \( x_{12}(\emptyset, c, p_\emptyset) = 0 \) implies that parties with realized platforms far away from the median voter always choose to spend 0. Then, we can restrict attention to spending of parties with platforms close to median voter.

**Step 4:** The system of differential equations.

Symmetry implies that \( c_{2LL} = c_{1HH} = c_f \) and \( c_{1HL} = c_{2HL} = c_c \). First, note that

\[
x_{12}(2, c, p_2) = -\frac{A_1 l \delta}{(\delta - l) B_1 + A_1 l}
\]

where

\[
A_1 = \rho(c_{2LL})
\]

and

\[
B_1 = \rho(c_{2LL}) + \rho(c_{2HL}) (1 - \rho(c_{1HL})).
\]
Symmetry implies that

$$x_{12}(2, c, p_2) = -\frac{\rho(c_f)l\delta}{(\delta - l)[\rho(c_f) + \rho(c_c)(1 - \rho(c_c))] + \rho(c_f)l} = -x_{12}(1, c, p_1).$$

Optimally chosen symmetric strategy profiles are described by the following equations:

$$\nu \rho'(c_c) \left\{ \begin{array}{l}
\rho(c_c) \frac{1}{2} + [1 - \rho(c_c)] \int_{-l}^{x_{12}(1,c,p_1)} f(y)dy \\
\rho(c_c) \int_{-l}^{x_{12}(2,c,p_2)} f(y)dy - [1 - \rho(c_c)] \int_{-l}^{x_{12}(2,c,p_2)} f(y)dy \end{array} \right\} - 1 = 0$$

and

$$\nu \rho'(c_f) \int_{x_{12}(2,c,p_2)}^{x_{12}(1,c,p_1)} f(y)dy - 1 = 0.$$

Let

$$C(c^*_m) = \frac{1}{\left[\rho(c^*_m)\right]^2} \frac{\delta l [\delta - l] \rho'(c^*_m) \rho(c^*_m)}{(\delta - l)(2 - \rho(c^*_m) + l)^2},$$

$$\bar{f}(c^*_m) = f(x_{12}(1, c^*, p_1))$$

and

$$S(c^*_m) = \nu \rho''(c^*_m) \int_{0}^{x_{12}(1,c^*,p_1)} f(y)dy.$$

Note that $S(c^*_m)$ is the second-order condition of the maximization problem of the parties.

Using a first-order Taylor series expansion at $c^*_m$ of the equilibrium conditions, gives the system of differential equation as

$$\begin{bmatrix}
\frac{dc_c}{dt} \\
\frac{dc_f}{dt}
\end{bmatrix} =
\begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix}
\begin{bmatrix}
c_c(t) - c^*_m \\
c_f(t) - c^*_m
\end{bmatrix}.$$ (11)

where

$$b_{11} = S(c^*_m) - \nu \rho'(c^*_m) \bar{f}(c^*_m) [(1 - \rho(c^*_m)) - \rho(c^*_m)] C(c^*_m),$$

$$b_{12} = \nu \rho'(c^*_m) \bar{f} [1 - \rho(c^*_m)] C(c^*_m),$$

$$b_{21} = \nu \rho'(c^*_m) \bar{f}(c^*_m) [(1 - \rho(c^*_m)) - \rho(c^*_m)] C(c^*_m),$$

$$b_{22} = \nu \rho''(c^*_m) \bar{f} [1 - \rho(c^*_m)] C(c^*_m).$$
\[ b_{21} = -\nu \rho'(c^*_m) \bar{f} [ (1 - \rho(c^*_m)) - \rho(c^*_m) ] C(c^*_m) \]

and

\[ b_{22} = S(c^*_m) + \nu \rho'(c^*_m) \bar{f} [ 1 - \rho(c^*_m) ] C(c^*_m) . \]

**Step 5:** Stability of system in Step 4.

For \( i = 1, 2 \), let \( \lambda_i \) denote the eigenvalues of this matrix. We have

\[ \lambda_1 = S(c^*_m) \]

and

\[ \lambda_2 = S(c^*_m) + \rho(c^*_m) C(c^*_m) \nu \rho'(c^*_m) \bar{f}(c^*_m) . \]

The first eigenvalue is negative by the second-order condition and the second by Condition 10, since \( \frac{\partial z}{\partial c^*_m} = \rho(c^*_m) C(c^*_m) \nu \) and \( \bar{f}(c^*_m) = f(z(c^*_m, \delta)) \).

Thus, the real parts of all eigenvalues are negative. Thus, any solution \( c^*_m \) found in Step 2 is weakly stable.

Note that \( \lambda_2 \geq 0 \) for any fixed point such that \( k'(b) \geq 1 \). Thus, the linear approximation of the differential equations used above is not stable for any such fixed point. ■

Now consider the effects of changes in \( \delta \) on campaign spending. Let \( \kappa : \Delta \rightarrow [0, \omega] \)
denote the correspondence that associates to each \( \delta \) the set of symmetric equilibria.

We have the following Proposition that shows that spending decreases with polarization.

**Proposition 8** Let \( c^* \) be any stable symmetric INE, where \( c^*_m \in (0, \omega) \). We have \( \frac{dc^*_m}{d\delta} < 0 \).

**Proof.** Note that a stable equilibrium is locally isolated. Consider the effects of a small change in \( \delta \). At a fixed point we have

\[ 0 = \nu \rho'(c^*_m) \int_0^{z(c^*_m, \delta)} f(y)dy - 1 . \]  

Consider a linear approximation of \( \kappa \) at \( (c^*_m, \delta) \). The linear approximation is given by \( c_m = \alpha c^*_m \delta + \beta c^*_m \). By using a linear approximation at an isolated symmetric equilibrium

19
we get
\[ \alpha_{c_m, \delta} = -\frac{\nu \rho'(c_m^*) \tilde{f}(c_m^*) \frac{\partial z}{\partial \delta}}{\rho''(c_m^*) \int_0^{\frac{l\delta}{\delta - \rho(c_m)}} f(y) dy + \rho'(c_m^*) \tilde{f}(c_m^*) \frac{\partial z}{\partial c_m}}. \]

Since \( z(c_m^*, \delta) = -\frac{l\delta}{l - \delta + \delta - \rho(c_m)} = \frac{l\delta}{\delta (2 - \rho(c_m^*)) - l(1 - \rho(c_m^*))} \), we have
\[ \frac{\partial z}{\partial \delta} = -\frac{l^2 (1 - \rho(c_m^*))}{(\delta (2 - \rho(c_m^*)) - l (1 - \rho(c_m^*)))^2}. \] (13)

Thus, \( \frac{\partial z}{\partial \delta} < 0. \)

Since \( \frac{\partial z}{\partial c_m} = \rho(c_m^*) C(c_m^*) \nu \), we have \( \lambda_2 = \rho''(c_m^*) \int_0^{\frac{l\delta}{\delta - \rho(c_m)}} f(y) dy + \rho'(c_m^*) \tilde{f}(c_m^*) \frac{\partial z}{\partial c_m} \). Weak stability implies that \( \lambda_2 < 0. \) If \( c^* \) is weakly locally stable, then \( \lambda_2 < 0 \) and \( \frac{\partial z}{\partial \delta} < 0 \) imply that \( \frac{\partial c_m^*}{\partial \delta} = \alpha_{c_m, \delta} < 0. \)

The reason behind the result is the following. Note that, for fixed voter beliefs about \( c_m^* \), the positive effect of reduced uncertainty on voters is unchanged. Consider voters that are informed by, say party 1 only, and assume that \( p_1 = p_{1H} \). For the indifferent voter, the expected platform of party 2 is closer to the voters’ peak than the actual platform of party 1. Since voters are risk-averse the voter would otherwise strictly prefer to vote for party 1.

If both parties move closer to each other, then the actual and expected platform moves closer to the indifferent voters peak by exactly the same distance. By strict concavity, the closer to the peak a platform is, the flatter preferences are. Then, since the actual and expected platform of the two parties move the same distance, the increase in payoff of voting for the party that informed is bigger than the increase for the other party. Thus, the voter with peak at \( x_{12}(1, c^*, p_{1H}) \) strictly prefers party 1. This makes parties gain more votes by informing if the parties are moderates, when the realized platform is close to zero. This can be seen in Figure 3, where the platform profile changes from \( p \) to \( p' \), by the distance \( \Delta p \).

Since the parties spend nothing when they are far away from zero this implies that both the mean and variability of spending increases when \( c_m^* \) increases. The increase in spending also reduce the posterior variance associated with each party. This reduces the incentives to inform voters. However, the reduction only partially counteracts the initial increase.
Figure 3. Effect on the Indifferent Voter Informed by Party 1 only when Parties Become more Moderate (for fixed voter beliefs).

This can also be seen in Figure 2. If \( \delta \) decreases, then \( x_{12}(1, c, p_{1H}) \) moves farther away from the mean of the voter distribution. Then the gain in votes increases and the function \( k \) shifts upward. The stable equilibria “move” upward and unstable “move” downward.

3.4 Voter Uncertainty

Reduction of voter uncertainty in this model comes from two different sources. First, an increase in spending leads to a bigger share of the electorate being informed. Then, since \( c^*_m \) increase as parties get closer to each other, the share of the electorate that is informed increases. The second source is the effect on beliefs of uninformed voters. Since

\[
Var[p_k | S, e^*, p_S] = 4 \rho^2 \frac{1 - \rho(c^*_m)}{(2 - \rho(c^*_m))^2}
\]
in a symmetric equilibrium we have
\[
\frac{\partial \text{Var} \ [p_k \mid S, c^*, p_S]}{\partial c_m^*} = \frac{\rho'(c_m^*)4\rho^2(c_m^*)}{(2 - \rho(c_m^*))^3} < 0.
\]

The fact that spending when a party is close to the median voter \((c_m^*)\) increases while spending when far away from the median voter is zero, makes it possible for uninformed voters to make a more precise prediction of the platform of the party. An uninformed voter is more sure that the party that has not informed him is extreme. This in turn leads to the posterior uncertainty associated with a particular party to decrease. Thus, the uncertainty associated with the parties decrease, also for the uninformed voters.

### 3.5 Stable equilibria

It can be shown that any stable equilibrium is also weakly stable. Thus, if a stable equilibrium exists, all comparative statics results described above holds for this equilibrium.

### 4 General Model

Since voters need to compute expected utility of voting for parties they have not been informed by, we need to restrict the strategies for the parties to integrable functions. Let \(L^2[X]\) denote the set of measurable functions on \(X\) such that \(\|f\| = \left[ \int_X |f|^2 \right]^{\frac{1}{2}}\) is bounded. Let the set of strategies for party \(k\) be

\[
T_k = \{ f \in L^2[X] \mid 0 \leq f(x) \leq \omega_k \text{ for a.e. } x \}.
\]

Thus, the strategy space for each party consist of all integrable functions that are smaller than the endowment almost everywhere and bounded in the \(L^2\) norm. A strategy for party \(k\) is denoted \(c_k\). Let \(T = \times_{k \in \mathcal{P}} T_k\).

Let \(\lambda_k(A)\) denote the Lebesgue measure of \(A \subseteq X_k\). For all \(S \subseteq \mathcal{P}\), let \(\lambda_S(A)\) denote the product measure of a set \(A \subseteq X_S\). Also, let \(\lambda(A)\) denote the product measure of a set \(A \subseteq X\).

Convexity and compactness of the strategy sets is shown in the following Lemma.
Lemma 9 For all $k \in P$, the strategy space $T_k$ is convex and compact.

Proof. Step 1: Convexity.

Let $c_k, d_k \in T_k$. Then there exists a set $X^c \subseteq X$ such that $\lambda(X^c) = 1$, where for all $x \in X^c$ we have $c_k(x) \leq \omega_k$. Similarly, there exists a set $X^d \subseteq X$ such that $\lambda(X^d) = 1$, where for all $x \in X^d$ we have $d_k(x) \leq \omega_k$. Let $e_k = \alpha c_k + (1 - \alpha) d_k$ for some $\alpha \in [0, 1]$. Note that $\lambda(X^c \cap X^d) = 1$. Clearly, for all $x \in X^c \cap X^d$ we have $0 \leq e_k(x) \leq \omega_k$.

It remains to show that $e_k \in L_2[X]$. Since $c_k$ and $d_k$ are measurable, $e_k$ is measurable. Also, $\int e_k^2 = \alpha^2 \int c_k^2 + 2\alpha(1 - \alpha) \int c_k d_k + (1 - \alpha)^2 \int d_k^2$. Since $\int c_k^2 < \infty$ and $\int d_k^2 < \infty$ and, moreover, $\int c_k d_k \leq \int \omega_k^2$, we have $\int e_k^2 < \infty$. Thus $e_k \in L_2[X].$

Step 2: Compactness.

Let $B = \{f \mid \|f\| \leq \omega_k\}$ be the ball of diameter $\omega_k$. This ball contains $T_k$. By Alaoglu's Theorem, (Royden, (1988)), it is compact. Thus, we need to show that $T_k$ is closed. Let $\{e_k^n\}_{n=1}^\infty$ be a sequence of functions such that $e_k^n \to e_k$ and, for all $n$, $e_k^n \in T_k$. Then, as a corollary of the Riesz-Fischer Theorem (Aliprantis and Burkinshaw (1990) p. 206), there exist a set $\hat{X} \subseteq X$ such that $\lambda(\hat{X}) = 1$ and a subsequence $\{e_k^n_l\}_{l=1}^\infty$ that converges pointwise on $\hat{X}$. Suppose $c_k \notin T_k$. Then there exists a set $U = \{z \in X \mid c_k(z) > \omega_k\}$ such that $\lambda(U) > 0$. Note that $\lambda(U \cap \hat{X}) > 0$ and $\int_{U \cap \hat{X}} c_k > \lambda(U \cap \hat{X}) \omega_k$. Since $\{e_k^n_l\}_{l=1}^\infty$ converges pointwise on $\hat{X}$, we have $\int_{U \cap \hat{X}} e_k^n_l \to \int_{U \cap \hat{X}} c_k > \lambda(U \cap \hat{X}) \omega_k$. This contradicts the fact that $e_k^n_l \in T_k$ for all $l$. ■

4.1 Voters Always Reached with Positive Probability

In this section, we only allow $\rho(0) > 0$. This assumption is relaxed in the next section.

Let $c = (c_k)_{k \in P}$. Since $\rho(0) > 0$, there is a positive probability that voters are informed by the parties in $S$ when $p_S$ is true. Then Bayes rule can be applied and we have the following result. For all $S \subseteq P$, all $k \notin S$, all $z \in X_k$, $p_S \in X_S$ and all $c \in T$, the posterior density, denoted $\Pr_k(z \mid S, c, p_S)$, is

$$\Pr_k(z \mid S, c, p_S) =$$

23
\[
\frac{\int_{y \in X_{P \setminus (S \cup \{\emptyset\})}} \prod_{j \in S} \rho(c_j(p_S, z, y)) \prod_{l \in P \setminus S} [1 - \rho(c_l(p_S, z, y))] g(p_S, z, y) dy}{\int_{x \in X_{P \setminus S}} \prod_{j \in S} \rho(c_j(p_S, x)) \prod_{l \in P \setminus S} [1 - \rho(c_l(p_S, x))] g(p_S, x) dx}.
\] (14)

If the expression above can be used to define a probability measure, the corresponding measure is denoted \( \Pr_k(S, c, p_S) \). If \( S = \emptyset \), then this is a well defined probability measure, since \( c \) is measurable on \( X \). Then, when \( S \neq \emptyset \), this is also a well defined probability measure (Billingsley (1986), section 33). When studying convergence of measures, we use the weak* topology.

To see how expression 14 is derived, let \( p(S \mid c, x) \) denote the probability that a voter is informed by exactly the parties in \( S \), given \( c \) and \( x \in X \). Bayes rule gives
\[
\Pr_k(z \mid S, c, p_S) = \frac{\int_{y \in X_{P \setminus (S \cup \{\emptyset\})}} p(S \mid c, (p_S, z, y)) g(p_S, z, y) dy}{\int_{x \in X_{P \setminus S}} p(S \mid c, (p_S, x)) g(p_S, x) dx}.
\] (15)

Since the electorate is large the share of the population of voters that is informed about the platforms for the parties in \( S \), given \( x \in X \), is
\[
p(S \mid c, x) = \prod_{j \in S} \rho(c_j(x)) \prod_{l \in P \setminus S} [1 - \rho(c_l(x))].
\] (16)

Combining expressions 15 and 16 gives expression 14.

The following Lemma shows that, for all \( S \subset P \), if \( c^n \to c \) then there exists a subsequence of \( \{c^n\}_{n=1}^\infty \) such that, for almost all \( p_S \in X_S \), the subsequence converges pointwise a.e. on \( X_{-S} \).

Note that, if \( \{c^n\}_{n=1}^\infty \) is a sequence such that \( c^n \to c \), then, by the Riesz-Fischer Theorem, there exists a set \( \hat{X} \subseteq X \) such that \( \lambda(\hat{X}) = 1 \) and a subsequence, \( \{c^{n_l}\}_{l=1}^\infty \), that converges pointwise on \( \hat{X} \).

**Lemma 10** Let \( \{c^n\}_{n=1}^\infty \) be a sequence such that \( c^n \to c \). For all \( S \subset P \), there exists a set \( \hat{X}_S \subseteq X_S \) such that \( \lambda_S(\hat{X}_S) = 1 \), a set \( \hat{X}_{-S}(p_S) = \{y \in X_{-S} \mid (p_S, y) \in \hat{X}\} \) such that, for all \( p_S \in \hat{X}_S \), \( \lambda_{-S}(\hat{X}_{-S}(p_S)) = 1 \) and a subsequence \( \{c^{n_l}\}_{l=1}^\infty \) such that, for all \( p_S \in \hat{X}_S \), all \( x \in \hat{X}_{-S}(p_S) \), \( c^{n_l}(p_S, x) \) converges to \( c(p_S, x) \).
Proof. Let \( \{c^n\}_{n=1}^{\infty} \) be a sequence such that \( c^n \to c \).

Then, let \( \hat{X}_-(p_S) = \{x \in X_- | (p_S, x) \in \hat{X}\} \). By Theorem 22.4 in Aliprantis and Burkinshaw (1990) there exists a set \( Y_S \subseteq X_S \) such that \( \lambda_S(Y_S) = 1 \) and, for all \( p_S \in Y_S \), \( \hat{X}_-(p_S) \) is measurable.

**Step 1:** Finding measurable sets \( \hat{X}_S \) and \( \hat{X}_S^c \) such that \( \lambda_S(\hat{X}_S) + \lambda_S(\hat{X}_S^c) = 1 \), and, for all \( p_S \in \hat{X}_S \), we have \( \lambda_S(\hat{X}_-(p_S)) = 1 \) and all \( p_S \in \hat{X}_S^c \), we have \( \lambda_S(\hat{X}_-(p_S)) < 1 \).

**a)** Finding \( \hat{X}_S \).

Consider the sets \( \hat{X}_S = \{p_S \in Y_S | \lambda_S(\hat{X}_-(p_S)) = 1\} \) and \( \hat{X}_S^c = \{p_S \in Y_S | \lambda_S(\hat{X}_-(p_S)) < 1\} \). If \( \hat{X}_S \) is measurable, let \( \hat{X}_S = \hat{X}_S \). Suppose \( \hat{X}_S \) is not measurable. Let \( \lambda' \) denote the outer measure associated with \( \lambda \). By Theorem 12.11 in Aliprantis and Burkinshaw (1990) there exists a measurable set \( E \supseteq \hat{X}_S^c \) such that \( E \subseteq Y_S \) where \( \lambda'(E) = \lambda'(\hat{X}_S) \). Then, let \( \hat{X}_S = Y_S \backslash E \subset Y_S \backslash \hat{X}_S = \hat{X}_S \). Since \( \hat{X}_S \subseteq \hat{X}_S \) we have, for all \( p_S \in \hat{X}_S \), \( \lambda_S(\hat{X}_-(p_S)) = 1 \).

Thus, \( \hat{X}_S \subseteq Y_S \) is a set of platforms for the parties in \( S \) such that, given any \( p_S \in \hat{X}_S \), \( \{c^n\}_{n=1}^{\infty} \) converges pointwise, for almost all \( x \in X_-\).

**b)** Finding \( \hat{X}_S^c \).

If \( \hat{X}_S^c \) is measurable, let \( \hat{X}_S^c = \hat{X}_S^c \). Suppose \( \hat{X}_S^c \) is not measurable. As in **a)**, there exists a measurable set \( E' \supseteq \hat{X}_S \) such that \( E' \subseteq Y_S \) where \( \lambda'(E') = \lambda'(\hat{X}_S) \). Let \( \hat{X}_S^c = Y_S \backslash E' \subset Y_S \backslash \hat{X}_S = \hat{X}_S \). Since \( \hat{X}_S \subseteq \hat{X}_S \) we have, for all \( p_S \in \hat{X}_S \), \( \lambda_S(\hat{X}_-(p_S)) < 1 \).

Thus, \( \hat{X}_S^c \subseteq Y_S \) is a set of platforms for the parties in \( S \) such that, given any \( p_S \in \hat{X}_S^c \), \( \{c^n\}_{n=1}^{\infty} \) converges pointwise on a set \( Z_-\subseteq X_- \) where \( \lambda_S(Z_-) < 1 \).

**c)** Proving that \( \lambda_S(\hat{X}_S) + \lambda_S(\hat{X}_S^c) = 1 \).

Note that since \( \lambda_S(Y_S) = 1 \) we have \( \lambda_S(\hat{X}_S) = \lambda'(Y_S \backslash E) = 1 - \lambda'(E) \). Also, we have \( \lambda_S(\hat{X}_S^c) = \lambda'(Y_S \backslash E') = 1 - \lambda'(E') \). Also, by definition of the outer measure, we have \( \lambda'(E) = \lambda'(\hat{X}_S) = 1 - \lambda'(\hat{X}_S) = 1 - \lambda'(E') \). Combining these conditions gives \( \lambda_S(\hat{X}_S) + \lambda_S(\hat{X}_S^c) = 1 \). Also, from **a)** and **b)**, we have \( \hat{X}_S \cup \hat{X}_S^c \subseteq Y_S \).

**Step 2:** Proving that \( \lambda_S(\hat{X}_S) = 1 \).

Suppose that \( \lambda_S(\hat{X}_S) > 0 \). Let \( \tilde{X} = \bigcup_{p_S \in Y_S} (\{p_S\} \times (X_- \backslash \hat{X}_-(p_S))) \). If \( \tilde{X} \) is measu-
surable, let $\tilde{X} = \check{X}$. If $\tilde{X}$ is not measurable, there exists a measurable set $E'' \supseteq \tilde{X}$ where $\lambda^*(E'') = \lambda^*(\tilde{X})$. Moreover, since $\tilde{X} \subseteq X \setminus \check{X}$ and $\check{X}$ is measurable we have $\lambda(E'') = \lambda^*(\tilde{X}) \leq \lambda^*(X \setminus \check{X}) = \lambda(X \setminus \check{X})$. Let $\tilde{X} = E''$. Then we get $\lambda(X \setminus \check{X}) = 1 - \lambda(\tilde{X}) \geq \lambda(\check{X})$.

Consider

$$\lambda(\tilde{X}) = \int_{z \in \check{X}} d\lambda(z) \geq \int_{ps \in \check{X}_S} \lambda_S(X_S \setminus \tilde{X}_S(p_S)) d\lambda_S(p_S)$$

$$+ \int_{ps \in \check{X}_S^c} \lambda_S(X_S \setminus \tilde{X}_S(p_S)) d\lambda_S(p_S) = \int_{ps \in \check{X}_S^c} \lambda_S(X_S \setminus \tilde{X}_S(p_S)) d\lambda_S(p_S).$$

where the inequality follows from Fubini’s Theorem, and the second equality since, for all $p_S \in \check{X}_S$, we have $\lambda_S(X_S \setminus \tilde{X}_S(p_S)) = 0$. Since, for all $p_S \in \check{X}_S$, we have $\lambda_S(X_S \setminus \tilde{X}_S(p_S)) > 0$, if $\lambda_S(\check{X}_S^c) > 0$ then $\lambda(\check{X}) > 0$. Since $1 - \lambda(\check{X}) \geq \lambda(\tilde{X})$, this contradicts the definition of $\check{X}$. Thus, we have $\lambda_S(\check{X}_S) = 1$. ■

As in section 3, we assume that parties do not change their relative ranking.

**Separability assumption** For all $j, k \in P$ such that $j \neq k$, for all $y \in X_j$ we have either $z < y$ for all $z \in X_k$ or $z > y$ for all $z \in X_k$.

Now consider a voter who is indifferent between two parties $j$ and $k$. Given $p_S \in X_S$ and $c \in T$, let $x_{jk}(S, c, p_S)$ denote the peak of the voter who is indifferent between voting for party $j$ and party $k$ when informed about the parties in $S$, if such a voter exists.

The following Lemma shows that, for all $S \subseteq P$, if $c^n \to c$ and $V$ is strictly concave, then there exists a subsequence, $\{c^n\}_{i=1}^\infty$, such that $x_{jk}(S, c^n, p_S)$ converges to $x_{jk}(S, c, p_S)$ for almost all $p_S \in X_S$.

To give a hint of the intuition behind the proof assume that $y < x^i < z$ for all $y \in X_j$ and $z \in X_k$. Consider voters informed by the parties in some $S \subseteq P$. Then, given that we have found some voter $i$ with peak $x^i$ that is indifferent between two parties, moving closer to the party with the highest platform always makes the voter like that party more. Also, moving away from the party with the lowest platform makes the voter lose. This implies
uniqueness of the indifferent voter. Also, a sequence $\{c^n\}_{i=1}^{\infty}$ that converges pointwise almost everywhere affects the voter payoff continuously almost everywhere. This in turn implies that the indifferent voter changes continuously. Note that strict concavity is not used in this informal description of the proof. Strict concavity is needed when it is not the case that $y < x^i < z$ for all $y \in X_j$ and all $z \in X_k$.

**Lemma 11** Let $S \subseteq P$ and $j, k \in P$ such that $j \neq k$ be given. Let $\{c^n\}_{n=1}^{\infty}$ be a sequence such that $c^n \rightarrow c$. Suppose $x_{jk}(S, c^n, p_S)$ exists for all $n$. If the Separability assumption is satisfied and $V$ is strictly concave, then there exists a set $\tilde{X}_S \subseteq X_S$ such that $\lambda_S(\tilde{X}_S) = 1$ and a subsequence $\{c^{n_l}\}_{l=1}^{\infty}$ such that for all $p_S \in \tilde{X}_S$, $x_{jk}(S, c^{n_l}, p_S)$ converges to $x_{jk}(S, c, p_S)$.

**Proof. Step 1:** The indifferent voter is unique.

Fix $c$. Suppose $j, k \notin S$. The indifferent voter is given by $x_{jk}(S, c, p_S) = x^i$ where $x^i$ is chosen such that

$$\int_{z \in X_k} V(z - x^i) \Pr_k(z \mid S, c, p_S) dz = \int_{y \in X_j} V(y - x^i) \Pr_j(y \mid S, c, p_S) dy. \quad (17)$$

Suppose without loss of generality that, for all $z \in X_k$ and all $y \in X_j$ we have $z > y$. Differentiating both sides of expression 17 with respect to $x^i$ gives the change in utility when voting for $k$ as

$$- \int_{z \in X_k} V'(z - x^i) \Pr_k(z \mid S, c, p_S) dz$$

and the change in utility when voting for $j$ as

$$- \int_{y \in X_j} V'(y - x^i) \Pr_j(y \mid S, c, p_S) dy.$$  

Consider any $x^i$, $y$ and $z$. Since $z > y$ we know that $z - x^i > y - x^i$. Strict concavity implies that $V'(w) > V'(w')$ whenever $w < w'$. This implies that

$$\int_{z \in X_k} V'(z - x^i) \Pr_k(z \mid S, c, p_S) dz < \int_{y \in X_j} V'(y - x^i) \Pr_j(y \mid S, c, p_S) dy.$$  

27
Then any voter \( h \) where \( x^h > x^i \) strictly prefers party \( k \). A similar argument works for any other \( S \).

**Step 2:** Continuity in \( c \).

By Fubini’s Theorem (Royden p. 307) there exists a set \( X_S^I \) where \( \lambda_S(X_S^I) = 1 \) such that, for all \( p_S \in X_S^I \), the function \( c_j(p_S, y) \) is integrable on \( X_{-S} \). Since \( \rho \) is continuous, \( \rho(c_j(p_S, y)) \) is integrable. Also, since \( \rho(0) > 0 \), \( \frac{1}{\rho(c_j(p_S, y))} \) is integrable. Then, let \( X_{-S}(p_S) = \{ x \in X_{-S} \mid (p_S, x) \in \tilde{X} \} \). Also, from Lemma 10, there exists a set \( \tilde{X}_S \) such that \( \lambda_S(\tilde{X}_S) = 1 \), a set \( \tilde{X}_S(p_S) \) such that \( \lambda_{-S}(\tilde{X}_S(p_S)) = 1 \) and a subsequence, \( \{ c^{n_l} \}_{l=1}^{\infty} \), such that for all \( p_S \in \tilde{X}_S \) and all \( y \in \tilde{X}_{-S}(p_S) \), \( c^{n_l}(p_S, y) \) converges to \( c(p_S, y) \).

Consider any \( p_S \in \tilde{X}_S \cap X_S^I \). From the assumptions on \( \rho \), there exists some \( \bar{b}_k \in \mathbb{R}_+ \) such that, for all \( z \in X_k \), we have \( \Pr_k(z \mid S, c^{n_l}, p_S) \leq \bar{b}_k < \infty \). Since \( \Pr_k(z \mid S, c^{n_l}, p_S) \) is measurable, by the Lebesgue Convergence Theorem (Royden p. 267) we have

\[
\int_{z \in X_k} V(z - x^i) \Pr_k(z \mid S, c^{n_l}, p_S)dz \to \int_{z \in X_k} V(z - x^i) \Pr_k(z \mid S, c, p_S)dz
\]

for all \( p_S \in \tilde{X}_S \cap X_S^I \). Note that expression 18 implies that \( x_{jk}(S, c, p_S) \) exists. Since \( \int_{z \in X_k} V(z - x^i) \Pr_k(z \mid S, c, p_S)dz \) is continuous in \( x^i \), the solution for \( x^i \) in 17 is continuous, for all \( p_S \in \tilde{X}_S \cap X_S^I \).

Note that, if there does not exist a voter that is indifferent between party \( j \) and party \( k \) for all \( l \) there does not exist a voter that is indifferent between party \( j \) and party \( k \) when the strategy profile is \( c \). Too see this, suppose that the expression \( \int_{z \in X_k} V(z - x^i) \Pr_k(z \mid S, c^{n_l}, p_S)dz \) is larger than the expression \( \int_{z \in X_j} V(z - x^i) \Pr_k(z \mid S, c^{n_l}, p_S)dz \) for all \( l \). Then, by expression 18, the first expression must be larger than the second when the strategy profile is \( c \).
To compute the votes we first define

\[ U(k, x^i \mid S, c, p_S) = \begin{cases} 
-V(p_k - x^i) & \text{if } k \in S \\
- \int_{z \in X_k} V(z - x^i) \Pr_k(z \mid S, c, p_S)dz & \text{otherwise.}
\end{cases} \]

Given \( p_S \) and \( c \), \( U(k, x^i \mid S, c, p_S) \) is the expected payoff for a voter with peak \( x^i \) if he is informed by the parties in \( S \subset P \) and votes for party \( k \).

For any \( c \in T \), let

\[ O_k(S, c, p_S) = \{ x^i \in Y \mid U(k, x^i \mid S, c, p_S) \geq U(j, x^i \mid S, c, p_S) \forall j \in P \}. \]

Given \( S \subset P \), \( p_S \in X_S \) and \( c \in T \), \( O_k(S, c, p_S) \) is the set of ideal points of the voters who, if informed by just the parties in \( S \), would prefer party \( k \). Note that the expression \( \int_{x \in O_k(S, c, p_S)} f(x)dx \) then is the mass of voters that prefer party \( k \).

Also, note that the separability assumption implies that \( O_k(S, c, p_S) \) is an interval. This follows, since if any two voters prefer to vote for party \( k \), by strict concavity of \( V \), any voter in between these two voters strictly prefer to vote for party \( k \).

Assume that \( O_k(S, d, p_S) \) is nonempty. Furthermore, suppose \( \min \{ x \in O_k(S, d, p_S) \} \) and \( \max \{ x \in O_k(S, d, p_S) \} \) exist. Then there are parties \( j, l \in P \) such that, given \( S \), there is a voter that is indifferent between voting for party \( k \) and voting for parties \( j \) and \( l \), respectively. Note that these parties might be different for different \( S \).

Then, for some \( j \in P \), we have \( x_{jk}(S, d, p_S) = \min \{ x \in O_k(S, d, p_S) \} \) and for some \( l \in P \) we have \( x_{lk}(S, d, p_S) = \max \{ x \in O_k(S, d, p_S) \} \). Then

\[ \int_{x \in O_k(S, d, p_S)} f(x)dx = \int_{x_{lk}(S, d, p_S)}^{x_{lk}(S, d, p_S)} f(x)dx. \]

If \( \min \{ x \in O_k(S, d, p_S) \} \) or \( \max \{ x \in O_k(S, d, p_S) \} \) do not exist, replace \( x_{jk}(S, d, p_S) \) or \( x_{lk}(S, d, p_S) \) with \( x \) and \( \bar{x} \), respectively. If \( O_k(S, d, p_S) \) has measure 0 or is empty, then

\[ \int_{x \in O_k(S, d, p_S)} f(x)dx = 0. \]
Then, let \( v_k(c_k, c_{-k} \mid d, p) \) denote the votes for party \( k \) if party \( k \) chooses \( c_k \), the other parties choose \( c_{-k} \), and voters update beliefs according to \( d \in T \), for some \( p \in X \). We have

\[
v_k(c \mid d, p) = \rho(c_k(p)) \sum_{S \subseteq P, j \neq k} \prod_{j \in S} \rho(c_j(p)) \prod_{l \in P \setminus S} [1 - \rho(c_l(p))] \int_{O_k(S, d, p_S)} f(x)dx
+ [1 - \rho(c_k(p))] \sum_{S \subseteq P, j \neq k} \prod_{j \in S} \rho(c_j(p)) \prod_{l \in P \setminus S} [1 - \rho(c_l(p))] \int_{O_k(S, d, p_S)} f(x)dx. \tag{19}
\]

Note that, by separability, two parties cannot be perceived as identical by voters.

Let \( w_k(c \mid d, p) = u_k(c_k, v_k(c \mid d, p)) \).

Since \( v_k(c_k, c_{-k} \mid d, p) \) can be rewritten as \( \rho(c_k)A(c_{-k} \mid d, p) + B(c_{-k} \mid d, p) \) and \( \rho \) is concave, \( w_k(c \mid d, p) \) is quasi-concave in \( c_k \). Too see this, fix \( p \) and first assume that \( A(c_{-k} \mid d, p) > 0 \).

By concavity of \( \rho \) we have

\[
\frac{\partial^2 w_k(c_k(p), c_{-k} \mid d, p)}{\partial c_k(p)^2} = \nu A(c_{-k} \mid d, p) \frac{\partial^2 \rho(c_k(p))}{\partial c_k(p)^2} - 1 < 0.
\]

This implies that \( w_k \) is concave.

Second, suppose \( A(c_{-k} \mid d, p) \leq 0 \). Since \( \frac{\partial \rho(c_k(p))}{\partial c_k(p)} > 0 \) we have

\[
\frac{\partial w_k(c_k(p), c_{-k} \mid d, p)}{\partial c_k(p)} = \nu A(c_{-k} \mid d, p) \frac{\partial \rho(c_k(p))}{\partial c_k(p)} - 1 < 0.
\]

Since the function \( w_k \) is decreasing in \( c_k(p) \), it is quasi-concave.

Recall the definition of equilibrium. The definition has to be modified slightly, since the strategy spaces are defined over a continuum of platform profiles.

**Definition 12** An **Informational Nash Equilibrium** is a \( c^* \in T \) such that

i) for all \( k \), \( c_k^*(z) = \arg \max_{c \in [0, \omega_k]} \omega_k - e + \nu v_k(e, c_{-k} \mid c^*, z) \) for almost all \( z \in X \),

\[
(20)
\]

ii) for all \( z \in X_k \), all \( S \subseteq P \), all \( k \notin S \) and all \( p_S \in X_S \) posterior beliefs are given by \( \Pr_k(z \mid S, c^*, p_S) \).
The first condition means that \( c^* \) is a best response almost everywhere. The second means that voters use Bayesian updating, according to the profile \( c^* \).

Now consider the best-reply correspondence for party \( k \in P \), given that the other parties use the strategy profile \( c_{-k} \) and voters update their beliefs according to \( c \). Thus, for all \( c \in T \), let

\[
h_k^*(c) = \{ f_k \in L_2(X) \mid f_k(z) = \arg \max_{e \in [0, \omega_k]} \omega_k - e + \nu v_k(e, c_{-k} \mid c, z) \text{ a.e. on } X \}.
\]

Clearly \( h_k^* : T \to T_k \). Let \( h^* = (h_k^*)_{k \in P} \). Then \( h^* : T \to T \).

Since expression \( \omega_k - e + \nu v_k(e, c_{-k} \mid c, z) \) is quasi-concave in \( e \), convex-valuedness of \( h^* \) follows.

The following Lemma shows that \( h^* \) has a closed graph.

**Lemma 13** If the Separability assumption is satisfied and \( V \) is strictly concave, \( h^* \) has a closed graph.

**Proof.** Step 1: \( v_k(f_k, c_{-k} \mid c, p) \) continuous in \( (f_k, c) \) a.e. on \( X \).

Again let \( \{c^n\}_{n=1}^{\infty} \) be a sequence such that \( c^n \to c \) and let \( \{f^n_k\}_{n=1}^{\infty} \) be a sequence such that \( f^n_k \in h_k^*(c^n) \) for all \( n \) and \( f^n_k \to f_k \). Then, by the Riesz-Fischer Theorem, there exists a subsequence \( \{c^{n_r}\}_{r=1}^{\infty} \) that converges pointwise a.e. on \( X \). Corresponding to \( \{c^{n_r}\}_{r=1}^{\infty} \) there is a subsequence \( \{f_k^{n_r}\}_{r=1}^{\infty} \). Again, by the Riesz-Fischer Theorem, there exists a further subsequence \( \{f_k^{n_{r'}}\}_{r'=1}^{\infty} \) that converges pointwise a.e. on \( X \). Then, let \( \tilde{X} \subseteq X \) denote the set of platforms such that \( c^{n_{r'}} \) and \( f_k^{n_{r'}} \) converge pointwise. Clearly, \( \lambda(\tilde{X}) = 1 \).

As in Lemma 11 there exists a set \( X^f_S \) where \( \lambda_S(X^f_S) = 1 \) such that, for all \( p_S \in X^f_S \) and all \( j \in P \setminus S \), the function \( c_j(p_S, y) \) is integrable on \( X_{-S} \). Let \( \tilde{X}(S) = \{(p_S, y) \in X \mid p_S \notin \tilde{X}_S \cap X^f_S \} \). From Lemma 10 we have \( \lambda(\tilde{X}(S)) = 0 \). Let \( \tilde{X}^* = \tilde{X} \setminus \{S \subseteq P \mid \tilde{X}(S) \} \).

Since \( \lambda(\tilde{X}(S)) = 0 \) for all \( S \subseteq P \) and \( P \) is finite, \( \lambda(\tilde{X}^*) = 1 \). Then, since \( c^{n_{r'}} \) converge pointwise, \( x_{jk}(S, c^{n_{r'}}) \) converges to \( x_{jk}(S, c, p_S) \) by Lemma 11. Since \( \rho \) is continuous \( v_k(f^{n_{r'}}, c_{-k} \mid c^{n_{r'}}, p) \) converges to \( v_k(f_k, c_{-k} \mid c, p) \) for all \( p \in \tilde{X}^* \) such that \( \lambda(\tilde{X}^*) = 1 \).

**Step 2:** \( f \in h^*(c) \).
Step 1 implies that
\[
f_k(p) = \arg \max_{e \in [0, \omega_k]} \omega_k - e + \nu v_k(e, c_{-k} \mid c, p)
\]
for all \(k\) and all \(p \in \tilde{X}^* \setminus U\) where \(U \subset \tilde{X}^*\) and \(\lambda(U) = 0\). Suppose \(\lambda(U) > 0\). Then, for all \(p \in U\),
\[
f_k(p) \neq \arg \max_{e \in [0, \omega_k]} \omega_k - e + \nu v_k(e, c_{-k} \mid c, p).
\]
Clearly \(f^n_{k'}\) and \(c^{n'}\) converge pointwise on \(U \cap \tilde{X}^*\) where \(\lambda(U \cap \tilde{X}^*) > 0\) By Egoroff’s Theorem, there exists a set \(\hat{U} \subseteq U \cap \tilde{X}^*\) such that \(\lambda(\hat{U}) > 0\) where \(f^n_{k'}\) and \(c^{n'}\) converge uniformly on \(\hat{U}\). Thus, for all \(p \in \hat{U}\), there exists a number \(\hat{r}\) such that, for all \(r' > \hat{r}\) we have
\[
f^n_{k'}(p) \neq \arg \max_{e \in [0, \omega_k]} \omega_k - e + \nu v_k(e, c_{-k}^{n'} \mid c^{n'}, p),
\]
contradicting the definition of \(\{f^n_{k'}\}_{r=1}^\infty\). Thus, \(f \in h^*(c)\). Thus \(h^*\) has a closed graph.  

By using Lemma 13 we are now able to show existence.

**Proposition 14** If the Separability assumption is satisfied and \(V\) is strictly concave, there exists an Informational Nash Equilibrium \(c^*\).

**Proof.** Note that by Lemmas 9, 13 above and Corollary 14.50 in Aliprantis and Border (1994), a fixed point \(c^* = h^*(c^*)\) exists. Then, for all \(p \in X\), all \(S \subseteq P\) and all \(k \in P\), by construction of \(h^*\), the beliefs are given by
\[
\Pr(z \mid S, c^*, p_S) = \frac{\int \prod_{y \in X_{P \setminus (S, k)}} \rho(c^*_j(p_S, z, y)) \prod_{l \in P \setminus S} [1 - \rho(c^*_l(p_S, z, y))] g(p_S, z, y) dy}{\int \prod_{v \in X_{P \setminus S}} \rho(c^*_j(p_S, v)) \prod_{l \in P \setminus S} [1 - \rho(c^*_l(p_S, v))] g(p_S, v) dv}.
\]
Also, given these beliefs and by the construction of the mapping \(h^*\), \(c^*\) satisfies i) for all \(k \in P\). Thus \(c^*\) is an Informational Nash Equilibrium.  

32
The main idea behind the proof is to start with some arbitrary strategy profile for the parties and let voters revise their beliefs according to this strategy profile. Given this profile and the revised beliefs, each party chooses an optimal strategy. Thus, we construct a mapping from the set of strategy profiles to itself. Then a fixed point theorem can be applied to show existence.

4.2 Allowing \( \rho(0) = 0 \)

Note that, if \( \rho(0) = 0 \), the probability measures defined in expression 14 need not be well defined.

Let

\[
\rho^n(y) = \rho(y) + \frac{1 - \rho(\omega_{\text{max}})}{2n}
\]

for all \( y \in [0, \omega_{\text{max}}] \).

Since \( \rho(\omega_{\text{max}}) < 1 \) we have \( 0 < \rho^n(y) < 1 \) for all \( y \in [0, \omega_{\text{max}}] \). Also, \( \frac{\partial \rho^n(y)}{\partial y} = \rho'(y) \geq 0 \) for all \( y \in [0, \omega_{\text{max}}] \). Then \( \rho^n \) converges pointwise to \( \rho \) as \( n \to \infty \). Since \( \rho \) is concave, \( \rho^n \) is concave.

Say that \( c \) is an INE a.e. if the profile \( c \) satisfies the requirement ii) in expression 20 almost everywhere. Thus, we require that beliefs are updated according to \( c^* \) almost everywhere.

**Definition 15** The profile \( c^* \in T \) is an Informational Nash Equilibrium a.e. if

\[
i) \text{ for all } k, \ c^*_k(z) = \arg \max_{e \in [0, \omega_k]} \omega_k - e + \nu v_k(e, c^*_e | c^*, z) \text{ for almost all } z \in X,
\]

\[
ii) \text{ for all } z \in X_k, \text{ all } S \subseteq P, \text{ all } k \notin S \text{ there exists } Z_S \subseteq X_S \text{ where } \lambda_S(Z_S) = 1 \text{ such that for all } p_S \in Z_S, \text{ posterior beliefs are given by } \Pr_k(z \mid S, c^*, p_S) \text{ when Bayes rule applies.}
\]

The following result shows existence of an INE a.e.
Theorem 16  If $\rho(0) = 0$ there exists a $c^* \in T$ such that $c^*$ is an INE a.e.

Proof. Step 1: The candidate equilibrium profile and beliefs.

Consider a sequence $\{\rho^n\}_{n=1}^{\infty}$. Since $\rho^n$ is concave, for each $n$, an equilibrium $c^{*n} \in T$ exists by Proposition 14. Let $\{c^{*n}\}_{n=1}^{\infty}$ denote a sequence of such equilibria. Since $T$ is compact there exists a convergent subsequence, $\{c^{*n_q}\}_{q=1}^{\infty}$ with limit $c^* \in T$. Also, for all $n$, all $S \subset P$ and all $k \notin S$, there exists a probability measure, $\Pr_k(S, c^{*n}, p_S)$ defined as in expression 14. Note that, since $X$ is compact, any such measure is tight. Also, note that, by the Riesz-Fischer Theorem, there exists a further subsequence $\{c^{*n_h}\}_{h=1}^{\infty}$ that converges pointwise a.e. on $X$.

Step 2: Existence of limit probability measures.

Consider the subsequence $\{c^{*n_h}\}_{h=1}^{\infty}$. Fix $S \subset P$ and $k \notin S$. Take the subsequence of measures corresponding to this subsequence. Denote the elements of this subsequence $\Pr_k(S, c^{*n_h}, p_S)$. Then, by Helly’s Theorem (Billingsley (1986), p. 392) there exists a further subsequence of measures that converges weakly to some well defined measure, $\Pr^*_k(S, p_S)$. Let $\{c^{*n_h}\}_{h=1}^{\infty}$ denote the subsequence of equilibria corresponding to the subsequence of measures. Clearly, this sequence is a subsequence of $\{c^{*n_h}\}_{h=1}^{\infty}$. Thus, $\Pr_k(S, c^{*n_h}, p_S)$ converges weakly to $\Pr^*_k(S, p_S)$. By repeating this argument for all $S \subset P$ and all $k \notin S$ there exists a subsequence $\{c^{*n_h}\}_{h=1}^{\infty}$ such that, for all $S \subset P$, all $k \notin S$, $\Pr_k(S, c^{*n_h}, p_S)$ converges weakly to $\Pr^*_k(S, p_S)$.

Step 3: Proof that $c^*$ is a best response, given limit beliefs.

Consider the subsequence $\{c^{*n_h}\}_{h=1}^{\infty}$. This subsequence converges pointwise on $\tilde{X}$, such that $\lambda(\tilde{X}) = 1$. Note that, since $c^{*n_h}(v)$ converges to $c^*(v)$ for any $v \in \tilde{X}$ then $\rho^k(c^{*n_h}(v))$ converges to $\rho(c^*(v))$. Since the sequence of measures converges weakly, for all $S \subset P$, for all $k \notin S$ and all $p_S \in X_S$, we have, by the weak* topology

$$\int_{z \in X_k} V(z - x^i) \Pr_k(z \mid S, c^{*n_h}, p_S) dz \to \int_{z \in X_k} V(z - x^i) \Pr_k(z \mid S, p_S) dz$$

for all $x^i \in Y$. 

34
When voter beliefs are given by $\Pr^*_k(S, p_S)$ for all $j, k \in P$, all $S \subset P$ and all $p_S \in X_S$, let $x^*_jk(S, p_S)$ denote the peak of the indifferent voter, when informed by just the parties in $S \subset P$ and when $p_S \in X_S$ is true.

By a continuity argument similar to step 2 in Lemma 11, for all $S \subset P$ and all $p_S \in X_S$, $x^*_jk(S, e^{n_b \alpha}, p_S)$ converges to $x^*_jk(S, p_S)$ for all $j, k \in P$. Also, for all $v \in \tilde{X}$, since $e^{n_b \alpha}(v)$ converges to $e^*(v)$ and $\rho^n$ converges pointwise to $\rho$ it follows that $\rho^n(e^{n_b \alpha}(v))$ converges to $\rho(e^*(v))$. Then $v_k(e^{n_b \alpha}, e^{n_b \alpha}, v)$ converges to $v_k(e^* | e^*, v)$ for all $v \in \tilde{X}$. By an argument identical to the argument in Step 2 in Lemma 13, $e^* \in h(e^*)$.

**Step 4:** Proof that limit beliefs result from $e^*$.

Consider the sequence $\{e^{n_b \alpha}\}_{q=1}^\infty$. For any $S \subset P$, $k \notin S$, let

$$C(S, e^*, p_S) = \{v-S \in X-S \mid c^*_k(p_S, v_S) > 0 \forall k \in S\}. $$

Thus, $C(S, e^*, p_S)$ consists of all the platforms of the parties not in $S$ such all parties in $S$ spends a positive amount, given $e^*$ and $p_S$.

Fix some $S \subset P$ and $p_S \in X_S$.

**Case 1:** $\lambda_{-S}(C(S, e^*, p_S)) = 0$.

Since $\lambda_{-S}(C(S, e^*, p_S)) = 0$, Bayes rule do not apply and any beliefs can be applied. Then, for all $S \subset P$ and $p_S \in X_S$ such that $\lambda_{-S}(C(S, e^*, p_S)) = 0$, we assume that beliefs are given by $\Pr^*_k(S, p_S)$.

**Case 2:** $\lambda_{-S}(C(S, e^*, p_S)) > 0$.

Let $\tilde{X}_{-S}(p_S) = \{v \in X_{-S} \mid (p_S, v) \in \tilde{X}\}$. Furthermore, as in Lemma 11, we let $\tilde{X}_{S} = \{p_S \in X_S \mid \lambda_{-S}(\tilde{X}_{-S}(p_S)) = 1\}$. Clearly $\lambda_{S}(\tilde{X}_{S}) = 1$. Then, for any $p_S \in \tilde{X}_{S}$ such that $\lambda_{-S}(C(S, e^*, p_S)) > 0$ we have $\Pr_k(z \mid S, e^{n_b \alpha}, p_S)$ converges to $\Pr_k(z \mid S, e^*, p_S)$, for all $k \notin S$ a.e. on $X_{-S}$. Then, by Scheffé's Theorem (Billingsley (1968), p. 224), $\Pr_k(S, e^*, p_S) = \Pr^*_k(S, p_S)$ for all $p_S \in \tilde{X}_{S}$.

Thus, for all $S \subset P$, all $k \notin S$ and all $p_S \in \tilde{X}_{S}$ we have that beliefs derived from $e^*$ are given by limit beliefs. ■
The idea behind the proof is the following. For each \( n \), an equilibrium \( c^n \in T \) exists from Proposition 14. Since \( T \) is compact there exists a convergent subsequence, \( \{c^{n_h}\}_{h=1}^{\infty} \) with limit \( c^* \in T \). Then, by repeatedly using Helly’s Theorem, there exists a subsequence \( \{c^{n_{h_q}}\}_{q=1}^{\infty} \), such that, for all \( S \subseteq P \), all \( k \notin S \) we have \( \Pr_k(S, c^{n_{h_q}}, p_S) \) converges weakly to some measure \( \Pr_k^*(S, p_S) \). Then the profile \( c^* \in T \) and the beliefs given by the limit measure is an INE a.e. If this were not the case, then for \( q \) sufficiently large, \( c^{n_{h_q}} \) cannot be an equilibrium.

5 Conclusions

The model described in this paper analyses political campaigning. We are able to show existence of equilibrium under mild assumptions for an arbitrary number of parties. In general, it is difficult to explicitly solve for an equilibrium. However, in a symmetric example with two parties we can show that voters are more informed when parties are moderate, than when parties are extreme.

The motivation is the following. First, note that if a party informs a voter, then the party knows the platforms of that party with certainty. Thus, the risk of voting for that party is eliminated. Also, to win an election, parties have to win voters in between the parties. Consider voters that are informed by one party only, say party 1, and assume that the platform of party 1 is close to the median voter. For the indifferent voter, the expected platform of party 2 is closer to the voters’ peak than the actual platform of party 1. Otherwise, since voters are risk-averse, the voter would strictly prefer to vote for party 1. If both parties move closer to each other, then the actual and expected platform moves closer to the indifferent voters peak by the same distance. By strict concavity, the closer to the peak a platform is, the flatter preferences are. Since the actual and expected platform of the two parties move the same distance, the increase in payoff of voting for the party that informed is bigger than the increase in payoff of voting for the other party. Thus, the previously indifferent voter now strictly prefers party 1. Hence more voters now prefer party
1. The effect makes parties gain more votes by informing when parties are moderate. Since spending increases, voters are more informed when parties are moderates. Thus extremism is bad in the sense that it leaves voters uninformed.
References


