On Metric Sorting for Successive Cancellation List Decoding of Polar Codes

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Abstract—We focus on the metric sorter unit of successive cancellation list decoders for polar codes, which lies on the critical path in all current hardware implementations of the decoder. We review existing metric sorter architectures and we propose two new architectures that exploit the structure of the path metrics in a log-likelihood ratio based formulation of successive cancellation list decoding. Our synthesis results show that, for the list size of $L = 32$, our first proposed sorter is 14% faster and 45% smaller than existing sorters, while for smaller list sizes, our second sorter has a higher delay in return for up to 36% reduction in the area.

I. INTRODUCTION

Polar codes [1] are a recently introduced class of provably capacity achieving channel codes with efficient encoding and decoding algorithms. Successive cancellation list (SCL) decoding [2] is a decoding algorithm which improves upon the conventional successive cancellation decoding of polar codes in terms of frame error rate, while only increasing the decoding complexity linearly by a factor of $L$, where $L$ is the list size of the decoder. The key step of SCL decoding is to choose $L$ paths with the smallest path metric out of $2L$ possible paths.

While theoretically this problem can be solved by finding the median of the $2L$ metrics whose computational complexity is $O(L)$ [3] Section 9.3, in practice, since $L$ is relatively small, it is easier to just sort the $2L$ numbers and pick the first $L$.

The metric sorter (which implements the above-mentioned sorting step) turns out to be the crucial component of all the SCL decoder hardware architectures proposed in the literature [4, 5, 6]. In [4, 5], the metric sorter lies on the critical path of the decoder for $L \geq 4$, so it determines the maximum clock frequency of the decoder. In [6], the sorter is pipelined in order to remove it from the critical path, but the number of cycles required to decode each codeword is increased.

Contribution: In this work, we review and compare the metric sorter architectures used in the existing SCL decoder architectures in [4, 5, 6]. We then leverage the properties of the LLR-based path metric introduced in [5] in order to introduce two new sorter architectures: a pruned bitonic sorter and a bubble sorter. By comparing the synthesis results of the new sorters and the existing sorters we highlight an advantage of the LLR-based formulation of the SCL decoder which leads to an optimized implementation of the metric sorter.

II. PROBLEM STATEMENT

Let $m = [m_0, m_1, \ldots, m_{2L-1}]$ denote the $2L$ real path-metrics to be sorted. In an LLR-based implementation of SCL decoding (e.g., in [4, 6]), $m$ contains arbitrary real numbers and a general sorting problem of size $2L$ needs to be solved. In an LLR-based implementation [5, 7], however, $m$ has a particular structure that can be exploited to simplify the sorting task. More specifically, let $\mu_0 \leq \mu_1 \leq \ldots \leq \mu_{L-1}$ be the $L$ sorted path metrics from the previous step of SCL decoding. Then, in an LLR-based implementation, the $2L$ new path metrics in $m$ are computed as

$$m_{2\ell} := \mu_\ell \quad \text{and} \quad m_{2\ell+1} := \mu_\ell + a_\ell, \quad \ell = 0, 1, \ldots, L-1,$$

where $a_\ell \geq 0$, for all $\ell \in \{0, 1, \ldots, L-1\}$. Thus, the sorting problem is to find a sorted list of $L$ smallest elements of $m$ when the elements of $m$ have the following two properties

$$m_{2\ell} \leq m_{2\ell+1}, \quad (1a)$$

$$m_{2\ell} \leq m_{2\ell+1} \leq m_{2\ell+2}, \quad (1b)$$

for $\ell \in \{0, 1, \ldots, L-2\}$.

Note that (1a) and (1b) imply that out of \((2L)/2 = L(2L-1)\) unknown pairwise relations between the elements of $m$, $L^2$ are known (every even-indexed element is smaller than all its following elements). Hence, we expect the sorting complexity to be reduced by a factor of 2.

Remark. For SCL decoding, in order for the assumptions on the list structure to hold, besides the above mentioned problem, a general sorting problem of size $L$ needs to be solved infrequently [7]. We note that this problem can be solved by using a sorter that finds the $L$ smallest elements of a list with properties (1a) and (1b). $L - 1$ times in a row.

III. EXISTING METRIC SORTER ARCHITECTURES

A. Radix-2L and Pruned Radix-2L Sorters

In our previous work of [4, 5] we used a radix-2L sorter [8], which compares every pair of elements $(m_\ell, m_{\ell+1})$ and then combines the results to find the $L$ smallest elements. This solution requires $L(2L-1)$ comparators together with $L 2L$-to-1 multiplexers. In an LLR-based implementation of SCL decoding, the full radix-2L sorter can be pruned by removing the comparators corresponding to $L^2$ known relations and observing that $m_{2L-1}$ is never among the $L$ smallest elements. Eliminating these unnecessary comparators, the number of required comparators is reduced to $(L-1)^2$ [7].

1Let $a_0, a_1, \ldots, a_{L-1}$ be arbitrary real numbers. For $\ell = 0, 1, \ldots, L-2$, set $m_{2\ell} := -\infty$ and $m_{2\ell+1} := a_\ell$. Finally set $m_{2L-2} := a_{L-1}$ and $m_{2L-1} := +\infty$. It is easy to check that (1a) and (1b) hold for the list $m$ and the ordered $L$ smallest elements of this list are $[-\infty, -\infty, \ldots, \min_{0 \leq \ell \leq L-1} a_\ell]$. Thus, we can find the minimum of up to $L$ arbitrary real numbers using such a sorter. Consequently, using the sorter $L - 1$ times in a row we can sort an arbitrary set of $L$ real numbers.
The authors of [6] used a bitonic sorter [9]. A bitonic sorter that can sort \( s_2 \) units in red dotted lines can be removed. Finally, we can remove the CAS units that can be pruned for \( L/2 \) elements. Moreover, the result of all comparators whose one input is amongst the smallest elements of the list, all comparisons involving \( m_{2L-1} \) are irrelevant and the corresponding CAS units can be removed completely due to (1b). In all remaining super-stages except the last one, the 2 CAS units per stage that are connected to \( m_0 \) and \( m_{2L-1} \) can be removed since \( m_0 \) is always the smallest element and \( m_{2L-1} \) is never among the \( L \) smallest elements. In the last super-stage, we can remove the CAS units connected to \( m_0 \) plus all the CAS units in the second half of the last \( \log L \) stages since they contribute in sorting the \( L \) largest elements of the list which we are not interested in. Hence, the total number of CAS units required for the pruned bitonic sorter can be shown to be equal to

\[
e^{\text{PBT}} = \frac{L}{2} (\log L + 1)(\log L + 2) + 1. \tag{5}\]

By examining the ratio between \( e^{\text{PBT}} \) and \( e^{\text{BT}} \) we can conclude that, similarly to the maximum delay, the relative reduction in the number of comparators also diminishes with increasing list size \( L \).

IV. PROPOSED SORTERS

A. Pruned Bitonic Sorter

As was the case with the radix-2\( L \) sorter, the known relations between the elements can be exploited in order to simplify the bitonic sorter. More specifically, due to (1b), the results of all sorters in stage 1 are already known. Thus, stage 1 can be removed completely from the sorting network. Moreover, the result of all comparators whose one input is \( m_0 \) are also known, since \( m_0 \) is, by construction, always the smallest element of \( m \). Furthermore, since \( m_{2L-1} \) is never amongst the \( L \) smallest elements of the list, all comparisons involving \( m_{2L-1} \) are irrelevant and the corresponding CAS units can be removed. Finally, we can remove the \( L/2 \) last CAS units of the \( \log L \) final stages of super-stage \( \log L + 1 \), since they are responsible for sorting the last \( L \) elements of \( m \) and we are only interested in the first \( L \) elements of \( m \). The CAS units that can be pruned for \( 2L = 8 \) numbers are illustrated with dotted red lines in Fig. 1.

Since only stage 1 can be removed completely from the sorting network, the number of sorting stages in the pruned bitonic sorter is given by

\[
s^{\text{PBT}} = s^{\text{BT}} - 1 = \frac{1}{2}(\log L + 1)(\log L + 2) - 1. \tag{4}\]

This means that the delay of the pruned bitonic sorter is only slightly smaller than the delay of the full bitonic sorter, especially for large list sizes \( L \).

To compute the number of CAS units in a pruned bitonic sorter, we note that the first super-stage can be eliminated completely due to (1b). In all remaining super-stages except the last one, the 2 CAS units per stage that are connected to \( m_0 \) and \( m_{2L-1} \) can be removed since \( m_0 \) is always the smallest element and \( m_{2L-1} \) is never among the \( L \) smallest elements. In the last super-stage, we can remove the CAS units connected to \( m_0 \) plus all the CAS units in the second half of the last \( \log L \) stages since they contribute in sorting the \( L \) largest elements of the list which we are not interested in. Hence, the total number of CAS units required for the pruned bitonic sorter can be shown to be equal to

\[
e^{\text{PBT}} = \frac{L}{2} (\log L + 1)(\log L + 2) + 1. \tag{5}\]

By examining the ratio between \( e^{\text{PBT}} \) and \( e^{\text{BT}} \) we can conclude that, similarly to the maximum delay, the relative reduction in the number of comparators also diminishes with increasing list size \( L \).

B. Bubble Sorter

While bubble sort [3, Chapter 2] is in general an inefficient sorting algorithm, it turns out to be a suitable candidate for our particular problem. More precisely, properties (1a) and (1b) result in a specific data dependency structure of the algorithm enabling an efficient hardware implementation of the sorter. Furthermore, since we only require the sorted list of \( L \) smallest elements of \( m \) (rather than sorting the entire list \( m \)) we can simplify the sorter by only implementing the first half of the rounds of the algorithm.

1) Data Dependency: The bubble sort algorithm is formalized in Alg. 1. It is clear that Alg. 1 sorts the full list. By restricting the while condition as “exists \( \ell \in \{0, 1, \ldots, L-1\} \) such that \( m_\ell > m_{\ell+1} \)” one can simplify the algorithm to only output the first \( L \) ordered elements of the list \( m \).

Algorithm 1: The Bubble Sort Algorithm

\begin{algorithm}[H]
\caption{The Bubble Sort Algorithm}
\begin{algorithmic}[1]
\While{exists \( \ell \) such that \( m_\ell > m_{\ell+1} \)}
\For{\( \ell = 2L - 1 \) to 1}
\If{\( m_\ell < m_{\ell-1} \)}
\State Swap \( m_\ell \) and \( m_{\ell-1} \);
\EndIf
\EndFor
\EndWhile
\State return \( m \)
\end{algorithmic}
\end{algorithm}

Lemma 1. Let \( m_\ell \) denote the element at position \( \ell \) of the list at the beginning of round \( t \) of the while loop in Alg. 1 and,

\[
B_t \triangleq \{ \ell \in \{1, 2, \ldots, 2L-1\} : m_\ell < m_{\ell+1} \}. \tag{6}\]

Then (1a) and (1b) imply that for all \( t \geq 1 ,

\( i \) \) \( B_t \) does not contain adjacent indices,

\( ii \) \( \) the if body (line 3) is executed at round \( t \) iff \( \ell \in B_t \),

\( iii \) \( B_{t+1} \subseteq B_t + 1 \), where \( A + a \triangleq \{ x + a : x \in A \} \).
Proof: To prove the lemma, we will prove that for all $t \geq 1$,
\[ m_t^1 \geq m_{t-2}^1 \quad \text{for all } \ell \in B_t, \]  

(7)

We first show that (7) implies (i)–(iii) and then prove (7).

(i) Suppose $\ell \in B_t$, hence, $m_t^1 < m_{t-1}^1$ and (7) implies $m_t^1 \geq m_{t-2}^1$. Thus $m_{t-1}^1 > m_{t-2}^1$ which implies $\ell-1 \notin B_t$.

(ii) Note that the element at position $\ell$ of the list is changed iff line 4 is executed for indices $\ell$ or $\ell+1$. We use strong induction on $\ell$ to prove (7). Clearly line 4 is executed for the first time for an index $\ell = \max B_t$. Assume line 4 is executed for some index $\ell$. This implies $m_\ell < m_{\ell-1}$ before execution of this line when $m_{\ell-1} = m_{\ell-2}^1$ (since line 4 has not been executed for $\ell$ nor for $\ell-1$ so far). Now if $\ell+1 \notin B_t$, then $m_\ell^1 = m_\ell$ as well by the induction assumption (since line 4 is not executed for index $\ell+1$ hence $m_{\ell+1}^1 < m_{\ell+1}^1$, which means $\ell \in B_t$. Otherwise, $\ell+1 \in B_t$, implies line 4 is executed for $\ell+1$. Since $B_t$ does not contain adjacent elements, line 4 is not executed for $\ell+2$. This implies $m_{\ell+1} = m_{\ell+1}^1 \geq m_{\ell-1} = m_{\ell-1}^1$ by (7) which contradicts the assumption of loop being executed for $\ell$.

Conversely, assume $\ell \in B_t$. Since $\ell+1 \notin B_t$, line 4 is not executed for $\ell+1$ by assumption. Hence the for loop is executed for index $\ell$, $m_\ell = m_{\ell+1}^1$ and (as we justified before) $m_{\ell-1} = m_{\ell-1}^1$. Therefore, $m_\ell < m_{\ell-1}$ and line 4 is executed for $\ell$.

(iii) Using (i) and (ii), we can explicitly write the time-evolution of the list as:

\[ m_{\ell+1}^t = \begin{cases} m_{\ell-1}^t, & \text{if } \ell \in B_t, \\ m_\ell^t, & \text{if } \ell \notin B_t \text{ and } \ell + 1 \notin B_t, \\ m_{\ell+1}^t, & \text{if } \ell + 1 \in B_t. \end{cases} \]

(8)

Therefore, if $\ell \in B_t$, $m_{\ell+1}^t = m_{\ell}^t > m_{\ell-1}^t = m_{\ell+1}^t$, and $\ell \notin B_{t+1}$. Pick $\ell \in B_{t+1}$. We shall show this requires $\ell-1 \notin B_t$. Since $\ell \notin B_t$ as we just showed in (iii), (8) yields:

\[ m_{\ell+1}^t = \begin{cases} m_\ell^t, & \text{if } \ell + 1 \notin B_t, \\ m_{\ell+1}^t, & \text{if } \ell + 1 \in B_t, \end{cases} \]

(9)

\[ m_{\ell-1}^t = \begin{cases} m_{\ell-2}^t, & \text{if } \ell - 1 \in B_t, \\ m_{\ell-1}^t, & \text{if } \ell - 1 \notin B_t. \end{cases} \]

(10)

Equation (9), together with (7) imply $m_{\ell+1}^t \geq m_{\ell}^t$. Now if $\ell - 1 \notin B_t$, by (10), $m_{\ell+1}^t - 1 = m_{\ell-1}^t \leq m_{\ell+1}^t$. Hence $\ell \notin B_{t+1}$.

It remains to show (7) holds for all $t \geq 1$ by induction. The claim holds for $t = 1$ by construction; $B_1 \subseteq \{2, 4, \ldots, 2L - 2\}$ (because of (1b)) and (1a) is equivalent to (7) for $t = 1$.

Pick $\ell \in B_{t+1}$. Assuming (7) holds for $t$, we know $m_{\ell+1}^t \geq m_{\ell-1}^t$ (as we just showed). Furthermore, since $\ell - 1 \notin B_t$ (and $\ell - 1 \notin B_t$ due to (1b)), (8) yields $m_{\ell-2}^t = m_{\ell-1}^t \leq m_{\ell-1}^t$.

Property (ii) means we can replace the condition of the if block by $m_{\ell}^t \leq m_{\ell-1}^t$ without changing the algorithm. In other words, to determine whether we need to swap adjacent elements or not we can take a look at the values stored at that positions at the beginning of each round of the outer while loop. Furthermore, property (i) guarantees that each element, at each round, participates in at most one swap operation. As a consequence the inner for loop can be executed in parallel.

Finally, property (iii) together with the initial condition $B_1 \subseteq \{2, 4, \ldots, 2L - 2\}$ implies that at odd rounds CAS operations take place only between the even-indexed elements and their preceding elements while at even rounds CAS operations take place only between the odd-indexed elements and their preceding elements.

2) Implementation of Full Bubble Sorter: Given the above considerations, we can implement the sorter in hardware as follows: The sorter has $2L - 2$ stages each of them implementing a round of bubble sort (i.e., an iteration of the while loop in Alg. 1). At round $t$ of bubble sort the first $t$ elements are unchanged and the sorter will have a triangular structure. Since in our setting, round $1$ is already eliminated, each stage $t = 1, 2, \ldots, 2L - 2$ only moves the elements at indices $t, t+1, \ldots, 2L - 1$. Each stage implements the execution of the inner for loop in parallel using the required number of CAS units. In Fig. 2 we show the structure of the sorter for $2L = 8$. Using simple counting arguments we can show that the full bubble sort requires $L(L-1)$ CAS blocks.

3) Implementation of Simplified Bubble Sorter: So far we have only discussed about the implementation of a sorter that sorts the entire list. However, we only need the first $L$ ordered elements of the list. Hence, we can simplify the full sorter as follows. The first obvious simplification is to eliminate all the stages $L, L + 1, \ldots, 2L - 2$ since we know that after round $L - 1$ of the bubble sort, the first $L$ elements of the list correspond to an ordered list of the $L$ smallest elements of the original list. Thus, the total number of required stages for this simplified bubble sorter is

\[ s_{\text{total}} = L - 1. \]

(11)

Furthermore, we note that (due to property (i)) each element of the list at each round of the algorithm is moved at most by one position. Consider the elements at positions $2L - t, 2L - t + 1, \ldots, 2L - 1$ at round $t$. Since at most $L - t$ rounds of bubble sort are executed (including the current round), these elements remain in their position for the remaining $L - t$ rounds.

*In general the bubble sort terminates in up to $2L - 1$ rounds but in our particular problem instance, since $m_0 = \mu_0$ is the smallest element of the list, the first round is eliminated.*
elements cannot be moved to the first half of the list. Hence, we can eliminate the CAS units involving elements at indices \(2L - t, 2L - t + 1, \ldots, 2L - 1\) at each stage \(t = 1, 2, \ldots, L - 1\) as well. The simplified bubble sorter thus requires

\[ c_{\text{tot}}^B = \frac{1}{2} L(L - 1) \]  \hspace{1cm} (12)

CAS units. In Fig. 2 the parts of the sorters that can be eliminated are drawn with red dotted lines.

V. SYNTHESIS RESULTS

In this section, we provide synthesis results for all discussed sorters. All results are obtained for the 90 nm TSMC technology using the typical timing library (25\(^{\circ}\) C, 1.2 V). The list elements are assumed to be \(Q = 8\) bits wide, as in [4].

In Table I we present synthesis results for the radix-2L sorter of [6] and the pruned radix-2L sorter of [7]. We observe that the pruned radix-2L sorter is at least 63\% smaller and at least 56\% faster than the full radix-2L sorter.

In Table II we present synthesis results for the bitonic sorter of [8] and the pruned bitonic sorter presented in this paper. We observe that, as discussed in Section IV-A, the improvement in terms of both area and operating frequency are diminishing as the list size \(L\) is increased. Nevertheless, even for \(L = 32\) the pruned bitonic sorter is 5\% faster and 14\% smaller than the full bitonic sorter.

In Table III we present synthesis results for the simplified bubble sorter described in Section IV-B. We observe that, for \(L \leq 8\), the simplified bubble sorter has a lower delay than the pruned bitonic sorter. This happens because, as can be verified by evaluating (2) and (11), for \(L \leq 8\) the bubble sorter has fewer stages than the pruned bitonic sorter while for \(L > 8\) the situation is reversed. Similar behavior can be observed for the area of the sorters, where the bubble sorter remains smaller than the pruned bitonic sorter for \(L \leq 16\).

We also observe that, for \(L \leq 16\), the pruned radix-2L sorter is faster than the other two sorters and similar in area to the pruned bitonic sorter, while the simplified bubble sorter is significantly smaller. Thus, for \(L \leq 16\) the pruned bitonic sorter is not a viable option, while trade-offs between speed and area can be made by using either the pruned radix-2L sorter or the simplified bubble sorter. For \(L = 32\), however, the pruned bitonic sorter has a higher operating frequency and a smaller area than the other two sorters.

VI. CONCLUSION

In this work, we presented a pruned bitonic and a bubble sorter that exploit the structure of the elements that need to be sorted in SCL decoding of polar codes. Our results indicate that the bitonic sorter used in [5], even with the pruning proposed in this work, is not a suitable choice for list sizes \(L \leq 16\). Our simplified bubble sorter, on the other hand, provides a meaningful trade-off between speed and area with respect to the pruned radix-2L sorter used in [7] (which still remains the fastest sorter for \(L \leq 16\)). For \(L = 32\), however, the superior delay and area scaling of the pruned bitonic sorter make it 14\% faster and 45\% smaller than the second-fastest radix-2L sorter. Moreover, both the pruned bitonic and the bubble sorter have stages that are identical in terms of delay, thus enabling much simpler pipelining than the radix-2L sorter.

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