Flicker Noise Up-Conversion due to Harmonic Distortion in Van der Pol CMOS Oscillators
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Abstract—Harmonic content modulation of the oscillator output voltage waveform can contribute to flicker noise up-conversion in LC-tuned oscillators. The paper reports a quantitative analysis of the effect in Van der Pol oscillators using the framework of the impulse sensitivity function (ISF). It is shown that most of the up-conversion efficiency results from the first harmonic of the ISF, which is not perfectly in quadrature to the output voltage waveform, and from the first harmonic of the transistor current, which is slightly lagging the voltage waveform. A closed-form expression of $1/f^3$ phase noise in voltage-limited LC-tuned oscillator is derived that is in good agreement with circuit simulations. The paper also shows that the values of both phase shifts are determined by the non-linearity of the active element and are linked to the relevant oscillator parameters, i.e., excess gain and tank quality factor.

Index Terms—Distortion, flicker noise, impulse sensitivity function (ISF), oscillator non-linearity, phase noise, total harmonic distortion (THD), up-conversion, voltage controlled oscillator.

I. INTRODUCTION

PHASE noise of voltage controlled oscillators (VCOs) is a key figure in the design of wireless communication systems and in the last decade extensive efforts have been devoted to completely understand how noise sources reflect into phase noise of fully integrated VCOs. Close-in phase noise in CMOS LC-tuned VCOs is dominated by up-conversion of the flicker noise. The effect is exacerbated by the adoption of deeply scaled sub-micron CMOS technologies that feature noise corner frequencies in the MHz-range. Even when the VCO is inside a phase-locked loop (PLL), the $1/f^3$ phase noise remains an issue since the loop bandwidth is typically in the 100–200 kHz range to minimize the noise contributions of charge-pump, reference and $3$-$\Delta$ modulator [1]. For example, transmitters of the so-called 3 GPP (GSM/GPRS/EDGE/WCDMA) systems require a phase noise of about $-114$ dBc/Hz at 400 kHz [1]. In practice, considering a safety margin for large scale production, a stringent figure of about $-124$ dBc/Hz at 400-kHz offset should be targeted. To reach such a low level, flicker noise up-conversion has to be accurately minimized [2]–[19]. Three major up-conversion mechanisms have been identified so far, namely:

1) amplitude to phase noise conversion due to non-linear varactors [3]–[6];
2) modulation of the current flowing through the tail capacitance in a current-biased VCO topology [7], [8];
3) modulation of the harmonic content of the output voltage waveform [8]–[14].

Along these years some minimizing solutions have been proposed. The adoption of a bank of digitally switchable capacitors can drastically reduce the AM-PM conversion due to the non-linear capacitances, without impairing the overall VCO tuning range [17]. The second cause can be minimized by either removing the bias current generator [5], [17]–[19] and/or adopting a resonant filter at the tail [15]. The up-conversion due to the modulation of the harmonic content is instead more difficult to handle since it is strictly related to the non-linear nature of oscillators.

The effect is known since 1933 when Janusz Groszkowski published his pioneering work on frequency stability in oscillators [20]. Groszkowski found that the steady-state oscillation frequency does not perfectly match the resonance frequency of the tank. Since the active element drives the tank with a non-harmonic current signal, a frequency shift of the oscillation frequency arises. The shift is needed to guarantee that the average reactive power delivered to the resonant tank in the oscillation period is zero. In fact, the high frequency harmonics of the current signal flow into the tank capacitor, which is a low impedance load on the high frequency side of the resonance, delivering a net capacitive reactive power. The first harmonic of the current should therefore lag the voltage in order to deliver the inductive power needed to keep the balance. In [20] the oscillation frequency was found to be

$$\omega_0^2 - \frac{\omega_R^2}{\sum_{k=1}^{\infty} \frac{V_k^2}{k^2}},$$

where $\omega_R/(2\pi)$ is the tank resonance frequency and $V_k$ is the amplitude of the $k$-th voltage harmonic. Due to (1), any time noise modulates the amplitude of the voltage harmonics, a frequency modulation is generated ultimately resulting into phase noise.

The concept that harmonic distortion may cause phase noise degradation in oscillators has been highlighted many times in literature. For example Vittoz et al. [12], describing the adoption of an automatic gain control circuit (AGC) in a crystal oscillator, state that non-linearities have a “devastating effect on stability” and propose “to limit the amplitude of oscillation” to reduce the effect of distortion. The same idea can be found in [11], where the authors adopt an AGC loop to make possible a quasi-linear operation of the VCO, thus reducing harmonic distortion and improving phase noise performance. Among most
recent papers, Jer
gen and Sodini [9] propose to reduce the device width of the differential-pair transistors in a current-biased oscillator to increase the overdrive voltage and to extend the linear range of the switching devices. In this way the harmonic distortion is reduced with benefits in terms of phase noise arising from both the switching pair and the flicker noise up-conversion of the bias current. These authors refer to the effect as a form of “indirect frequency modulation”. Also in [10] damping resistors at the source side of the differential pair transistors are introduced to linearize the transconductor and to suppress 1/f noise up-conversion.

Despite distortion has been always considered a fundamental cause of phase noise generation, only recently Bevilacqua and Andreani [14] started to develop quantitative frameworks by studying the 1/f noise up-conversion of the bias current due to non-linearity in both Colpitts and differential-pair LC-tuned oscillators. Moving further along this investigation path, this paper provides a quantitative analysis of the flicker noise up-conversion arising from the cross-coupled pair in a Van der Pol oscillator. For the first time a quantitative link between 1/f noise up-conversion and non-linearities is addressed taking also into account the cyclo-stationary nature of the noise sources.

The paper is organized as follows. Section II describes the motivations to study a Van der Pol topology. Section III is devoted to disclose the parameters involved into the flicker noise up-conversion mechanism in the circuit topology under investigation. The quantitative model is validated against circuit simulations in Section IV. Section V explains why and how the up-conversion factor is due to the distortion generated by the active element, thus providing quantitative links to key design parameters of the oscillator. In Section VI a closed-form expression of 1/f^3 phase noise is derived matching accurately the simulation results for low degree of distortion. Finally, conclusions are drawn in Section VII.

II. THE CASE STUDY

Fig. 1 shows a differential LC-tuned oscillator implementing a Van der Pol topology that has been taken as case study in this work. With respect to the well-known current-biased differential topology, the circuit does not feature a bias current generator thus removing dominant contributions to both 1/f^2 and 1/f^3 phase noise [4], [21]. Moreover the varactors have been replaced by a linear capacitor to avoid further AM-PM conversion. Since the topology has very few noise sources, it provides the proper environment to study the up-conversion mechanism only due to non-linearity of the switching active elements without the presence of other terms. In designing the circuit, NMOS and PMOS transistors have been sized to set the output nodes to half the supply voltage. The choice maximizes the oscillation swing. By assuming the same threshold voltage for both types of transistor (V_{TH} = V_{TH}^0 = V_{ID}), the condition leads to set the same transconductance for both NMOS and PMOS transistors, which translates into W_n/W_n = $\mu_n/\mu_p = 2.2$. In the following the width W will always refer to the width of the NMOS transistors while the width of the PMOS transistors will be always considered adjusted accordingly to retain the 2.2-ratio. The circuit was simulated in a 65-nm CMOS technology with 1.2-V supply. The oscillator central frequency was set to 2 GHz by using a tank with L = 1.2 nH, C = 5.3 pF and a quality factor Q = 10 (i.e., R = 150 $\Omega$). Once the tank has been sized, the only free parameter left to the designer is the small signal loop gain, or excess gain [11], [22], defined as

$$G_X = g_m R_c$$

(2)

$g_m$ being the small signal transconductance of the double cross-coupled transistor. The excess gain is usually set between 2 and 4 to guarantee oscillator start-up even in presence of process spread and variability of the tank quality factor [11]. In practice, when this topology is adopted to synthesize a VCO, the excess gain is set by choosing the transistors’ widths, while the lengths may be kept at the minimum value not to narrow the achievable tuning range. Assuming a square-law transistor characteristic and considering that at the start-up the overdrive voltage is $V_{DD}/2 - V_{TH}$, the overall transconductance is given by the sum of the NMOS and the PMOS pair transconductance, that is

$$g_m = g_{m,n} + g_{m,p} \approx \mu_{n} C_{ox} \left(\frac{W}{L}\right) C \left(\frac{V_{DD}}{2} - V_{TH}\right).$$

(3)

It follows that the excess gain varies linearly with the transistor width, as shown in Fig. 2. As W ranges from 25 $\mu$m to 145 $\mu$m the excess gain spans from 1.5 to about 9. On the other hand, the larger the transistor width, the larger the excess gain and the higher the voltage harmonic distortion.

Fig. 3 shows the simulated Total Harmonic Distortion (THD) [22], [32] of the differential output voltage, together with the oscillation frequency, as a function of the excess gain. By increasing the harmonic distortion of the output voltage, the oscillation frequency shifts down with respect to the resonance
Triangles refer to the oscillation frequency estimated taking into account the effect of distortion (see (1)).

Fig. 3. Oscillation frequency and Total Harmonic Distortion [22],[23] for the VCO in Fig. 1 varying the excess gain $G_X$. Triangles refer to the oscillation frequency estimated taking into account the effect of distortion (see (1)).

Frequency, in agreement with the Groszkowski effect. In Fig. 3 triangles refer to the estimate derived from (1), having care to take into account also the slight dependence of the resonance frequency on transistor width due to the variations of the gate and drain parasitic capacitances.

Let us now consider the circuit phase noise, quoted as Single Sideband to Carrier Ratio (SSCR) [24] in the following. Fig. 4 shows the dependence of the $1/f^2$ and the $1/f^3$ phase noise on the excess gain, $G_X$, evaluated at 10 MHz and 1 kHz, respectively. While at 10 MHz phase noise is dominated by contributions of the stationary noise source (i.e., the tank loss resistance) and of the cyclostationary white noise sources of the transistors, at 1-kHz their contributions are negligible, independently of the excess gain, thus making possible to isolate the term due to flicker noise up-conversion. Results using both BSIM4 and SPICE2 flicker noise model.

Fig. 4. Phase noise at 10-MHz (a) and 1-kHz (b) frequency offset as a function of the excess gain $G_X · 1/f^3$ phase noise at 1 kHz was evaluated adopting both BSIM4 and SPICE2 flicker noise model.

The contribution of each single noise source to the oscillator phase noise may be computed using the Impulse Sensitivity Function (ISF) [26]. For each noise generator the ISF can be defined as a dimensionless function, $\Gamma(\omega_0 t)$, that provides the asymptotic shift of the oscillator waveform phase caused by the injection of a current noise pulse with unity amplitude, at the generator terminals, when the oscillation phase is $\omega_0 t$. Under this frame, the phase fluctuation generated by a noise source, $i_n(t)$, can be written as

$$\phi(t) = \frac{1}{q_{m,\text{max}}} \int_{-\infty}^{t} \Gamma(\omega_0 \tau) \cdot i_n(\tau) \, d\tau,$$

where $q_{m,\text{max}}$ is the maximum charge displacement taking place during the oscillation period across the tank capacitor. Let us now refer to the circuit in Fig. 1. The noise injected by each switching transistor (e.g., the n-MOSFET at the bottom left) will affect the oscillator phase noise via $\Gamma_I(\omega_0 t)$, the latter being the ISF for a noise generator applied between its drain node and ground. On the other hand, the noise source in (4), $i_n(t)$, will be the cyclostationary noise generated by the dependence of the transistor flicker noise current on its periodically varying operating point.

Before deriving both functions by circuit simulations, some simplified arguments may help in highlighting some of their key features that will be then confirmed by numerical results. Let us assume that the output voltage waveform is given by $V(t) \approx A_1 \cos(\omega_0 t)$. In this case, the ISF of a noise source placed across the tank can be well approximated by a harmonic function in quadrature to the differential voltage [26], [27]

$$\Gamma_I(\omega_0 t) \approx \cos(\omega_0 t + \frac{\pi}{2}).$$

On the other hand, a noise current pulse injected by the noise generator across the n-MOSFET at the bottom left in Fig. 1 will cause some signal to flow through the tank. More precisely, since the tank capacitor behaves as a short-circuit and the impedances of the two branches ($M_3 - M_4$) are almost the same for any operating bias point along the oscillator cycle, about half of current pulse injected at the drain node is expected to flow through the tank. It follows that $\Gamma_D(\omega_0 t)$ should be well approximated by half the ISF across the tank, that is

$$\Gamma_D(\omega_0 t) \approx \frac{1}{2} \Gamma_I(\omega_0 t) \approx \frac{1}{2} \cos(\omega_0 t + \frac{\pi}{2}).$$

Fig. 5 shows the ISF derived by simulating the circuit in Fig. 1 for a particular value of the excess, $G_X = 3.3$ (see Appendix A for details about ISF simulation methodology). The comparison with the approximation given by (6) is good. However, some slight discrepancies may be noticed. First of all the simulated function is distorted. This result is not a surprise since any distortion of the output voltage waveform will also reflect on the ISF [28]. In addition it can be seen that the simulated ISF slightly leads the harmonic estimate given by (6). The same phase shift...
appears on $\Gamma_D(\omega_0 t)$ (not shown). A better approximation for the first harmonic of $\Gamma_D(\omega_0 t)$ can therefore be given by:

$$\Gamma_D(\omega_0 t) \approx \frac{1}{2} \cos \left( \omega_0 t + \frac{\pi}{2} + \varphi_e \right), \tag{7}$$

where $\varphi_e$ is a positive phase shift. This term is by far the most dominant one. However, in the following sections the impact of the higher order harmonics arising from distortion will be also addressed.

Let us now consider the noise generator, $i_n(t)$, in (4). As far as the transistor operates in small signal regime, noise processes are described by a power spectral density that depends on some device parameters. A simple model equation for the $1/f$ term is provided by [29] and adopted by the SPICE2 flicker noise model. According to this model the power spectral density of the $1/f$ current noise may be written as

$$S_{ix}(\omega) = \frac{K_F}{C_{ox}F_{MOS}^2} \frac{2\pi}{\omega}, \tag{8}$$

where $K_F$ is a process dependent constant, $I_{DS}$ the transistor channel current, $I_{MOS}$ the MOS channel length and the exponent $\alpha$ a constant typically ranging between 1 and 2. More refined models, as BSIM3 or BSIM4, are numerically implemented in circuit simulators. In all cases the power spectral density varies according to some device parameters whose value depends on the transistor bias point. In a large signal time-variant regime, these parameters are periodically modulated and the noise processes become cyclostationary. In numerical simulations cyclostationary processes are therefore described starting from a stationary noise and then considering the intermixing of the noise components arising from the large signal variations of the device parameters. The process is schematically depicted in Fig. 6. Referring to the flicker noise model given by (8), the noise components are considered as generated by the product [30]:

$$i_n(t) = x(t) \cdot m(t), \tag{9}$$

where

$$x(t) = x \cdot \cos(\omega_m t + \varphi_m) \tag{10}$$

is a noise tone with frequency $\omega_m/(2\pi)$ and phase $\varphi_m$ randomly distributed over the 0-2$\pi$ interval, while the amplitude, $x$, is

chosen to account for the stationary part of the unilateral power spectral density:

$$S_x(\omega) = \frac{K_F}{C_{ox}F_{MOS}^2} \frac{2\pi}{\omega}. \tag{11}$$

The periodic modulating function is instead given by

$$m(t) = |I_{DS}(t)|^{1/2}. \tag{12}$$

Even if this approach seems to work well to describe the behavior of cyclostationary white noise sources (shot or thermal noise) in linear time-variant circuits, there is a general belief that for long-term correlation noises, like flicker noise, this approach may be questionable [31]. Moreover, experimental measurements have demonstrated that stationary-based noise models, like SPICE2 and BSIM3, are not accurate enough in cyclostationary regime [32]–[34]. However, since the purpose of this paper is to grasp a physical insight of the up-conversion mechanism and to quantitatively justify the SpectreRF simulation results, we follow the procedure described above sticking to the simple model in (8).

Referring to the noise generator in Fig. 1, the contribution to the output phase arising from the low-frequency tone $x(t)$ can therefore be written as

$$\phi(t) = \frac{1}{q_{\text{max}}} \int_{-\infty}^{t} \Gamma_D(\omega_0 \tau) m(\tau) x(\tau) d\tau. \tag{13}$$

Note that the product $m(t) \cdot x(t)$ generates correlated noise components $i_n$ at offsets $k\omega_0 + \omega_m$ as schematically depicted in Fig. 7. These components translate via the ISF, $\Gamma_D(\omega_0 t)$, to phase noise.

Let us now try to gain a more quantitative insight into the outcome of (13). Since $I_{DS}(\tau)$ is a periodic function, $m(t)$ can be expanded in a Fourier series as

$$m(t) = \sum_{k=-\infty}^{\infty} m_k \cdot \cos(k\omega_0 t + \varphi_k), \tag{14}$$

with Eulerian components given by

$$m_k = \frac{1}{2\pi} \int_{0}^{2\pi} m(t) \cdot e^{-jk\omega_0 t} dt. \tag{15}$$
Taking for $\Gamma_D(\omega_n)$ the first dominant harmonic term given by (7), it turns out that the current tones significantly contributing to phase modulation are only those at $\omega_n \pm \omega_m$.

$$
\begin{align*}
I_n(\omega_n + \omega_m) &= \frac{z m_n}{2} e^{i(\omega_n^1 t + \varphi_n)} \\
I_n(\omega_n - \omega_m) &= \frac{z m_n}{2} e^{i(\omega_n^1 t - \varphi_n)}
\end{align*}
$$

(16)

These tones are multiplied by the first harmonic of $\Gamma_D(\omega_0 t)$,

$$
\Gamma_D e^{i(\nu/2 + \varphi_n)} \approx \frac{1}{2} e^{i(\nu/2 + \varphi_n)},
$$

(17)

and the integral in (13) leads to a phase modulation at a frequency $\omega_m$ given by

$$
\phi(\omega_m) \Rightarrow \frac{x \cdot m_1 \cdot e^{i\varphi_1}}{2|q_{\text{max}}\omega_m} \left[ e^{i(\omega_m^1 t - \varphi_1)} + e^{-i(\omega_m^1 t - \varphi_1)} \right].
$$

(18)

Taking $q_{\text{max}} \approx CA_1$, the amplitude of the phase modulation is therefore written as

$$
\phi \approx \frac{x \cdot m_1}{4CA_1\omega_m} \left| \sin \left( \omega_m^1 t - \varphi_1 \right) \right|.
$$

(19)

Since the Single Sideband to Carrier Ratio is given by $(\phi/2)^2$, it results:

$$
\text{SSCR}(\omega_m) \approx \frac{S_x}{2} \left( \frac{m_1 \sin \left( \varphi_1 - \omega_m^1 \right)}{4CA_1\omega_m} \right)^2.
$$

(20)

Considering the contribution of both PMOS and NMOS transistors, the power spectral density of the process $x(t)$ results:

$$
S_x(\omega_m) = 2 \left( \frac{K_p}{C_{ox} L_n^2} \frac{2\pi}{\omega_m} + \frac{K_p}{C_{e2} L_p^2} \frac{2\pi}{\omega_m} \right),
$$

(21)

where the initial factor of 2 derives from adding the uncorrelated 1/f noise contributions of the two transistors within each cross coupled pair. Thus, the overall phase noise can be written as

$$
\text{SSCR}(\omega_m) = \left( \frac{K_p}{C_{ox} L_n^2} + \frac{K_p}{C_{e2} L_p^2} \right) \left( \frac{2\pi}{\omega_m} \right) \cdot \left( \frac{m_1 \sin \left( \varphi_1 - \omega_m^1 \right)}{4CA_1\omega_m} \right)^2.
$$

(22)

Equations (21) and (22) assume that the modulating function $m(t)$ is the same for NMOS and PMOS transistors. The extension to a more general case can be easily derived. From a conceptual standpoint, (22) highlights that the 1/f noise up-conversion depends on the values of two key phase shifts:

- the phase $\varphi_x$ shifting the ISF first harmonic from being precisely in quadrature to the voltage waveform;
- the phase $\varphi_m(1)$ of the first harmonic of the modulating function, $m(t)$, with respect to the voltage waveform.

In the next section both phase shifts will be evaluated and the phase noise estimate given by (22) will be compared to simulation results.

**IV. CIRCUIT SIMULATIONS AND THEORY VALIDATION**

The close-in 1/f noise phase of the oscillator topology in Fig. 1 has been simulated for different excess gain $G_X$. On the same circuits the two phases $\varphi_x$ and $\varphi_m(1)$ have been evaluated through linear time variant SpectreRF simulations. The phase $\varphi_x$ has been estimated by periodic AC (PAC) analysis. The extraction relies on the following procedure. As a current tone is injected across the tank resonator at $\omega_0 = \omega_m$, the output voltage shows two correlated tones at $\omega_0 \pm \omega_m$. They are generated by a phase modulation that can be written as:

$$
\phi(f) = \frac{1}{q_{\text{max}}} \int_{-\infty}^{\infty} \Gamma_T(\omega_0 t) \cdot i_n \cos(\omega_m t + \varphi_1) dT 
$$

(23)

where, in the last step, $\Gamma_T(\omega_0 t)$ has been taken equal to $\cos(\omega_0 t + \pi/2 + \varphi_x)$. It follows that the phases of the output voltage tones are given by:

$$
\{ \varphi_V(\omega_0 + \omega_m) = \frac{\pi}{2} + \varphi_x \\
\varphi_V(\omega_0 - \omega_m) = \frac{\pi}{2} - \varphi_x \}
$$

(24)

More often, in the simulation environment, the output voltage $V(t)$ at the fundamental frequency turns out to have an initial phase, $\varphi_n$, with respect to the origin of the time scale, depending on the start-up dynamics. In this more general case the phases of the two output tones are given by:

$$
\{ \varphi_V(\omega_0 + \omega_m) = 2\varphi_0 + \frac{\pi}{2} + \varphi_x \\
\varphi_V(\omega_0 - \omega_m) = \frac{\pi}{2} - \varphi_x \}
$$

(25)

Therefore, $\varphi_x$ can be extracted from the phase of the output voltage tone at $\omega_0 = \omega_m$. Fig. 8 shows the dependence of $\varphi_x$ on the excess gain. By increasing $G_X$ using larger transistors, the phase $\varphi_x$ grows.

Regarding the phase $\varphi_m(1)$, it can be derived from a periodic steady state (PSS) simulation. Since the modulating function $m(t)$ is $|I_{DS}(t)|^{\alpha/2}$ (with $\alpha = 1$ in the considered technology) it can be evaluated using a Verilog-A block computing $|I_{DS}(t)|^{\alpha/2}$ starting from the simulated waveform of the current flowing through a transistor (e.g., $M_1$ in Fig. 1). Some care must be devoted to estimate the actual transistor channel current. In fact, the only accessible node is the transistor drain but the drain current also includes components due to drain-substrate (or drain-well) and gate-drain parasitic capacitances that cannot be neglected at 2-GHz frequency. To overcome this limitation the oscillator has been scaled in order to lower the operating frequency, by retaining the same quality factor, the same
Fig. 9. Phase of the first harmonics of $-I_{DS}(t)$, $|I_{DS}|$ and modulating function, $m(t)$. Triangles refer to the estimate of $-I_{DS}(t)$ first harmonic phase using (39).

Fig. 10. $1/f^2$ phase noise at 1-kHz offset. Dashed line and triangles refer to the analyses based on first harmonic (see (22)) and all harmonics of $\Gamma_P$, respectively. Diamonds refer to the rough estimate given by (51).

tank resistance and thus the same current waveform for the same transistor width. Since

$$Q = \frac{1}{\omega_R L} - \sqrt{\frac{C}{L}}$$  \hspace{1cm} (26)

scaling is performed by increasing the tank capacitance and inductance of the same factor. By lowering the operating frequency in the MHz range the capacitive components of the drain current become negligible thus making possible to get the time dependence of the channel current from the drain node. The procedure is accurate enough provided that phase delay due to finite gate resistance and finite cut-off frequency ($f_T$) of the transistors is negligible. In the case under investigation this is reasonable since $f_T$ is larger than 300 GHz and the oscillation frequency is 2 GHz, while the gate delay was estimated to be less than 200 fs, independently of the considered transistor width.

Fig. 9 shows the phases of $1^\text{st}$ harmonics of $-I_{DS}(t)$, $|I_{DS}|$ and modulating function, $|I_{DS}|^{1/2}$, versus the oscillator excess gain. The reason to consider $-I_{DS}(t)$, rather than $I_{DS}(t)$, is related to the relationship between tank and transistors’ current. In fact, referring to Fig. 1, it is

$$I = +I_{DS,3} - I_{DS,1}$$  \hspace{1cm} (27)

where $I_{DS,3}$ and $I_{DS,1}$ are the currents flowing in the PMOS $M_3$ and in the NMOS $M_1$, respectively. Clearly, the phase of $-I_{DS,1}(\omega_0)$, which is plotted in Fig. 9, is negative since $I_{DS,1}(\omega_0)$ lags the voltage $v_{ds}$ and thus $-I_{DS,1}(\omega_0)$ lags the differential output $V = v_{ds} - v_{ds}$. An excess gain greater than 4, corresponding to NMOS transistor width larger than 65 $\mu m$, the oscillation amplitude is higher than the power supply, thus making the transistor current $I_{DS}$ to be negative for a fraction of the period (the current flows from source to drain in the NMOS $M_1$). This is why the phase of first harmonic of $|I_{DS}|$ deviates from the phase of $-I_{DS}(\omega_0)$ while the power raising operator has a minor effect.

Fig. 10 (dashed line) shows the phase noise at 1-kHz offset estimated by using (22), taking into account the contributions of all devices. The $1/f$ noise coefficients are $K_p = 10^{-27}$ A/Hz and $K_f = 2 \cdot 10^{-28}$ A/Hz for NMOS and PMOS transistors, respectively, and $C_{ox} \approx 19$ fF/\mu m$. The result is in good agreement, within 3 dB, with the values obtained from a SpectreRF periodic noise (PNOISE) simulation. Note that the maximum discrepancy occurs for large transistor widths where the contribution of the higher order harmonics of the ISF is expected to be larger.

In order to refine the estimate, $\Gamma_P$ can be written as [35]

$$\Gamma_P(\omega) = \sum_{k=0}^{+\infty} \Gamma_{k} \cos \left( k \omega_0 + \varphi_k^{(k)} \right).$$  \hspace{1cm} (28)

Thus, a slow noise tone $x(t)$ as in (10) generates a phase modulation of the differential output voltage with an amplitude equal to

$$\phi \triangleq \frac{x \cdot m_0}{CA_{\omega_m}} \Gamma_0 \cos \left( \varphi_m^{(0)} - \varphi_T^{(0)} \right) + \sum_{k=1}^{+\infty} \frac{x \cdot m_k}{2CA_{\omega_m}} \Gamma_k \cos \left( \varphi_m^{(k)} - \varphi_T^{(k)} \right)$$  \hspace{1cm} (29)

where $A$ is the differential oscillation amplitude, which, for a non-harmonic output voltage, differs from $A_1$ (see (19)). Equation (29) reduces to (19) if $\Gamma_P$ can be approximated by its first harmonic term and taking $\varphi_k^{(1)} = \pi/2 + \varphi_1$. From (29) the $1/f^3$ phase noise can be expressed as:

$$SSCR(\omega_m) = \frac{S_z(\omega_m)}{2} \left[ \frac{m_0 \Gamma_0}{CA_{\omega_m}} \cos \left( \varphi_m^{(0)} - \varphi_T^{(0)} \right) + \sum_{k=1}^{+\infty} \frac{m_k \Gamma_k}{2CA_{\omega_m}} \cos \left( \varphi_m^{(k)} - \varphi_T^{(k)} \right) \right]^2$$  \hspace{1cm} (30)

where $S_z(\omega_m)$ is the power spectral density of the noise stationary process $x(t)$, as in (21). The phases $\varphi_m^{(k)}$ and $\varphi_T^{(k)}$ can be evaluated by means of the same simulation methods adopted to estimate the first harmonic terms of $\Gamma_P$ and of the modulating function.

Fig. 10 (triangles) shows the $1/f^3$ phase noise estimation taking into account all the $\Gamma_P$ harmonic terms and the contribution of both NMOS and PMOS pairs. The discrepancy is now reduced to less than 0.5 dB with respect to the simulated phase noise at 1-kHz offset. Fig. 11 shows the contribution to the output noise due to the most relevant terms in the brackets of (30). In addition to the dominant term generated by the first harmonic of $\Gamma_P$, the contribution due to the third harmonic is important. Its rising dependence is not due to the phase $\varphi_m^{(3)} - \varphi_T^{(3)}$, which is always around zero degrees thus giving
V. LINKING UP-CONVERSION TO DISTORTION

A Van der Pol oscillator can be schematically represented as an LC tank with a loss resistor R and a third order non-linear transconductor, \( I(V) = g_{m} V - g_{3} V^{3} \), providing the energy to balance, at steady state, its losses (Fig. 12).

In the circuit in Fig. 1 the transconductor is synthesized by a double cross-coupled pair. Its non-linear \( I-V \) curves are reported in Fig. 13 for two different values of the transistor width together with the fitting third order polynomial curves. The first coefficient, \( g_{1} \), is the small signal transconductance, that is \( g_{1} = g_{m} \). It can be also shown that \( g_{3} = g_{m}/V_{D}^{2} \).

In order to simplify the analysis let us consider only the first and third harmonic of output voltage. The voltage waveform will therefore be taken as:

\[
V(t) \simeq A_{1} \cos(\omega_{0}t) - A_{3} \sin(3\omega_{0}t). \tag{31}
\]

where \( A_{1} \) and \( A_{3} \) are both positive. The phase relationship between first and third voltage harmonic will be justified in the following. Starting from (31) the output current of the transconductor at the fundamental frequency, \( I(\omega_{0}) \), is computed as

\[
I_{1} \approx g_{1} A_{1} - \frac{3}{4} g_{3} A_{3}^{2} \cos(\omega_{0}t) + \frac{3}{4} g_{3} A_{3}^{2} A_{3} \sin(3\omega_{0}t),
\]

where \( I_{1}^{I} \) and \( I_{1}^{Q} \) are the in-phase and in-quadrature terms, respectively, of the first harmonic of the current. Due to the quadrature component a phase shift \( \varphi \) is generated between \( I_{1} \) and the first harmonic of the voltage \( V_{1} \). More precisely \( I_{1}^{Q} \) lags with respect to \( V_{1} \). Since the ratio \( V_{1}/I_{1} \) is the tank impedance evaluated at the oscillation frequency, (32) suggests that oscillation frequency deviates from resonance, being \( \omega_{0} < \omega_{R} = 1/\sqrt{LC} \). By equating \( I_{1} \) resulting from (32) to the current flowing into the tank, both for the in-phase part and for the in-quadrature term, the amplitude \( A_{1} \) and the shift of the oscillation frequency respect to \( \omega_{R} \) can be derived as:

\[
A_{1} = \sqrt{\frac{4}{3} \left( \frac{g_{1}}{g_{3}} - \frac{1}{2} \right) g_{3} \omega_{0}}, \quad \Delta\omega_{0} \approx -\frac{3}{8} \frac{g_{3} A_{1} A_{3}}{C}. \tag{33}
\]

Note that this is a simplified version of the rigorous analysis already performed by Groszkowski [20]. In order to derive a closed form expression for \( A_{3} \), the transconductor output current at \( 3\omega_{0} \) is needed. By taking \( A_{3} \ll A_{1} \), this term turns out to be:

\[
I_{3} \simeq -\frac{1}{4} g_{3} A_{1}^{3} \cos(3\omega_{0}t). \tag{35}
\]

Equation (35) justifies the approximation of the output voltage given in (31). Since the third harmonic of the current flows mainly into the tank capacitor, whose impedance is \( 1/(j3\omega_{0}C) \),
it results in a third harmonic of the output voltage in quadrature given by

\[ -A_3 \sin(3\omega_0 t) \equiv - \frac{g_2 A_1^2}{12\omega_0 C} \sin(3\omega_0 t). \]  

(36)

More detailed balance equations are given in Appendix B, where it turns out that \( A_2 \) is better approximated by a slightly different value as in (59). It follows that the total harmonic distortion of the oscillator is:

\[ \text{THD} \approx \frac{A_2}{A_1} \approx \frac{1}{8} \frac{(G_X - 1)}{Q}. \]  

(37)

Equation (37) suggests that the harmonic distortion is only function of excess gain and quality factor. From (34) and from the amplitude of the third harmonic in (36), the shift of the oscillation frequency results

\[ \Delta \omega \approx -\left( \frac{G_X - 1}{Q} \right)^2 \omega_R. \]  

(38)

Moreover, from (32), the phase of the first harmonic of the current \( I \) is given by:

\[ \phi \approx -arctg \left( \frac{I_0^1}{I_0} \right) \approx -arctg \left[ \frac{(G_X - 1)^2}{8Q} \right]. \]  

(39)

The compact equations derived above have been compared to the steady-state solution of the behavioral model in Fig. 12 obtained by a PSS simulation in a Cadence environment. Figs. 14–16 show the oscillation frequency, the amplitudes and the phase \( \phi \) of the first harmonic of the current \( I(\omega_0) \) as a function of the excess gain. The small signal transconductance, \( g_1 \), has been changed from 10 mA/V to 60 mA/V, while the other model parameters have been taken as: \( L = 1.2 \, \text{nH}, C = 5.3 \, \text{pF}, R = 150 \, \Omega \) and \( g_3 = 3 \, \text{mA/V}. \) Note that although the analysis was carried out only considering the first and the third harmonic of the output voltage, the accuracy is very good even up to an excess gain of 9.

The next step is to link the tank current to the transistor current and then to the modulating function. To this purpose note that the current entering the noise model is the MOS channel current. Since NMOS and PMOS transistors are sized to be electrically equivalent and the phase of the transistor current at \( \omega_0 \) is only function of the voltage distortion, while other sources of delay are negligible, it results

\[ I_{DS,1}(\omega_0) \approx -I_{DS,3}(\omega_0). \]  

(40)

and then from (27)

\[ I(\omega_0) \approx 2I_{DS,1}(\omega_0) \approx 2I_{DS}(\omega_0). \]  

(41)

Equation (41) states that the phase of the first harmonic of \( -I_{DS}(t) \) can be approximated by \( \phi \). Clearly, it is not possible to find a closed-form expression of \( \varphi^{(1)}_{\text{in}} \), since the modulating function \( I_{DS}(t) \) is related to MOS channel current by two non-linear operators: absolute value and power raising. However, for low excess gain, i.e., for quasi-harmonic current, \( \varphi^{(1)}_{\text{in}} \) is well approximated by the phase of \( -I_{DS}(\omega_0) \), as evident in Fig. 9. The following analysis will therefore be limited to this range, that is adequate for most of the applications.

Let us now derive a compact expression for the phase \( \varphi \), which plays a key role in the up-conversion process (see (22)). The first step will be to demonstrate the following statement: if a small current tone at the oscillation frequency of the unperturbed system, \( \omega_0 \), is injected into the tank with an initial phase \( \varphi_n \), a particular phase, \( \varphi_{\text{in}} \), exists so that the output voltage does not exhibit phase variation (while the oscillation frequency remains \( \omega_0 \) whatever the phase \( \varphi_n \)) (36), [37]); this phase is equal to the phase shift, \( \varphi_{\text{in}} \), of the first harmonic of the ISF.

\[ ^1 \text{In a real oscillator the parameter} \, g_3 \, \text{is function of} \, g_1, \, \text{while it was kept constant in the behavioral simulations. However, this does not invalidate the obtained results, since simulations confirm that THD and current phase are only function of the excess gain and thus of} \, g_1. \]
Let us consider a Van der Pol oscillator whose steady-state solution has a first harmonic given by $V_1 = A_1 \cos(\omega_0 t)$. Therefore, the first harmonic of the ISF may be taken as:

$$\Gamma(\omega_n t) \approx \cos\left(\omega_0 t + \frac{\pi}{2} + \varphi_e\right).$$ \hspace{1cm} (42)

Let us now inject across the tank a current tone at the oscillation frequency of the unperturbed system, $i_n \cos(\omega_0 t + \varphi_n)$ (Fig. 12). The first harmonic of the output voltage is perturbed both in amplitude and phase, thus becoming $A(t) \cos(\omega_0 t + \phi(t))$ where $A(t) = A_1 + \Delta A_1$ and $\phi(t) = \phi$ at steady state. Since the tone is injected at an offset $\omega_m t = 0$ from the carrier, the condition $\phi(t) \ll \omega_m t$ no longer valid as well as (4). This is a particular case of injection locking condition and the output voltage phase has to be evaluated solving the following differential equation [37]:

$$\phi(t) = \frac{1}{q_{\text{max}}} \Gamma(\omega_n t + \varphi(t)) \cdot i_n \cos(\omega_n t + \varphi_n) \approx$$

$$\approx \frac{i_n}{2q_{\text{max}}} \cos\left[\frac{\pi}{2} + \varphi_e + \phi(t) - \varphi_n\right].$$ \hspace{1cm} (43)

Since the output phase has to be constant, i.e., $\phi(t) = 0$, it follows that

$$\phi = \varphi_n - \varphi_e.$$ \hspace{1cm} (44)

If $\varphi_n = \varphi_e$, the phase of the first harmonic of the output voltage is left unperturbed. In this case, only amplitude variation takes place and the output waveform can be written as

$$V_1 = (A_1 + \Delta A_1) \cos(\omega_0 t).$$ \hspace{1cm} (45)

By writing the proper harmonic balance equations for the voltage and the current components, $\Delta A_1$ and $\varphi_n = \varphi_e$ can be linked to the parameters of the Van der Pol oscillator. The results are detailed derived in Appendix B. They lead to

$$\Delta A_1 \approx \frac{R_i n \cos(\varphi_n)}{2(G_X - 1)},$$ \hspace{1cm} (46)

$$\varphi_e \approx \arctan\left[\frac{G_X - 1}{4Q}\right] \approx \frac{G_X - 1}{4Q}. $$ \hspace{1cm} (47)

Equation (46) represents the effect of the non-linearity on the amplitude variation. The larger the excess gain the larger the non-linearity and the smaller the amplitude variation due to the injected current tone. Equation (47) is the excess phase of ISF first harmonic. Note that the term $(G_X - 1)/Q$ is equal to $\epsilon/4$, $\epsilon$ being the Van der Pol parameter [38]. This is the reason why the phase shift has been denoted as $\varphi_e$. Equation (47) suggests that the phase $\varphi_e$ is only function of the excess gain, $G_X$, and of the tank quality factor, Q. Fig. 17 compares the excess phase evaluated through SpectreRF periodic AC simulations on the behavioral oscillator of Fig. 12 with the same method described in Section IV and the result given by (47). The estimates and the simulation results are in good agreement and only for high excess gain the discrepancy becomes evident. In the same way, (39) and (47) can be used to estimate the transistor current phase and $\varphi_e$ for the circuit in Fig. 1. Figs. 8 and 9 show that the model provides an accurate prediction of circuit performance.

### VI. DISCUSSIONS

In Section III it was demonstrated that $1/f^3$ close-in phase noise is linked to the phases $\varphi_e$ and $\varphi_m^{(1)}$, both depending on the amount of non-linearity of the oscillator. Equations (20), (39) and (47) highlight that flicker noise up-conversion is reduced in case of:

1) low excess gain, i.e., low distortion of the voltage output;
2) high tank quality factor (for a fixed excess gain), i.e., strong attenuation of high order voltage harmonics.

Under these conditions the transistor flicker noise generates only amplitude noise since $\varphi_e \approx 0$ and $\varphi_m^{(1)} \approx 0$. In practice distortion cannot be neglected since some excess gain is needed to guarantee the oscillation start-up and the quality factor of fully integrated resonators is not large enough to filter out high order harmonics, causing $\varphi_e > 0$ and $\varphi_m^{(1)} < 0$.

It’s worth nothing that up-conversion arises since noise current tones modulate the amplitude of the voltage harmonics, changing the oscillation frequency as in (34) and thus causing phase noise. This induced phase modulation increases with the excess gain as can be qualitatively explained with the help of the phasor diagram in Fig. 18 representing the first harmonic of the voltage waveform, $V_1$, and the first harmonic of the transconductor current, $I_1$. In the more general case, due to the Groszkowski effect, the current lags the voltage by a phase $\delta$. The oscillation frequency is shifted from the tank resonance and $\theta$ is also the phase of the tank impedance at that frequency.

Let us now consider the injection of two current noise tones at an offset $+\omega_m$ from the fundamental frequency with the same phase of the current $I_1$ (Fig. 18(a)). This situation represents the conversion of a slow frequency tone of the flicker noise into correlated current noise around the fundamental frequency due to the cyclostationary regime of the transistors, when $\varphi^{(1)}$ is well approximated by the phase of the current $I_1$, $\theta$. These two tones generate a modulation of the current in-phase to the output voltage, i.e., by $i_2 = 2i_n \cos(\theta) \cos(\omega_m t)$. The in-quadrature modulation is instead (Fig. 18(b)). However, this is not the steady state condition. It should be taken into account that, due to oscillator non-linearities, the system reacts to reduce the amplitude modulation of the voltage $V_1$,
By taking $A_1 \simeq V_{DD}$ and increasing the excess gain from 1.5 to 4, the phase noise of the reference oscillator is expected to increase from $-62.4$ dBc/Hz to $-40.85$ dBc/Hz. This estimate matches very closely the simulation results (see rough estimate in Fig. 10). Equation (51) also suggests that $1/f^3$ phase noise can be reduced by decreasing the excess gain and/or improving the quality factor. Note that the phase noise is proportional to $1/Q^3$ even if both $\varphi_i$ and $\varphi_m^{(1)}$ are proportional to $1/Q$, thus suggesting a $1/Q^3$ dependence of phase noise. The reason for this additional dependence on $1/Q$ is that also the transistor channel current depends on the tank loss resistance $R = \omega_0 L Q$. The larger the tank quality factor, the smaller the current and also the flicker noise intensity.

Finally, the reader may wonder whether this flicker noise up-conversion mechanism is dominant or not in a real oscillator that makes use of varactors to tune the oscillation frequency. The AM-PM conversion due to non-linear capacitances may be quantified by using a conversion coefficient, $K_{AM}^{FM}$, representing the sensitivity of the oscillation frequency to amplitude variations of the output voltage [3]–[5]. The phase noise resulting from this AM-PM conversion mechanism can be written as

$$SSCR (\omega_m) = \frac{K_{AM}^{2} - F, S_{AM}^{2} (\omega_m)}{2 \omega_m^2}$$

where $S_{AM} (\omega_m)$ is the power spectral density of the amplitude noise at an offset $\omega_m$ from the carrier. This amplitude noise is mainly due to current flicker noise that is up-converted around the fundamental frequency by the time-varying operating point of the transistors. The value of the conversion coefficient can be estimated for both CMOS and bipolar varactors following the arguments in [3]–[5]. In both cases the coefficient is proportional to the oscillator gain, $K_{VCO} = \partial \omega / \partial V_{TUNE}$, where $V_{TUNE}$ is the control voltage of the VCO. It turns out that the AM-PM conversion due to varactors is expected to be the dominant for large VCO gains. Fig. 19 shows a Van der Pol VCO using diode varactors to tune the frequency. The $C_S$ capacitors are adopted in order to remove another phase noise generation mechanism, i.e., the common mode voltage to phase modulation effect (CM-PM, see [5]). The VCO gain is changed by varying the voltage $V_{TUNE}$. For a fair comparison the oscillation frequency is always kept at 2 GHz. Fig. 20 shows the simulated phase noise at 1-kHz offset for two different values of VCO gain as a function of the excess gain. The up-converted flicker noise is now determined by both AM-PM conversion due to varactors and harmonic content modulation. The former contribution is dominant for small values of $K_X$ and large oscillator gains, and it decreases for high excess gain since amplitude noise is reduced. Moreover, the presence of non-linear capacitances in the tank slightly increases the voltage harmonic distortion causing a small increment of $1/f^3$ phase noise also for large excess gains (see Fig. 20). However, while the varactor contribution may be reduced by minimizing the varactor sensitivity ($K_{VCO}$) and using a bank of digitally switched capacitors not to impair the tuning range, the contribution arising from the transconductor non-linearity is unavoidable since it is intrinsically related to the non-linear nature of oscillators.
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Fig. 19. Van der Pol VCO with variable capacitors.

Fig. 20. Phase noise at 1-kHz offset for two different values of the oscillator gain ($K_{VOC}$) as function of the excess gain.

VII. CONCLUSIONS

In this work, the up-conversion of flicker noise due to voltage harmonic content variation in Van der Pol oscillator has been discussed and quantitatively assessed. The phase noise up-conversion mechanism has been quantified and linked to the oscillator non-linearity. Design equations have been derived and compared to simulation results adopting a stationary-based flicker noise model.

APPENDIX A

The appendix describes the simulation method implemented in a Cadence environment to compute the ISF of a current noise source. The traditional approach adopts transient simulations and follows the ISF definition as introduced by Hajimiri et al. [26]. A short current pulse, with amplitude small enough to just perturb the oscillation steady-state, is applied at the nodes where the current noise source is injecting its noise. When the initial transient is over, the variation of the oscillation phase is read. Repeating the procedure by changing the pulse injection time over the oscillation period, the periodic ISF is computed. Despite its simplicity, this procedure is time-consuming. In addition, since the injected charge has to be small, the results may be affected by numerical noise. The alternative adopted in this work is to derive the ISF by using PAC (periodic AC analysis) or PXF (periodic transfer function) SpectreRF simulations.

Let us consider the ISF to be evaluated

$$\Gamma \left( \omega_n \right) = \sum_{k=0}^{+\infty} \Gamma_k \cos \left( k \omega_n + \varphi_k \right).$$

where $\Gamma_k$ and $\varphi_k$ are the magnitude and the phase of the $k$-th harmonic term, respectively. Let us also write the output voltage component at the fundamental frequency as $V(t) = A_1 \cos(\omega_0 t + \phi_0)$. In this frame, a small current tone at an offset $\omega_m$ around the $k$-th harmonic of the output voltage, $i_n \cos [(k \omega_m t + \psi_0)]$, causes a phase modulation at $\omega_m$ of the output voltage at the fundamental frequency. In fact, the corresponding output phase results

$$\Delta \phi = \frac{i_n \Gamma_k}{2 \Omega_{\text{peak}} \omega_m} \sin \left( \omega_m t + \varphi_k \right) = \Delta \phi \sin \left( \omega_m t + \varphi_k \right).$$

Note that only the $k$-th harmonic term of the ISF contributes to determine the output phase $\phi(t)$. Thus, the output voltage at the fundamental frequency, $V(t) = A_1 \cos(\omega_0 t + \phi_0 + \phi(t))$, shows two correlated terms at $\omega_0 \pm \omega_m$, that is

$$V(t) \approx A_1 \cos(\omega_0 t + \phi_0) + - \frac{A_1 \Delta \phi}{2} \sin \left( \omega_0 - \psi_0 \right).$$

The magnitude and the phase of the resulting voltage tone at $\omega_m$ are

$$|V(\omega_n - \omega_m)| = A_1 \frac{\Delta \phi}{2} = \frac{i_n \Gamma_k}{4 C \omega_m},$$

$$\angle V(\omega_n - \omega_m) = \pi + \phi_0 - \psi_0.$$

These equations pose the basis of the numerical methodology to derive the ISF function by PAC analysis. A current tone at 1-kHz offset from the $k$-th harmonic of the oscillation frequency is injected at the nodes of interest. PAC simulation is performed and $\Gamma_k$ and $\varphi_k$ can be easily derived from the resulting magnitude and phase of the output voltage at $\omega_0 - \omega_m$ using (56). In the above mentioned equations $\varphi_0$ is the phase of the output voltage and can be estimated through a PSS (Periodic Steady State) simulation, while $C$ is the total tank capacitance. Finally, note that the PAC analysis is a linear time-variant analysis. Thus, a current tone of 1 A can be injected without altering the oscillator behavior but simplifying the evaluation of $\Gamma_k$ and $\varphi_k$. Usually, in a LC tank oscillator, the evaluation of the first 5-6 terms of the ISF is enough to get a good accuracy of the phase noise estimate.

APPENDIX B

This appendix describes the mathematical derivation of the excess phase, $\varphi_e$, for a third order Van der Pol oscillator. Let us suppose that the oscillator is unperturbed and the output voltage

\begin{align*}
\varphi_e &= \frac{1}{i_n \Gamma_k} \int_{-\infty}^{t} \Gamma(t) \cdot \sin \left( k \omega_m t + \psi_0 \right) dt \equiv \frac{i_n \Gamma_k}{2 \Omega_{\text{peak}} \omega_m} \sin \left( \omega_m t + \varphi_k \right) \\
&= \Delta \phi \sin \left( \omega_m t + \varphi_k \right).
\end{align*}

\begin{align*}
\Delta \phi &= \frac{i_n \Gamma_k}{2 \Omega_{\text{peak}} \omega_m} \\
\psi_0 &= \frac{i_n \Gamma_k}{4 C \omega_m}.
\end{align*}

\begin{align*}
\angle V(\omega_n - \omega_m) &= \pi + \phi_0 - \psi_0.
\end{align*}
is as in (31). Balancing both the in-phase and in-quadrature currents of the resonant tank and of the transconductor at the fundamental frequency, we get

\[
\begin{align*}
&g_1 A_1 - \frac{3}{4} g_3 A_1^3 = \frac{A_1}{R} \\
&\frac{3}{4} g_3 A_1^2 A_3 = \left( \frac{1}{\omega_0 C} \right) A_1
\end{align*}
\]  
\tag{57}
\]

\[
A_3 \text{ being the amplitude of the third harmonic voltage that can be evaluated balancing the current at } 3\omega_0
\]
\[
-\frac{1}{4} g_3 A_1^3 = -3\omega_0 C A_3 + \frac{A_3}{3\omega_0 L} 
\tag{58}
\]

It results
\[
A_3 \approx \frac{3}{32} \frac{g_3}{\omega_0 C} A_1^2. 
\tag{59}
\]

Note that (59) is a refinement of (36). Now, let us perturb the oscillator with a current tone \(i_n \cos(\omega_I t + \varphi_n)\) and suppose that it perturbs only the first harmonic amplitude \(A_1\) (and consequently \(A_3\)) but not the phase, as discussed in Section V. The solution of the system for the first harmonic would be \(V_1 = (A_1 + \Delta A_1) \cos(\omega_0 t)\) as in (45). Balancing again in-phase and in-quadrature currents at the fundamental frequency and taking into account (59), it results
\[
\begin{align*}
g_1 (A_1 + \Delta A_1) - \frac{3}{4} g_3 (A_1 + \Delta A_1)^3 + i_n \cos \varphi_n = A_1 + \Delta A_1 \\
\frac{9g_3}{128\omega_0 C} (A_1 + \Delta A_1)^2 - i_n \sin \varphi_n
\end{align*}
\tag{60}
\]

Since we are dealing with a small current tone, it is possible to linearize (60) and derive
\[
\begin{align*}
g_1 (A_1 + \Delta A_1) - \frac{3}{4} g_3 (A_1^3 + 3A_1^2 \Delta A_1) + i_n \cos \varphi_n = A_1 + \Delta A_1 \\
\frac{9g_3}{128\omega_0 C} (A_1^2 + 5A_1 \Delta A_1) - i_n \sin \varphi_n
\end{align*}
\tag{61}
\]

Removal from (61) the unperturbed part given by (57), it turns out
\[
\begin{align*}
g_1 \Delta A_1 - \frac{3}{4} g_3 A_1^2 \Delta A_1 + i_n \cos \varphi_n - \Delta A_1 R \\
\frac{45g_3}{128\omega_0 C} A_1^2 \Delta A_1 - i_n \sin \varphi_n = \frac{1}{\omega_0 L} \Delta A_1
\end{align*}
\tag{62}
\]

Finally, from (62) the expression for \(\varphi_n\) and \(\Delta A_1\) can be derived as in (46) and (47).

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