The minimum broadcast range assignment problem on linear multi-hop wireless networks

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Abstract

Given a set $N$ of radio stations located on an Euclidean space, a source station $s$ and an integer $h$ ($1 \leq h \leq |N| - 1$), the minimum bounded-hop broadcast range assignment problem consists in finding a range assignment for $N$ of minimum total power consumption that allows broadcast operations from $s$ to every station in $N$ in at most $h$ hops. The problem is known to be NP-hard on $d$-dimensional spaces for any $d \geq 2$ (18th Annual Symp. on Theoretical Aspects of Computer Science (STACS’01), Lecture Notes in Computer Science, Vol. 1770, 2000, pp. 651–660.) and some efficient approximation algorithms have been given in Clementi et al. and Wann et al. (18th Annual Symp. on Theoretical Aspects of Computer Science (STACS’01), Lecture Notes in Computer Science, Vol. 1770, 2000, pp. 651–660, IEEE INFOCOM’01, 2001). In this paper, we address the case in which the stations are arbitrarily located along a line (i.e., the linear case). We provide the first polynomial-time algorithm that returns an optimal solution for any instance of the linear case. The algorithm works in $O(h|N|^2)$ time.

Keywords: Ad hoc wireless networks; Energy consumption; Dynamic programming

1. Introduction

Wireless networking technology will play a key role in future communications and the choice of the network architecture model will strongly impact the effectiveness of
the applications proposed for the mobile networks of the future. Broadly speaking, there are two major models for wireless networking: single-hop and multi-hop. The single-hop model [17], based on the cellular network model, provides one-hop wireless connectivity between mobile hosts and static nodes known as base stations. This type of network relies on a fixed backbone infrastructure that interconnects all base stations by high-speed wired links. On the other hand, the multi-hop model [13] requires neither fixed, wired infrastructure nor predetermined interconnectivity. Ad hoc networking [11] is the most popular type of multi-hop wireless networks because of its simplicity: indeed, an ad hoc wireless network is constituted by a homogeneous system of mobile stations connected by wireless links. In ad hoc networks, a transmission range is assigned to every station: the overall range assignment determines a (directed) communication graph since a station \( s \) can transmit to another station \( t \) if and only if \( t \) is within the transmission range of \( s \).

The range transmission of a station depends, in turn, on the energy power supplied to the station: the power \( P_s \) required by a station \( s \) to correctly transmit data to another station \( t \) must satisfy the inequality

\[
\frac{P_s}{d(s,t)^\alpha} > \gamma, \tag{1}
\]

where \( d(s,t) \) is the Euclidean distance between \( s \) and \( t \), \( \alpha \geq 1 \) is the distance-power gradient, and \( \gamma \geq 1 \) is the transmission-quality parameter. In an ideal environment (i.e., in the empty space) it holds that \( \alpha = 2 \) but it may vary from 1 to more than 6 depending on the environment conditions of the place the network is located (see [15]). A fundamental problem underlying any phase of a dynamic resource allocation algorithm in ad hoc wireless networks is the following: find a transmission range assignment such that (1) the corresponding communication graph satisfies a given property \( \pi \), and (2) the overall energy power required to deploy the assignment (according to Eq. (1)) is minimized.

A well-studied case of the above problem consists in choosing \( \pi \) as follows: the communication graph has to be strongly connected. In this case, it is known that: (a) the problem is not solvable in polynomial time (unless \( P=NP \)) [6,12]; (b) it is possible to compute a range assignment which is at most twice the optimal one (that is, the problem is 2-approximable) for multi-dimensional wireless networks [12]; (c) there exists a constant \( r > 1 \) such that the problem is not \( r \)-approximable (unless \( P=NP \)), for \( d \)-dimensional networks with \( d \geq 3 \) [6], and (d) the problem can be solved in polynomial time for one-dimensional (i.e. linear) networks [12]. Another analyzed case consists in choosing \( \pi \) as follows: the diameter of the communication graph has to be at most \( h \). In this case, while non-trivial negative results are not known, some tight bounds (depending on \( h \)) on the minimum energy power have been proved in [7] and a polynomial-time 2-approximation algorithm for the linear case has been given in [5]. Other trade-offs between connectivity and energy consumption have been obtained in [14,16,18].

In this paper, we address the case in which \( \pi \) is defined as follows: given a source station \( s \) and an integer \( 1 \leq h \leq n-1 \) (where \( n = |N| \)), the communication graph must contain a directed spanning tree rooted at \( s \) having depth not larger than \( h \). This case
has been posed as an open question by Ephremides in [9,20] and its relevance is due to the fact that any communication graph satisfying the above property allows the source station to perform a broadcast operation in at most \( h \) hops. Broadcast is a task initiated by the source station which transmits a message to all stations in the wireless network. This task constitutes a major part of real life multi-hop radio networks [20,1,2].

Notice that, if stations are located on an Euclidean space, the case in which the distance-power gradient \( \varepsilon \) is equal to 1 is optimally solved by assigning a range to the source sufficient to reach any other station in one hop. On the other hand, when \( \varepsilon > 1 \) the problem is far to be easily solved mainly because the triangular inequality does not hold. The planar unbounded case (where stations are located on the plane and \( h = n - 1 \)) is known to be NP-hard [4]. On the other hand, the same authors provide an efficient approximation algorithm based on the computation of an MST for the distance graph. A better analysis of the approximation ratio achieved by the MST-based algorithm has been independently presented in [19].

This paper focuses on the linear case, i.e., networks that can be modelled as sets of stations located along a line. Linear radio networks have been the subject of several recent studies [5,10,12,14]. As pointed out in [14], rather than a simplification, this version of the problem results in a more accurate analysis of the situation arising, for instance, in vehicular technology applications. The one-dimensional space is in fact the most suitable framework in order to study road traffic information systems [3,8,12,14]. Vehicles follow roads, and messages are to be broadcasted along lanes. Typically, the curvature of roads is small in comparison to the transmission range (half a mile up to some few miles). In the sequel, the above described optimization problem restricted to the linear case will be denoted as MIN BH-BRA.

In this paper, we provide the first polynomial-time algorithm for the MIN BH-BRA problem. The algorithm works in \( O(hn^2) \) time and it relies on dynamic programming. Informally speaking, the main technical issue lies in the presence of bridges in the communication directed graph yielded by an optimal solution. A bridge is an edge that connects a station on the left of the source \( s \) to a station on the right of \( s \) or vice versa. Such bridges may be functional, i.e., their removal may cause the infeasibility of the solution. A straightforward application of dynamic programming in order to optimally locate such bridges would yield an algorithm working in \( \Omega(n^h) \) time. Our main technical contribution here is a strong characterization of the structure of any optimal solution. Indeed, we prove that any optimal solution contains at most one functional bridge. This structural result allows to exponentially reduce the size of the search space of our dynamic programming algorithm. We also emphasize that the performance of our algorithm does not depend on \( \varepsilon \).

Dynamic programming has been also used in [5] to optimally solve the all-to-one version of the min-range assignment problem, i.e., the case in which \( \pi \) consists in requiring that, for each station \( v \), the range assignment must guarantee the existence of a path of length at most \( h \) from \( v \) to a given sink \( s \). Their algorithm works in \( O(hn^3) \) time. Informally speaking, this is the opposite version of our problem. Even though, at a first look, the problems seem to be strongly related, this is not the case: in particular, our problem cannot be (at least easily) reduced to the all-to-one version. Among the others, we emphasize two key differences: (i) In the all-to-one version, any feasible
solution must assign a positive range to every station: this does not clearly hold for the broadcast version. (ii) Bridges are not useful for the all-to-one version while, as discussed above, it is a crucial issue in our technique. So, the dynamic programming used in our algorithm relies on a combinatorial structure which is significantly different from that used in [5] for the all-to-one version.

2. Preliminary definitions

Let \( N = \{x_1, \ldots, x_n\} \) be a set of \( n \) consecutive stations located on a line. So \( x_i \) represents the rational coordinate of the \( i \)th station with respect to a fixed origin. We denote by \( d(i,j) \) the distance between station \( x_i \) and \( x_j \). For the sake of simplicity, we denote \( x_i \) as \( i \). Finally, we denote the set of stations \( \{i, \ldots, j\} \) as \( [i;j] \).

A range assignment for \( N \) is a function \( A : N \rightarrow \mathbb{R}^+ \). Given a range assignment \( A \), we say that \( i \) directly reaches \( j \) or, equivalently, \( i \) reaches \( j \) within one hop (in short \( i \rightarrow j \)) if \( A(i) \geq d(i,j) \). Similarly, \( i \) reaches \( j \) within at most \( h \) hops (in short \( i \rightarrow_h j \)) if there exist \( h-1 \) stations \( i_1, \ldots, i_{h-1} \) such that \( i \rightarrow i_1 \rightarrow i_2, \ldots, \rightarrow i_{h-1} \rightarrow j \).

A range assignment \( A \) uniquely determines a communication directed graph \( G_A(N,E_A) \) where \( (i,j) \in E_A \) iff \( i \) directly reaches \( j \).

Given a station \( s \in N \), a range assignment \( B \) for \( N \) is a broadcast range assignment with respect to \( s \) if the communication graph \( G_B \) contains a spanning source tree rooted at \( s \). We say that \( B \) is an \( h \)-broadcast range assignment \( (1 \leq h \leq n-1) \) if \( G_B \) contains a spanning source tree rooted at \( s \) whose depth is at most \( h \).

For any \( 1 \leq i < j \leq n \) and \( s \in [i,j] \), the cost of an \( h \)-broadcast range assignment \( B \) for \( [i,j] \) with respect to \( s \) is defined as

\[
c_B([i,j],s,h) = \sum_{k=i}^{j} B(k)^h.
\]

An optimal \( h \)-broadcast range assignment is denoted as \( B^*[i,j],s,h] \) and its cost as \( c^*[i,j],s,h) \). We will omit the indication of \( s, [i,j] \) and \( h \) when they will be clear from the context.

The minimum \( h \)-broadcast range assignment problem for linear wireless networks (in short, Min BH-BRA) is then defined as follows: given a set \( [1,n] \) of stations located on a line, a station \( s \in [1,n] \), and an integer \( h > 0 \), find a minimum cost \( h \)-broadcast range assignment for \( [1,n] \) with respect to \( s \). In what follows, an instance of Min BH-BRA will be denoted as \( ([1,n],s,h) \).

3. The structure of optimal broadcast range assignments

In order to investigate the structure of optimal solutions for Min BH-BRA, we need to introduce the notion of bridge.

Given an \( h \)-broadcast range assignment \( B \) for \( N \) with respect to the source \( s \in N \) (yielding the communication graph \( G_B(N,E_B) \)), a left-bridge is a directed edge \( (l,r) \in E_B \) such that \( 1 \leq l < s < r \leq n \) (the left-bridge is denoted as \( lr \)). Similarly, a right-bridge
is a directed edge \((r,l) \in E_B\) such that \(1 \leq l < s < r \leq n\) (the right-bridge is denoted as \(\overrightarrow{lr}\)). In the sequel, we shall always assume that, in a left-bridge \(\overleftarrow{lr}\) (right-bridge \(\overrightarrow{lr}\)), \(r\) is the rightmost station directly reached by \(l\) (\(l\) is the leftmost station directly reached by \(r\)).

A left-bridge \(\overleftarrow{lr}\) is functional for \(B\) if there exists \(i \in \mathbb{N}\) such that its hop-distance from \(s\) in \(G(N,E_B\backslash\{\overrightarrow{lr}\})\) its greater than \(h\). We define functional right-bridges similarly. An \(h\)-broadcast range assignment is left-bridge feeded if it yields (at least) one functional left-bridge. A right-bridge feeded \(h\)-broadcast range assignment is defined similarly. An \(h\)-broadcast range assignment which is not (left- or right-) bridge feeded is said bridge free.

The aim of this section is to show that any instance of \textsc{Min BH-BRA} admits optimal solutions (namely, optimal \(h\)-broadcast range assignments) which are either bridge free or contain a single functional bridge. As first step towards this aim, we provide the following:

**Theorem 1.** Let \(N = [1,n]\), \(s \in N\), and let \(h\) be an integer such that \(1 \leq h \leq n-1\). Then, any optimal \(h\)-broadcast range assignment \(B^*\) for \(([1,n],s,h)\) is not both left-bridge feeded and right-bridge feeded.

**Proof.** The proof is by contradiction. Let us consider an optimal \(h\)-broadcast range assignment \(B^*\) for \(([1,n],s,h)\) having cost \(c^*\) which is both left-bridge and right-bridge feeded. Let \(l_1r_1\) and \(l_2r_2\) be, respectively, a functional left-bridge and a functional right-bridge of \(B^*\).

Four cases may arise:

1. \([l_2<l_1\text{ and }r_2<r_1]\) (see Fig. 1). Let \(d_1 = d(s,l_1)\) and \(d_2 = d(s,r_2)\). Consider the two minimal integers \(h_1\) and \(h_2\) such that \(s \rightarrow_{h_1} l_1\) and \(s \rightarrow_{h_2} r_2\) by means of \(B^*\). It must be \(h_1 \leq h_2 + 1\): indeed, it is possible to reach \(l_1\) in \(h_2 + 1\) hops by using \(l_2r_2\). Similarly, it must hold that \(h_2 \leq h_1 + 1\). The two inequalities imply that \(h_1 \in \{h_2 - 1, h_2, h_2 + 1\}\).

Suppose first \(h_1 = h_2 - 1\). Since \(l_2r_2 \in E_{B^*}\), it holds that

\[
B^*(r_2) \geq d(r_2, l_2) = d(r_2, l_1) + d(l_1, l_2).
\]

We then consider the new range assignment \(B'\) defined as follows: \(B'(r_2) = 0\), \(B'(l_1) = \max\{B^*(l_1), d(l_1, l_2)\}\), \(B'(i) = B^*(i)\) for any \(i \neq r_2, l_1\). Since \(l_1r_1\) is a functional left-bridge for \(B^*\), \(r_2\) is “useless” for any station at its right; hence, the new
assignment is an $h$-broadcast range assignment. Furthermore its cost is smaller than $c^\ast$.

The case $h_1 = h_2 + 1$ is symmetric.

Finally, suppose $h_1 = h_2$. Let $l_x$ be the leftmost station such that $l_1 \rightarrow l_x$. If $l_2 < l_x$ then $l_1 r_1$ is not functional, otherwise $l_2 r_2$ is not functional.

(2) $[l_1 < l_2$ and $r_1 < r_2]$. This case is symmetric to the previous one.

(3) $[l_2 < l_1$ and $r_1 < r_2]$ (see Fig. 2). Without loss of generality, we consider two bridges $l_1 r_1$ and $l_2 r_2$ such that no further bridge $l_3 r_3$ (or $l_3 r_3$) occurs such that $l_2 \leq l_3 \leq l_1$ and $r_1 \leq r_3 \leq r_2$.

Then, a new assignment $B'$ can be defined as follows: $B'(s) = d(s, r_1)$, $B'(i) = 0$ for $i \in [l_1, s - 1]$, $B'(j) = B^\ast(j)$ for all the other stations $j$. The range assignment $B'$ does not contain the left-bridge $l_1 r_1$ and it is still an $h$-broadcast range assignment for $N$ with respect to $s$. Indeed, since $l_1 r_1$ is functional in $B^\ast$, then $B^\ast(s) < d(s, r_1)$ and, also, all stations in $[l_2, s - 1]$ are covered by $l_2 r_2$. Furthermore, it can be easily verified that

$$c_{B'} \leq c^\ast + B'(s)^x - B^\ast(l_1)^y - B^\ast(s)^y.$$ 

Since $B^\ast(l_1) \geq d(l_1, r_1)$ and $d(s, r_1) < d(l_1, r_1)$, then $c_{B'} < c^\ast$ thus contradicting the optimality of $B^\ast$.

(4) $[l_1 < l_2$ and $r_2 < r_1]$. This case is symmetric to the previous one. □

The previous theorem allows us to search for optimal $h$-broadcast range assignments which are left-bridge free or right-bridge free. The next theorem furtherly reduces the search space by stating that at most one functional bridge may be contained in any optimal solution.

**Theorem 2.** Any optimal $h$-broadcast range assignment $B^\ast$ for $([1, n], s, h)$ contains at most one functional bridge.

**Proof.** Again, the proof is by contradiction. Thanks to Theorem 1, we can consider an optimal solution which is not both left-bridge feded and right-bridge feded. Hence, let $B^\ast$ be any optimal right-bridge-free $h$-broadcast range assignment that contains at least two functional left-bridges $l_1 r_1$ and $l_2 r_2$. A symmetric argument holds for left-bridge-free solutions. Our goal is to transform $B^\ast$ into an $h$-broadcast range assignment $B'$ that contains one functional bridge less than $B^\ast$ and has a smaller cost: a contradiction.
Let \( h_1 \) and \( h_2 \) be, respectively, the minimal integers such that 
\[
\begin{align*}
  s &\rightarrow_{h_1} l_1 \rightarrow r_1 \\
  s &\rightarrow_{h_2} l_2 \rightarrow r_2
\end{align*}
\]
by means of \( B^* \).

Without loss of generality, we assume that \( l_2 < l_1 \). This implies that \( h_2 \geq h_1 \). Then, since \( l_1 r_1 \) and \( l_2 r_2 \) are both functional, it must hold that \( r_2 \geq r_1 \) (informally speaking, the two bridges cannot cross each other). If \( l_1 r_1 \) is functional, there must exist a station \( x > r_2 \) for which all paths from \( s \) to \( x \) that do not contain \( l_1 r_1 \) have length greater than \( h \) (i.e., they are not feasible, see Fig. 3). Let us assume that \( s \rightarrow_{h_x} x \) for some \( h_x \leq h \) (by means of \( l_1 r_1 \)). Let us fix one of the “feasible” paths from \( s \) to \( x \)
\[
p = s \rightarrow_{h_x} l_1 \rightarrow r_1 \rightarrow_{h_y} y \rightarrow z \rightarrow_{h_x - h_1 - h_y - 2} x,
\]
where \( y < r_2 < z \).

Since the two bridges are both functional, it must hold that \( h_y = h_2 \). Indeed, the functionality of \( l_1 r_1 \) implies that \( h_y \leq h_2 \) (otherwise \( x \) could have been reached in less than \( h_x \) hops thus making \( l_1 r_1 \) non-functional), while the functionality of \( l_2 r_2 \) implies that \( h_y \geq h_2 \) (otherwise \( r_2 \) could have been reached in \( h_y + 1 \leq h_2 \) hops thus making \( l_2 r_2 \) not functional).

Let us now consider the new range assignment \( B' \) where \( B'(i) = B^*(i) \) for all \( i \neq y \) and \( i \neq r_2 \), while \( B'(y) = 0 \) and
\[
B'(r_2) = \max\{B^*(r_2), d(r_2, z)\}.
\]
Notice that \( l_1 r_1 \) is not functional in \( B' \). This new assignment is feasible and has cost less than \( c_{B^*} \). \( \square \)

By summarizing the results of this section, we can state that any optimal \( h \)-broadcast range assignment is either bridge-free or it contains a single functional (left- or right-) bridge.

4. The dynamic programming algorithm

In this section we provide the algorithm that solves the Min BH-BRA problem by using dynamic programming. The algorithm relies on the structure of optimal solutions that has been shown in the previous section. In particular, the search for optimal
h-broadcast range assignments can be restricted to solutions of either of the three following kinds:
(1) bridge-free solutions;
(2) left-bridge free but right-bridge feeded solutions, and
(3) right-bridge free but left-bridge feeded solutions.

In particular, given any instance \(\langle [1,n],s,h\rangle\) of MIN BH-BRA, we show how to compute its minimal bridge-free \(h\)-broadcast range assignment and its minimal bridge feeded ones. Then, by choosing among them the solution yielding minimum cost, we optimally solve MIN BH-BRA. This will prove our main result.

**Theorem 3.** An algorithm exists that solves MIN BH-BRA in \(O(hn^2)\) time.

### 4.1. Computing optimal bridge-free solutions

We first show a simple recursive method to optimally solve the MIN BH-BRA problem for instances in which \(s = 1\) (the case \(s = n\) is symmetric and hence it is omitted). Then, we exploit this method to solve the bridge-free case. In the sequel, an \(h\)-broadcast range assignment for the instance \(\langle [i,j],s,h\rangle\) will be denoted as \(B\langle [i,j],s,h\rangle\).

**Lemma 4.** Given any instance \(\langle [i,n],i,h\rangle\) of MIN BH-BRA with \(i \in [1,n-1]\), the optimal cost \(c^*(\langle [i,n],i,h\rangle)\) satisfies the following recursive equation:
\[
c^*(\langle [i,n],i,h\rangle) = \min\{d(1,j)^2 + c^*(\langle [j,n],j,h-1\rangle) : i < j \leq n\}.
\]

Hence, an algorithm exists that computes both the optimal solution \(B^*\langle [i,n],i,h\rangle\) and \(c^*(\langle [i,n],i,h\rangle)\) within \(O(hn^2)\) time.

**Proof.** The index \(j\) in the above equation denotes the rightmost station reached by \(s = i\) in one hop. Once \(j\) has been fixed, then all stations in the interval \([i+1,j-1]\) must have range 0 while the range for the stations in \([j,n]\) must be an optimal \((h-1)\)-broadcast range assignment for \([j,n]\) with source \(j\).

Eq. (2) yields a dynamic programming algorithm that works in \(O(hn^2)\) time. Indeed, assume that we have already computed the optimal solutions for some \(h'\) such that \(1 \leq h' \leq h - 1\) and for every interval \([j,n]\) with \(i \leq j \leq n\) (that is, we have computed \(B^*\langle [j,n],j,h'\rangle\) for every \(i \leq j \leq n\)). Then, thanks to Eq. (2), we can compute \(B^*\langle [i,n],i,h' + 1\rangle\) in \(O(n)\) time.

Let us define the set
\[
A = \{\delta \in \mathbb{R}^+ : \exists j \in N \text{ such that } j \neq s \text{ and } d(j,s) = \delta\}
\]
and, for each \(\delta \in A\), define the indices \(i_l(\delta)\) and \(i_r(\delta)\) such that \(i_l(\delta)\) \((i_r(\delta))\) is the left-most (rightmost) station such that \(d(s,i_l(\delta)) \leq \delta(d(s,i_r(\delta)) \leq \delta)\). Notice that \(|A| = n - 1\).
Lemma 5. The cost $c_{B_F}$ of an optimal bridge-free solution $B_F$ for any instance $\langle [1,n], s, h \rangle$ satisfies the following recursive equation:

$$c_{B_F} = \min \{ \delta^* + c^*([1, i_l(\delta)], i_l(\delta), h - 1) + c^*([i_l(\delta), n], i_l(\delta), h - 1) : \delta \in \Delta \}. \quad (3)$$

Furthermore, the time required to compute both $B_F$ and its cost is $O(hn^2)$.

Proof. The parameter $\delta$ in Eq. (3) stands for the range assigned to $s$. Once $\delta$ has been fixed, the station $i_l(\delta)$ ($i_r(\delta)$) is the leftmost (rightmost) station reached by $s$ in one hop. Then, all stations in the set $[i_l(\delta) + 1, s - 1] \cup [s + 1, i_r(\delta) - 1]$ must have range 0 (in any optimal solution) while, in the intervals $[1, i_l(\delta)]$ and $[i_r(\delta), n]$, the optimal solutions are, respectively, $B^*([1, i_l(\delta)], i_l(\delta), h - 1)$ and $B^*([i_r(\delta), n], i_r(\delta), h - 1)$.

The optimal bridge-free range assignment $B_F$ is computed in two phases.

During the first phase, by using Lemma 4, we compute $B^*([i, n], i, h - 1)$ and $B^*([1, j], j, h - 1)$ for any $s + 1 \leq i \leq n$ and $1 \leq j \leq s - 1$ and keep all such results. This can be done in $O(hn^2)$ time. Indeed, for $h' = 1$, we can compute all the above $B^*([i, n], i, 1)$'s and $B^*([1, j], j, 1)$'s in $O(n)$ time. Hence, assume that we have already computed the optimal solutions for some $h'$ such that $1 \leq h' \leq h - 1$ and for every interval $[i, n]$ with $s < i \leq n$ (that is, we have computed $B^*([i, n], i, h')$ for every $s < i \leq n$) (the $B^*([1, j], j, h - 1)$ can be computed similarly). Then, similar to Eq. (2), it holds that

$$c^*([i, n], i, h' + 1) = \min \{ d(i, i + k)^2 + c^*([i + k, n], i + k, h') : k \geq i \}. \quad (3)$$

It thus follows that the optimal solution for $h' + 1$ and for interval $[i, n]$ ($s < i \leq n$) can be obtained in $O(n)$ time.

As for the second phase, we apply Eq. (3). Notice that $B^*([1, i_l(\delta)], i_l(\delta), h - 1)$ and $B^*([i_r(\delta), n], i_r(\delta), h - 1)$ can be recovered in constant time from those computed during the first phase. Since $|\Delta| = n - 1$, the second phase is completed in $O(n)$ time.

4.2. Computing optimal bridge fed solutions

We now show how to compute an optimal left-bridge fed solution $B_L$ for an instance $\langle [1,n], s, h \rangle$ of $\text{MIN BH-BRA}$ (the computation of an optimal right-bridge fed solution $B_R$ is similar and thus it is omitted). From Section 3, we know that $B_L$ contains a single functional left-bridge and no functional right-bridges. Hence, we have to optimally choose a node $1 \leq b < s$, a range $\delta$ for $b$, and the number of hops $1 \leq h_b < h$ required to connect $s$ to $b$: such parameters determine the unique bridge of the optimal left-bridge fed $h$-broadcast range assignment. More formally, we define, for any $1 \leq b < s$,

$$\Delta_h = \{ \delta \in \mathbb{R}^+ \mid \exists j \in N \text{ such that } j \neq s \text{ and } d(b, j) = \delta \}$$

and, for each $\delta \in \Delta_h$, define the indices $i_l$ and $i_r$ such that $i_l(i_r)$ is the leftmost (rightmost) station such that $d(b, i_l) \leq \delta(d(b, i_r) \leq \delta)$. Notice that $|\Delta| \leq n - 1$.
Lemma 6. The cost of an optimal left-bridge feeded solution $B_L$ for any instance $(\{1,n\}, s, h)$ satisfies the following recursive equation:

$$
c_{B_L} = \min \{c^*([b,s], s, h_b) + \delta^* + c^*([1,i], i, h-h_b-1)$$
$$+ c^*([i,n], i, h-h_b-1) | b \in [1,s-1], \; \delta \in A_b, \; h_b \in [1,h-1] \}. \tag{4}$$

Furthermore, both $B_L$ and its cost can be computed in $O(hn^2)$ time.

Proof. Let us denote by $b^*, \delta^*, i^*_1, i^*_r$ and $h^*_b$ some optimal values for $b, \delta, i_1, i_r$, and $h_b$, respectively. The optimal range assignment required to reach $b^*$ in $h^*_b$ hops is $B^*([b^*,s], s, h^*_b)$ for all stations in $[b^*,s]$ and 0 to all other stations. Indeed, no stations in $[s,n]$ can yield a right functional bridge for the interval $[b^*,s]$. Furthermore, no station in $[1,b^*-1]$ is reached by $s$ in less than $h^*_b$ hops and, thus, it is not functional for $[b^*,s]$.

Since $i^*_1$ and $i^*_r$ are reached by $b^*$ in 1 hop, all stations in the intervals $[i^*_1+1,b^*-1]$ and $[s+1,i^*_r-1]$ must have range 0 (notice that, because of the functionality of the left bridge starting at $b^*$, it cannot exist a station in the interval $[s+1,i^*_r-1]$ that is “functional” for the interval $[i^*_r,n]$) and the range assignments for the intervals $[1,i^*_1]$ and $[i^*_r,n]$ are, respectively, $B^*[1,i^*_1], i^*_1, h-h^*_b-1]$ and $B^*[i^*_r,n], i^*_r, h-h^*_b-1]$.

It thus follows that the left-bridge feeded solution $B_L$ can always be split into six disjoint and independent components:

- The interval $[b^*,s]$; the optimal solution is computed according to Lemma 4 and yields cost $c^*([b^*,s], s, h^*_b)$. Notice that $b^*$ has range 0 in $B^*[b^*,s], s, h^*_b$.
- The bridge $b^*$ which has range $\delta^*$.
- The interval $[i^*_1+1,b^*-1]$; no positive range is required here since it is covered by the bridge $b^*$.
- The interval $[s+1,i^*_r-1]$; no positive range is required here since it is covered by the bridge $b^*$.
- The interval $[1,i^*_1]$; the optimal solution is computed according to Lemma 4 and yields cost $c^*([1,i^*_1], i^*_1, h-h^*_b-1)$.
- The interval $[i^*_r,n]$; the optimal solution is computed according to Lemma 4 and yields cost $c^*([i^*_r,n], i^*_r, h-h^*_b-1)$.

Hence, Eq. (4) follows.

As for the time complexity, we observe that we can compute all the $c^*([*,*],*,*)$’s in Eq. (4) during a pre-processing within $O(hn^2)$ time by using Lemma 4. □

References


