On Portfolio Optimization: 
Imposing the Right Constraints

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Abstract

We develop a shrinkage theory based framework for determining optimal portfolio weight constraints for minimum-variance portfolios in the presence of parameter uncertainty. We impose the set of constraints that yields the optimal trade-off between sampling error reduction and bias for the variance-covariance matrix. Our results show that our approach yields significantly lower variances and consistently higher Sharpe ratios than the equally or value weighted portfolio strategies. Moreover, we observe that our constrained minimum-variance portfolio strategy improves the popular short-sale constrained and factor model based minimum-variance portfolio on most data sets in terms of lower variance and higher Sharpe ratio.

JEL Classification: G11

Key words: Portfolio choice, optimal portfolio weights, shrinkage, mean squared error

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Abstract

We develop a shrinkage theory based framework for determining optimal portfolio weight constraints for minimum-variance portfolios in the presence of parameter uncertainty. We impose the set of constraints that yields the optimal trade-off between sampling error reduction and bias for the variance-covariance matrix. Our empirical results for five data sets show that our approach yields significantly lower variances and consistently higher Sharpe ratios than the equally or value weighted portfolio strategies. Moreover, we observe that our constrained minimum-variance portfolio strategy improves the popular short-sale constrained and factor model based minimum-variance portfolio on most data sets in terms of lower variance and higher Sharpe ratio.

JEL Classification: G11, G17

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1 Introduction

Since the foundation of modern portfolio theory by Markowitz (1952), the development of new portfolio strategies has become a horserace-like challenge among researchers. The sobering finding that theoretically optimal, utility maximizing portfolios perform poorly out-of-sample\(^1\) can be attributed to the error prone estimation of expected returns, which leads to unbalanced optimization results. This result directed researchers’ attention to the minimum-variance portfolio, the only portfolio on the efficient frontier that simply requires the variance-covariance matrix as input parameter for the optimization. For instance, Merton (1980), Jorion (1985), and Nelson (1992) remark that variance-covariance estimates are relatively stable over time and can, hence, be predicted more reliably than expected returns. Nevertheless, recently DeMiguel et al. (2009a) have argued that no single portfolio strategy from the existing portfolio selection literature outperforms the 1/N strategy consistently in terms of out-of-sample Sharpe ratio. In a related paper, DeMiguel et al. (2009b) propose a class of partial minimum-variance portfolios that accomplish this goal.\(^2\) However, the major drawback of this approach is that it yields a relatively high turnover, thus making its practical implementation difficult and costly.

Our paper is closely related to DeMiguel et al. (2009a and 2009b). We develop a portfolio strategy that achieves both objectives: it outperforms the 1/N strategy in terms of a higher out-of-sample Sharpe ratio while, at the same time, yielding a turnover that does not overly hamper the practical implementation of this portfolio strategy. Hence, we provide an alternative to DeMiguel et al. (2009a) that is on the one hand theoretically sound and on the other hand particularly appealing to practitioners. Specifically, we pro-

\(^1\)See Frost and Savarino (1986), Jorion (1986), Michaud (1989), and Black and Litterman (1992), among others.
\(^2\)Chevrier and McCulloch (2008) as well as Tu and Zhou (2009a) develop Bayesian portfolio selection frameworks that seem to deliver higher out-of-sample Sharpe ratios compared to the 1/N benchmark. Tu and Zhou (2009b) also propose a set of portfolio strategies which combine the 1/N strategy with several sophisticated portfolio selection approaches. Based on the Sharpe ratio, the results suggest a superior out-of-sample performance compared with the 1/N strategy on many data sets. Critically, however, robust statistical inference is not provided in any of these papers.
pose a minimum-variance portfolio strategy with flexible upper and lower portfolio weight constraints. Incorporating these constraints into the portfolio optimization process trades off the reduction of sampling error and loss of sample information (Jagannathan and Ma (2003)). On the one hand, weight constraints assure that the optimal portfolio weights are not too heavily driven by the sampling error inherent in historical data, which often leads to highly concentrated portfolios. On the other, portfolio weight constraints cause a misspecification of the optimization problem as the resulting portfolio weights are less driven by potentially useful sample information (Green and Hollifield (1992)). Consequently, incorporating portfolio weight constraints into the portfolio optimization problem is promising if the input parameters are error prone.

Specifically, we investigate how much structure, i.e. which portfolio weight constraints, yield an optimal trade-off between the reduction of sampling error and loss of sample information. Using this shrinkage theory based framework, we introduce portfolio weight constraints which depend on the error inherent in the empirical variance-covariance matrix estimate. In particular, we impose the set of upper and/or lower portfolio weight constraints that minimizes the mean squared error (MSE) of the variance-covariance matrix. The latter serves as a loss function quantifying the trade-off between the reduction of sampling error and loss of sample information.

The empirical results we obtain show that our constrained minimum-variance portfolio strategy generates a consistently higher Sharpe ratio than the $1/N$ strategy. At the same time, the average turnover of our proposed portfolio strategy is lower than in DeMiguel et al. (2009b), which is, to the best of our knowledge, the only portfolio strategy to date that significantly outperforms the $1/N$ benchmark. Crucially, our approach also outperforms the value weighted portfolio and existing minimum-variance portfolio strategies with a static structure on the variance-covariance matrix. In addition, our strategy outperforms the pop-

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3See Green and Hollifield (1992), Chopra (1993), and Chopra and Ziemba (1993) for evidence concerning this point.
ular short-sale constrained and factor model based minimum-variance portfolio in terms of variance and Sharpe ratio on most data sets used in the empirical analyses. Finally, our constrained minimum-variance approach achieves results comparable to those obtained with the single factor shrinkage approach introduced by Ledoit and Wolf (2003).

Concerning the importance of portfolio weight constraints, we observe that imposing flexible upper or lower weight constraints results in an equally effective risk reduction. The risk adjusted performance of both portfolios is equally good. Additionally, the simultaneous imposition of flexible upper and lower portfolio weight constraints leads to a further reduction of risk compared to the imposition of just one flexible (upper or lower) portfolio weight constraint. This is particularly pronounced in cases where the number of degrees of freedom for the variance-covariance matrix estimation is low.

In addition to providing an alternative to DeMiguel et al. (2009a), our study is related to, and contributes to, two lines of literature. First, we extend previous work on constrained portfolio optimization in the presence of arbitrarily chosen upper and lower portfolio weight constraints (Frost and Savarino (1988), Grauer and Shen (2000), Jagannathan and Ma (2003)). Contrary to these studies, however, we postulate a framework which is based on shrinkage estimation theory for the determination of optimal – in the sense of MSE minimizing – portfolio weight constraints. We thereby bridge the gap between standard shrinkage estimation of the variance-covariance matrix and the implementation of portfolio weight constraints. Specifically, we provide a framework that grants flexibility to portfolio weight constraints, which are the most important portfolio optimization parameters in practice.

Second, our approach represents a new approach to shrinkage estimation of the variance-covariance matrix. In contrast to the shrinkage estimator developed by Ledoit and Wolf (2003), our approach considers any potential structure on the empirical variance-covariance matrix estimate and takes this into account when determining the optimal level of structure to be imposed. This property guarantees the local optimality of our approach. It is also of particular interest to portfolio managers who face regulatory boundaries and/or
investment rules such that they are not allowed to hold an unconstrained minimum-variance portfolio. This may be due to limited short-sale possibilities, maximum holding constraints, etc.\textsuperscript{4} Furthermore, imposing structure via portfolio weight constraints has a more intuitive interpretation than standard shrinkage approaches, which combine the empirical variance-covariance matrix with a shrinkage target.

The remainder of the paper is organized as follows. Section two outlines our methodology and data. Section three contains the empirical results. Finally, section four concludes.

2 Methodology and Data

This section is devoted to a description of the methodology and the data we use in the empirical tests. First, we give a detailed illustration of our proposed approach to determining optimal portfolio weight constraints. This is followed by a brief review of the portfolio strategies to which we benchmark our portfolio optimization strategy. We then describe the rolling sample portfolio optimization setup we use throughout the paper as well as the portfolio performance metrics and inference used to evaluate the performance measures. Finally, we give a short description of the data sets on which we test our portfolio strategy.

2.1 Determining Optimal Portfolio Weight Constraints

Throughout the paper, we consider the standard myopic constrained minimum-variance portfolio optimization problem. In particular, the objective in each period $t$ is the minimization of the portfolio variance subject to the full investment as well as minimum and maximum portfolio weight constraints. Formally, this results in an optimization problem for each period $t$ of the following form:\textsuperscript{5}

\textsuperscript{4}Almazan et al. (2004) provide a comprehensive overview of regulatory constraints for mutual fund managers in the U.S.

\textsuperscript{5}The notation and variable explanation closely follow Jagannathan and Ma (2003). Throughout the paper bold face letters indicate vectors (lower case letters) and matrices (upper case letters).
The Kuhn-Tucker conditions (necessary and sufficient) are accordingly:

\[
\sum_j S_{ij} w_j - \lambda_i + \delta_i = \lambda_0 \geq 0, \text{ for } i = 1, 2, \ldots, N
\]  
\[
\lambda_i \geq 0 \text{ and } \lambda_i = 0 \text{ if } w_i > w_{\min}, \text{ for } i = 1, 2, \ldots, N
\]  
\[
\delta_i \geq 0 \text{ and } \delta_i = 0 \text{ if } w_i < w_{\max}, \text{ for } i = 1, 2, \ldots, N
\]  

The notation is as follows: \( w_t \) denotes the \( N \times 1 \) vector of portfolio weights at time \( t \), \( S \) the empirical estimator of the variance-covariance matrix which is given by \( S = \tau^{-1}X'X \) where \( X \) denotes the set of de-meaned iid\(^6\) in-sample observations of size \( \tau \times N \), \( \lambda \) the \( N \times 1 \) vector of Lagrange multipliers for the lower portfolio weight constraint, \( \delta \) the \( N \times 1 \) vector of multipliers for the upper portfolio weight constraints, \( \lambda_0 \) the multiplier for the portfolio weights to sum up to one, and \( w_{\min} \) as well as \( w_{\max} \) denote \( N \times 1 \) vectors with uniform elements such that every asset has the same upper and lower weight constraint.\(^7\)

Let \( w^{++} \) denote the solution to the constrained minimum-variance portfolio optimization problem in (1)-(4). We may then state the following proposition:

\(^6\)The assumption of iid returns is not needed at this point. To be consistent with the following, however, we already make the assumption here.

\(^7\)Note that it is, of course, easily possible to extend the univariate portfolio weight constraints to multivariate ones. Nevertheless, the overall estimation risk of the optimal portfolio weight constraints increases since the portfolio weight constraints are not known with certainty (see also Golosnøy and Okhrin (2009) on this). A grouping of assets and the derivation of optimal constraints for a group of portfolio weight constraints in the sense of Golosnøy and Okhrin (2009) would easily be possible. However, the computational burden associated with the determination of optimal multivariate portfolio weight constraints increases substantially and becomes too costly in terms of time.
Proposition 1 (Jagannathan and Ma (2003)): Let

\[ \tilde{S} = S + J, \text{ with } J = (\delta 1' + 1 \delta') - (\lambda 1' + 1 \lambda') \]  

(8)

then \( \tilde{S} \) is symmetric and positive semi-definite, and \( w^{++}(S) \) is one of its global minimum-variance portfolios.

The appealing feature of the variance-covariance matrix \( \tilde{S} \) is that a global minimum-variance optimization on it yields the same results as the constrained optimization on \( S \). Jagannathan and Ma (2003) interpret the modified matrix \( \tilde{S} \) as a variance-covariance matrix shrunk towards zero. The imposed portfolio weight constraints shrink the extremely large and small elements of the variance-covariance matrix, which are most likely affected by the estimation error, towards zero (Jagannathan and Ma (2003)). Effectively, each element of the empirical variance-covariance matrix, \( S_{i,j} \), is adjusted by the quantity \((\lambda_i + \lambda_j) + (\delta_i + \delta_j)\) whenever the portfolio weight constraints are binding. Hence, the imposed portfolio weight constraints impose a certain structure on the variance-covariance matrix estimate which reduces the impact of sampling variation and thus the variance of the new variance-covariance matrix estimator \( \tilde{S} \).

Critically, the reduction of the estimation error comes at a cost: The shrunk variance-covariance matrix \( \tilde{S} \) is no longer unbiased. This stems from the addition of the matrix \( J \) to the unbiased variance-covariance matrix estimate \( S \), which causes the new estimator \( \tilde{S} \) to be biased. Hence, the aim is to find the optimal trade-off between the introduced bias and the variance reduction of our variance-covariance estimate that we achieve by imposing portfolio weight constraints. This very basic principle stems from shrinkage estimation theory. It proposes to find a combination of an unbiased but estimation error prone and a biased but estimation error free parameter estimate to minimize the error of the resulting estimator.

Contrary to standard shrinkage estimation, however, we do not aim to find the optimal
linear combination of the empirical estimate with a predefined shrinkage target. We rather use a fixed shrinkage intensity since the new variance-covariance matrix \( \tilde{S} \) is constructed by simply adding \( S \) and \( J \). The flexibility of our approach stems from the variability of the matrix \( J \) which is determined by the tightness of the imposed portfolio weight constraints on the portfolio optimization process. Thus, a major difference between our approach and standard shrinkage estimators is the necessity of defining a shrinkage target. While standard shrinkage estimators require a definition of a shrinkage target, our proposed approach is free of any such requirement. Additionally, via the matrix \( J \) our approach directly takes account of any potential structure, which might, for instance, stem from regulatory constraints. In the presence of such exogenous constraints, our approach guarantees local optimality. In contrast, the shrinkage approach developed by Ledoit and Wolf (2003) would not take the additional structure imposed by exogenous constraints into account when determining the optimal shrinkage intensity. As a result, the determined shrinkage intensity would be suboptimal in the presence of portfolio weight constraints.

A natural candidate for a loss function that quantifies the trade-off between bias and variance of a parameter estimate is the MSE (Ledoit and Wolf (2003, 2004)). It is defined as the squared distance, measured by the squared Frobenius norm, \( \| \cdot \|_F^2 \), between the estimated variance-covariance matrix \( S \) and the true (but unknown) variance-covariance matrix \( \Sigma \):

\[
MSE(S) = E \left( \| S - \Sigma \|_F^2 \right) = E \left( \| S - E(S) \|_F^2 \right) + \| E(S) - \Sigma \|_F^2 
\]

(9)

The first term on the right hand side of equation (9) is the variance and the second term the squared bias of the empirical variance-covariance matrix.

In particular, we search for the set of portfolio weight constraints that yield the MSE minimizing matrix \( J \) through the resulting Lagrange multipliers of the portfolio optimization problem described in equations (1) - (4). Pursuing this idea, we have the following bounded optimization problem if we impose upper and lower portfolio weight constraints in
the minimum-variance portfolio optimization:

\[
\min_{w_{\text{min}}, w_{\text{max}}} \text{MSE} \left( \hat{S} \right) = \| S + J(w_{\text{min}}, w_{\text{max}}) - \Sigma \|_F^2 \quad \text{s.t.} \\
-\infty \leq w_{\text{min}} \leq \frac{1}{N} \\
\frac{1}{N} \leq w_{\text{max}} \leq \infty
\]  

(10)

(11)

(12)

The bounds of the optimization problem in equations (11) and (12) ensure that the portfolio weight constraints do not violate the adding up constraint (3) of the constrained minimum-variance portfolio optimization problem. Practitioners may, of course, find it necessary to choose smaller intervals for the lower and upper bound in equations (11) and (12) due to regulatory or investor specific constraints. Note that our approach takes the structure of any regulatory or investor specific constraints directly into account when determining the set of optimal constraints. The major disadvantage of our approach is that there exists no closed form solution. Since the Lagrange multipliers of the constrained optimization problem described in equations (1) - (4) may not be computed in closed form, the solution of the bounded optimization problem described in equations (10) - (12) has to be solved numerically.

Note that the bounds on the upper and lower portfolio weight constraints in (11) and (12) allow our portfolio strategy to nest (i) the unconstrained minimum-variance portfolio (if \( w_{\text{min}} \) and \( w_{\text{max}} \) are not binding), (ii) the short-sale constrained minimum-variance portfolio (if \( w_{\text{min}} = 0 \) and \( w_{\text{max}} \) is not binding or not imposed), and (iii) the naively diversified portfolio (if \( w_{\text{min}} = \frac{1}{N} \) and/or \( w_{\text{max}} = \frac{1}{N} \)).

Unfortunately, we face the problem that we do not know the true variance-covariance matrix \( \Sigma \). However, we are able to estimate the MSE of the empirical variance-covariance

\footnote{Note that we also consider only upper or lower portfolio weight constraints in the latter such that the bounded optimization problem is formulated in these cases without constraint (11) or (12) respectively.}
matrix, as defined by equation (9), consistently. Remember that \( X \) denotes the set of demeaned iid in-sample observations of size \( \tau \times N \) and let \( x_t \) denote an \( 1 \times N \) vector comprising the \( t \)-th row of \( X \). It then follows that the empirical variance-covariance matrix \( S = \tau^{-1}X'X \) can be rewritten as \( S = \tau^{-1}\sum_{t=1}^{\tau}x_t'x_t \), stating that the empirical variance-covariance matrix is the average of \( x_t'x_t \). Accordingly, the MSE of the empirical variance-covariance matrix captures how \( x_t'x_t \) varies around its mean \( S \). This intuition is manifested in the following lemma:

**Lemma 1** (Ledoit and Wolf (2004)): Define \( \bar{b}^2 = \frac{1}{\tau^2} \sum_{t=1}^{\tau} \|x_t'x_t - S\|^2_F \) and \( \beta^2 = E(\|S - \Sigma\|^2) \). Then \( \bar{b}^2 - \beta^2 \xrightarrow{q.m.} 0 \)

where \( \xrightarrow{q.m.} \) denotes convergence in quadratic mean. Since we are interested in the MSE minimizing portfolio weight constraints, the objective of the bounded optimization problem described in equation (10) can be rewritten using lemma 1:

\[
\min_{w_{\text{min}}, w_{\text{max}}} \left\| \tilde{S} - \Sigma \right\|^2_F = \frac{1}{\tau^2} \sum_{t=1}^{\tau} \|x_t'x_t + ((\delta 1' + \delta'1) - (\lambda 1' + 1\lambda')) - S\|^2_F \tag{13}
\]

The constraints of the bounded optimization problem described in equations (11) to (12) remain the same. The solution to this bounded optimization problem yields the MSE minimizing portfolio weight constraints. Imposing these constraints in the minimum-variance portfolio optimization problem described in equations (1) to (4) finally yields the optimal - in the sense of MSE minimal - local minimum-variance portfolio. In the remainder of this paper, we will analyze the performance of this portfolio strategy vis-à-vis alternative benchmark portfolio strategies using robust statistical inference.

### 2.2 Benchmark Portfolio Strategies

This section provides a brief overview of the portfolio selection literature from which we choose our benchmark portfolio strategies (see Table 1). We consider two groups of bench-
mark portfolios in our analysis: simple portfolio strategies which do not rely, or only sparsely rely, on historical data and minimum-variance portfolio strategies. The first group includes the equally-weighted \((1/N)\) and value-weighted portfolio strategy. The \(1/N\) strategy is a particularly important benchmark because to date only one class of portfolio strategy is known that significantly outperforms this strategy (DeMiguel et al. 2009a). Second, we include a group of minimum-variance portfolios which serve as further benchmarks for our constrained minimum-variance portfolio strategy.

The various minimum-variance strategies to which we benchmark our approach can be attributed to the different variance-covariance matrix estimators, \(\hat{\Sigma}\), employed in these strategies. All considered estimators differ in the way they tackle the problem of noisy data in estimating the variance-covariance matrix reliably. However, all portfolio weights of the following minimum-variance strategies are computed using the well known closed form solution for the global minimum-variance portfolio:

\[
    w = \hat{\Sigma}^{-1} 1(1'\hat{\Sigma}^{-1}1)^{-1}
\]

As a first estimator of the variance-covariance matrix, we choose the empirical variance-covariance matrix estimator, \(S\), such that:

\[
    \hat{\Sigma}_U = S
\]

A major drawback of this estimator is that it is highly error prone if the number of assets is not substantially lower than the number of observations in the variance-covariance matrix estimation period. The impact of the estimation error can, however, be alleviated by imposing structure on the empirical variance-covariance matrix estimate.

For instance, the short-sale constrained minimum-variance portfolio aims to reduce the estimation error by imposing structure on the empirical variance-covariance matrix estimator via the prohibition of short-sales. The resulting structured variance-covariance matrix
The intuition of imposing structure on the variance-covariance matrix estimate also underlies the factor model based approach. In the following, we use the implied variance-covariance matrix estimate from Sharpe’s (1963) single-index model as benchmark:

\[ \hat{\Sigma}_1 = s_M^2 \beta \beta' + \Psi \]  

(17)

The factor model based approach structures the empirical variance-covariance matrix estimator by eliminating all idiosyncratic components in the off-diagonal elements of the variance-covariance matrix, thereby reducing all covariances to their systematic value.

Finally, we include the shrinkage estimator of Ledoit and Wolf (2003), which is a weighted average of the empirical and the factor model based variance-covariance matrix estimator as described in equation (17). The combination intensity, \( \alpha \), depends on the trade-off between the variance of the empirical estimate and the bias of the shrinkage target, i.e. the factor model based estimate:

\[ \hat{\Sigma}_{\text{LW1F}} = \alpha \hat{\Sigma}_1 + (1 - \alpha) S \]  

(18)

The shrinkage estimator preserves as much of the sample information contained in the empirical variance-covariance matrix as possible while reducing the variance of the resulting variance-covariance matrix estimator. The same idea underlies our proposed constrained

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9 Where \( \lambda_C \) in equation (16) denotes the Lagrangian multiplier for the short-sale constraint in the minimum-variance portfolio optimization problem.

10 Chan et al. (1999) compared the out-of-sample performance of various factor model based variance-covariance matrix estimates in the context of minimum-variance portfolio optimization and found that including more than one factor does not notably affect the resulting performance.

11 Where \( s_M^2 \) denotes the sample variance of market returns, \( \beta \) an \( N \times 1 \) vector of slope coefficients and \( \Psi \) an \( N \times N \) diagonal matrix of residual variances.
minimum-variance portfolio approach.

2.3 Portfolio Optimization and Performance Measurement

We adopt the well established rolling sample approach to test the performance of our minimum-variance strategies in comparison to the benchmark portfolio strategies. Specifically, we revise the portfolio on a monthly frequency, taking the return observations of the last \( \tau = 120 \) months as input parameter for the variance-covariance matrix estimation. This has become standard in the portfolio optimization literature. We then compute the optimal portfolio weights for all portfolio strategies. After the optimization process, we hold the portfolio weights constant for one month and compute the resulting out-of-sample return. Repeating the aforementioned steps over the whole sample period gives us a time series of \( T - \tau + 1 \) monthly out-of-sample returns for all portfolio strategies, upon which we evaluate the performance of the strategies along the dimensions risk, return, and turnover.

We evaluate the resulting out-of-sample performance of the portfolio strategies using three performance measures: the portfolio variance, \( \hat{\sigma}^2 \); the Sharpe ratio, \( \hat{SR} \); and the turnover of each portfolio strategy:

\[
\hat{\sigma}_i^2 = \frac{1}{T - \tau - 1} \sum_{t=\tau}^{T-1} (w_{i,t}'r_{t+1} - \hat{\mu}_i)^2, \text{ with } \hat{\mu}_i = \frac{1}{T - \tau} \sum_{t=\tau}^{T-1} (w_{i,t}')r_{t+1}
\]

\[
\hat{SR}_i = \frac{\hat{\mu}_i - r_f}{\hat{\sigma}_i}
\]

\[
Turnover_i = \frac{1}{T - \tau - 1} \sum_{t=\tau}^{T-1} \sum_{j=1}^{N} (|w_{i,j,t+1} - w_{i,j,t}|)
\]

Here, \( T \) denotes the total number of months in the sample and \( r_t \) an \( N \times 1 \) vector of asset returns at time \( t \).

The (over the sample period) realized variance is of major importance for the as-
essment of our proposed constrained minimum-variance portfolio strategies because it is a measure of how well the objective of risk minimization is achieved. Second, we evaluate the risk adjusted performance of each portfolio strategy by means of the realized Sharpe ratio in order to evaluate whether the reduction of risk comes at the cost of a lower realized return per unit of risk. Finally, the assessment of the turnover associated with each portfolio strategy investigates how applicable each portfolio strategy is for practitioners.

In order to compare the benchmark strategies with our constrained minimum-variance strategies, we test whether the pairwise differences of the out-of-sample variance and Sharpe ratio are statistically different from zero. Since standard hypothesis tests do not control for time series characteristics in portfolio returns (e.g., autocorrelation, volatility clustering and departure from normally distributed returns), we employ bootstrapping to overcome these problems. Specifically, in line with DeMiguel et al. (2009b) we employ the stationary bootstrap of Politis and Romano (1994) to test for significance of the portfolio variance differences. Accordingly, we construct a two-sided confidence interval for the difference of the portfolio variances and then compute the resulting p-value for the null hypothesis that the variance of our portfolio strategy \( i \) is equal to that of the portfolio strategy \( n \): 

\[ H_0 : \hat{\sigma}_i^2 - \hat{\sigma}_n^2 = 0 \]

using remark 3.2 by Ledoit and Wolf (2008). For the Sharpe ratio, we use Ledoit and Wolf’s (2008) studentized circular block bootstrap. In particular, we report the two-sided p-value for the null hypothesis that the Sharpe ratio of portfolio strategy \( i \) is equal to that of portfolio strategy \( n \): 

\[ H_0 : \hat{SR}_i - \hat{SR}_n = 0 \]

For all bootstraps, we use a block length of \( b=5 \) and base our reported p-values on \( K=1,000 \) bootstrap iterations.

### 2.4 Data

We use five different data sets which have been used extensively in the portfolio optimization literature to analyze the performance of the proposed portfolio optimization approach. They represent different cuts of the U.S. as well as the international stock markets and contain monthly returns of value weighted portfolios and indices. The data sets for the U.S. stock
market are taken from Kenneth French’s website\(^\text{12}\) and span the time period July 1967 to December 2008. The international stock market data are taken from MSCI (Morgan Stanley Capital International) and span the period January 1970 to December 2008. For the risk free rate we take the 30-day T-bill rate which we extracted from Ken French’s website.

Table\(^2\) lists the five data sets. For the U.S. stock market, we employ the ten and thirty industry portfolios as well as the six and twenty-five Fama French portfolios of firms sorted by size and book-to-market. In our international data set we include eight international equity indices from Canada, France, Germany, Italy, Japan, Switzerland, the U.K., the U.S., and the World Index.

3 Empirical Results

We first discuss the out-of-sample variance, second the Sharpe ratio, and third the generated turnover of the portfolio strategies.

3.1 Discussion of the Variance

Table \(3\) shows the out-of-sample variances of all considered portfolio strategies and the corresponding p-values of the test that the variance of a particular portfolio strategy is different from that of our constrained minimum-variance portfolio with upper and lower weight constraints (Min-B).\(^\text{13}\) Assessing the variances within the group of the constrained minimum-variance portfolios (Min-Up, Min-L, Min-B), we see that imposing upper (Min-Up) or lower (Min-L) weight constraints yields very similar results. Simultaneously imposing upper and lower weight constraints leads to a further risk reduction relative to single (upper or lower) weight constraints for the 25 Fama French and 30 industry data set. This does not


\(^{13}\)We also computed all pairwise p-values and base the interpretation concerning the significance of our findings partly on these values. The detailed results are omitted to save space; they are available upon request.
come as a surprise since the necessity for a tighter structure is pronounced for the larger data sets, attributable to an increase in the variance of the empirical estimator due to a decrease in the number of degrees of freedom.

Comparing our constrained minimum-variance portfolios with the $1/N$ and value weighted (VW) benchmarks, we find that our portfolio strategies exhibit significantly lower variances. The differences are significant on the 1% level for all but the international data set. For the latter, we find that the constrained minimum-variance portfolios achieve a significantly lower variance than the $1/N$ strategy on the 5% level, while the risk reduction relative to the value weighted benchmark is not significant.

The comparison of our approach with the unconstrained minimum-variance portfolio (Min-U) reveals that the variance reduction in all constrained minimum-variance portfolios is significant on the 5% level for the 25 Fama French data set. For the 30 industry data set, only Min-B achieves a lower variance which is weakly significant. Compared to the short-sale constrained minimum-variance portfolio (Min-C), our approach yields lower variances, which are significant on the 1% level, in the cases of the 6 and 25 Fama French data sets. With respect to the factor model based minimum-variance portfolio (Min-1F), we see consistently lower variances for our portfolio strategies, although the differences are only highly significant in the cases of the Fama French data sets. Compared with the shrinkage approach of Ledoit and Wolf (LW1F), we observe a significantly lower variance of Min-Up and Min-B for the 6 Fama French data set on the 5% and 10% significance level, respectively. However, the opposite holds for the 10 and 30 industry data sets, for which our portfolios exhibit significantly higher variances. This suggests that the imposed structure via the portfolio weight constraints is in general less restrictive than that of the Ledoit and Wolf single factor model.
3.2 Discussion of the Sharpe Ratio

Table 4 displays the out-of-sample Sharpe ratios of the considered portfolio strategies.\textsuperscript{14} It can be seen that Min-B yields consistently higher Sharpe ratios than the $1/N$ strategy, while the Min-L and Min-Up portfolios achieve this in four out of the five considered data sets. Compared with the VW benchmark, we observe consistently higher Sharpe ratios for all constrained minimum-variance portfolios. On average, the Sharpe ratio increase of the Min-B strategy vis-à-vis the $1/N$ is 30\% and 60\% vis-à-vis the VW benchmark. Similar Sharpe ratio increases are found for the Min-L and Min-Up strategies. The Sharpe ratio differences between our constrained minimum-variance portfolios and the $1/N$ strategy are significant on the 5\% level for the 6 Fama French data set and on the 1\% level for the 25 Fama French data set. Compared with the VW strategy, the Sharpe ratio differences are significant on the 1\% level for the 6 and 25 Fama French data sets. Furthermore, all constrained minimum-variance portfolios achieve a higher Sharpe ratio, which is significant on the 5\% level, for the 10 industry data set. This is compelling evidence that, in terms of Sharpe ratio, the performance of our constrained minimum-variance portfolios is superior to that of simple portfolio strategies. So far, DeMiguel et al. (2009a) is the only other study that documents significant performance differences between simple portfolio strategies and more sophisticated approaches.

Within the group of constrained minimum-variance portfolios, we note statistically significantly different Sharpe ratios for three data sets. Specifically, we observe that Min-B outperforms Min-Up on the 30 industry portfolios, while the opposite holds for the 6 and the 25 Fama French portfolios, although the latter difference is only weakly significant. However, the overall picture that emerges is that all constrained minimum-variance strategies exhibit very similar Sharpe ratios.

The Sharpe ratios of the Min-U and our constrained minimum-variance strategies

\textsuperscript{14}We also computed all pairwise p-values and base the interpretation concerning the significance of our findings partly on these values. The detailed results are omitted to save space; they are available upon request.
are, on the whole, very similar. The Min-U portfolio achieves a significantly higher Sharpe ratio than the constrained minimum-variance portfolios only in the case of the 6 Fama French portfolios, while the opposite holds for the 30 industry data set. Evaluation of our constrained minimum-variance portfolios in comparison to the Min-C and Min-1F portfolios shows that the reduced risk of our portfolio strategies mostly does not come at the cost of less return. Specifically, our minimum-variance portfolio strategies yield higher Sharpe ratios than Min-C and Min-1F in three out of five cases. This difference is highly significant in two cases, the 6 and 25 Fama French portfolios. Finally, we observe no statistically significantly differences between the LW1F strategy and our constrained minimum-variance portfolios.

3.3 Discussion of the Turnover

Table 5 shows the turnover associated with each portfolio strategy. Unsurprisingly, the simple portfolio strategies exhibit the lowest turnover of all portfolio strategies. This observation is consistent throughout all data sets. Within the group of minimum-variance portfolios, the short-sale constrained minimum-variance portfolio has the lowest turnover across all data sets. Again, this is not surprising because ruling out short-sales limits the maximum possible turnover. Given the dynamic and flexibility of the portfolio weight constraints of our constrained minimum-variance strategies, we observe substantially higher turnovers associated with our strategies compared to that of the short-sale constrained minimum-variance portfolio. Generally, our portfolio strategies have a similar turnover to that of the unconstrained minimum-variance portfolio. On the other hand, it is higher than that of Min-1F and LW1F. However, it remains at an acceptable level for all data sets except for the 25 Fama French portfolios.

While our strategy does not yield the lowest turnover of the considered portfolio strategies, we claim that the turnover created by our portfolio strategies is considerably lower than the one reported in DeMiguel et al. (2009a), which is the only other study that develops a portfolio strategy that significantly outperforms the 1/N strategy. The results reported
there are based on the same data sets as the ones used in our study. They display that the average monthly turnover of the partial minimum-variance portfolios amounts to 200% across all data sets. This is five times the average turnover of our constrained minimum-variance portfolio, which generates an average turnover across all data sets of 40%. We thus argue that our constrained minimum-variance portfolios may be seen as an attractive alternative to DeMiguel et al. (2009a), particularly from the perspective of applicability.

4 Conclusion

In this paper, we develop a framework for determining optimal portfolio weight constraints for minimum-variance portfolios in the presence of parameter uncertainty. Our framework is based on standard shrinkage estimation theory and combines results from Jagannathan and Ma (2003) and Ledoit and Wolf (2004). Specifically, we impose the set of upper and lower portfolio weight constraints that minimize the MSE of the empirical variance-covariance matrix and thus yield an optimal trade-off between the reduction of sampling error and loss of sample information. The advantage of this approach compared to other shrinkage approaches of the variance-covariance matrix is the direct consideration of structure that may stem from regulatory or investor specific constraints. We claim that this is of particular interest to practitioners who often face such constraints. The major drawback of our approach is that no closed form solution exists. The computational effort is, however, manageable for reasonable portfolio sizes.

We then derive empirical results based on five data sets that were extensively used in the portfolio optimization literature. Our results document that the constrained minimum-variance portfolios outperform the natural benchmark strategy $1/N$ consistently in terms of Sharpe ratio. This result is particularly noteworthy in light of the results obtained by DeMiguel et al. (2009b), who document the difficulty of outperforming the $1/N$ strategy. Furthermore, our approach also outperforms the VW benchmark strategy. On aver-
age, we find that our constrained minimum-variance portfolios yield a 30% (60%) higher Sharpe ratio than the $1/N$ (VW) benchmark strategy. We further find that our constrained minimum-variance portfolios improve the popular short-sale constrained and factor model based minimum-variance portfolios in terms of both risk reduction and Sharpe ratio increase.

Compared with the only other known portfolio strategy that consistently outperforms the $1/N$ strategy, laid out in DeMiguel et al. (2009a), our approach has the advantage of generating a substantially lower turnover. Specifically, the average turnover of our portfolio strategies is 40% across all data sets, which is only one fifth of DeMiguel et al.’s partial minimum-variance portfolios (DeMiguel et al. (2009a)). We hence argue that our approach presents an attractive alternative that is both theoretically sound and particularly appealing from a practitioner’s perspective.
Table 1: List of considered portfolio strategies

The table lists the developed and the benchmark portfolio strategies. Panel A lists the new minimum-variance portfolio strategies with flexible portfolio weight constraints. Panel B shows the simple benchmark strategies which are considered following DeMiguel et al. (2009b). Panel C lists minimum-variance portfolio strategies proposed in the literature.

<table>
<thead>
<tr>
<th>#</th>
<th>Description</th>
<th>Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A. Developed portfolio strategies</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Minimum-variance portfolio with flexible minimum portfolio weight constraints</td>
<td>Min-L</td>
</tr>
<tr>
<td>2</td>
<td>Minimum-variance portfolio with flexible maximum portfolio weight constraints</td>
<td>Min-Up</td>
</tr>
<tr>
<td>3</td>
<td>Minimum-variance portfolio with flexible minimum and maximum portfolio weight constraints</td>
<td>Min-B</td>
</tr>
<tr>
<td><strong>Panel B. Simple benchmark strategies which do not require optimization</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Equally weighted (1/N) portfolio</td>
<td>1/N</td>
</tr>
<tr>
<td>5</td>
<td>Value weighted (market) portfolio</td>
<td>VW</td>
</tr>
<tr>
<td><strong>Panel C. Minimum-variance portfolios</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Minimum-variance portfolio without constraints</td>
<td>Min-U</td>
</tr>
<tr>
<td>7</td>
<td>Minimum-variance portfolio with short-sale constraints as in Jagannathan and Ma (2003)</td>
<td>Min-C</td>
</tr>
<tr>
<td>8</td>
<td>Minimum-variance portfolio with the market as single factor</td>
<td>Min-1F</td>
</tr>
<tr>
<td>9</td>
<td>Minimum-variance portfolio with the variance-covariance matrix as weighted average between the sample variance-covariance and the single factor variance-covariance matrix</td>
<td>LW1F</td>
</tr>
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</table>
The table lists the various data sets used for the evaluation of the portfolio performance, their abbreviations, the number of assets that each data set comprises, the time period over which we use data from each particular data set, and the data sources. All data sets comprise monthly data and apply the value weighting scheme to the respective constituents. The data sets from Kenneth French are taken from his website [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html) and represent different cuts of the U.S. stock market. The eight MSCI equity indices plus the World Index include the indices for Canada, France, Germany, Italy, Japan, Switzerland, the U.K., the U.S., and the MSCI World Index.

<table>
<thead>
<tr>
<th>#</th>
<th>Data set</th>
<th>Abbreviation</th>
<th>N</th>
<th>Time period</th>
<th>Source</th>
</tr>
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<tr>
<td>1</td>
<td>6 Fama and French portfolios of firms sorted by size and book-to-market</td>
<td>6 FF</td>
<td>6</td>
<td>07/1963-12/2008</td>
<td>K. French</td>
</tr>
<tr>
<td>2</td>
<td>8 MSCI country indices plus the MSCI World Index</td>
<td>8+1 MSCI</td>
<td>9</td>
<td>01/1970-12/2008</td>
<td>MSCI</td>
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<tr>
<td>3</td>
<td>10 industry portfolios representing the U.S. stock market</td>
<td>10 IND</td>
<td>10</td>
<td>07/1963-12/2008</td>
<td>K. French</td>
</tr>
<tr>
<td>5</td>
<td>30 industry portfolios representing the U.S. stock market</td>
<td>30 IND</td>
<td>48</td>
<td>07/1963-12/2008</td>
<td>K. French</td>
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</table>
Table 3: Variances of the portfolio strategies

The table reports the monthly out-of-sample variances of all considered portfolio strategies with an estimation period of $\tau=120$ months. The p-values (in italics) are computed for the null hypothesis that the portfolio variance of a particular portfolio strategy $i$ is equal to that of our constrained minimum-variance portfolio with flexible upper and lower portfolio weight constraints (denoted by Min-B): $H_0: \sigma^2_i = \sigma^2_{Min-B} = 0$. The p-values are computed using the stationary bootstrap by Politis and Romano (1994) with a block length of 5 and 1,000 iterations (cf. section 2.3).

<table>
<thead>
<tr>
<th>Strategy</th>
<th>6 FF</th>
<th>8+1 MSCI</th>
<th>10 IND</th>
<th>25FF</th>
<th>30 IND</th>
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<tr>
<td>Min-L</td>
<td>0.00163</td>
<td>0.00176</td>
<td>0.00137</td>
<td>0.00141</td>
<td>0.00152</td>
</tr>
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<td>0.281</td>
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<td>0.068</td>
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<td>0.00177</td>
<td>0.00136</td>
<td>0.00143</td>
<td>0.00156</td>
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<td>0.279</td>
<td>0.760</td>
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<td>Min-B</td>
<td>0.00161</td>
<td>0.00176</td>
<td>0.00137</td>
<td>0.00138</td>
<td>0.00151</td>
</tr>
<tr>
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<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>1/N</td>
<td>0.00240</td>
<td>0.00201</td>
<td>0.00188</td>
<td>0.00259</td>
<td>0.00231</td>
</tr>
<tr>
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<td>0.001</td>
<td>0.016</td>
<td>0.002</td>
<td>0.000</td>
<td>0.002</td>
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<td>0.00214</td>
<td>0.00212</td>
<td>0.00214</td>
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<td>0.130</td>
<td>0.000</td>
<td>0.000</td>
<td>0.004</td>
</tr>
<tr>
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<td>0.00137</td>
<td>0.00149</td>
<td>0.00157</td>
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<td></td>
<td>0.794</td>
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<td>0.419</td>
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<tr>
<td>Min-C</td>
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<td>0.00177</td>
<td>0.00136</td>
<td>0.00187</td>
<td>0.00138</td>
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<td>0.00316</td>
<td>0.00157</td>
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<td>0.551</td>
<td>0.341</td>
<td>0.009</td>
<td>0.634</td>
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<tr>
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<td>0.093</td>
<td>0.847</td>
<td>0.051</td>
<td>0.605</td>
<td>0.000</td>
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</table>
Table 4: Sharpe ratios of the portfolio strategies

The table reports the monthly out-of-sample Sharpe ratio of the portfolio strategies with an estimation period of $\tau=120$ months. We use the 30-day T-bill rate as the risk free rate. The p-values (in italics) are computed for the null hypothesis that the Sharpe ratio of a particular portfolio strategy $i$ is significantly different from that of our constrained minimum-variance portfolio with flexible upper and lower portfolio weight constraints (denoted by Min-B):

$$H_0 : \frac{\hat{\mu}_i - r_f}{\hat{\sigma}_i} = \frac{\hat{\mu}_{\text{Min-B}} - r_f}{\hat{\sigma}_{\text{Min-B}}} = 0.$$  

The p-values are computed using the studentized circular block bootstrap of Ledoit and Wolf (2008) with a block length of 5 and 1,000 iterations (cf. section 2.3).

<table>
<thead>
<tr>
<th>Strategy</th>
<th>6 FF</th>
<th>8+1 MSCI</th>
<th>10 IND</th>
<th>25FF</th>
<th>30 IND</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min-L</td>
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<td>0.105</td>
<td>0.144</td>
<td>0.217</td>
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<tr>
<td></td>
<td>0.113</td>
<td>0.969</td>
<td>0.944</td>
<td>0.355</td>
<td>0.238</td>
</tr>
<tr>
<td>Min-Up</td>
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<td>0.105</td>
<td>0.145</td>
<td>0.244</td>
<td>0.098</td>
</tr>
<tr>
<td></td>
<td>0.003</td>
<td>0.963</td>
<td>0.702</td>
<td>0.069</td>
<td>0.035</td>
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<tr>
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<td>0.111</td>
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<tr>
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<td>0.129</td>
<td>0.012</td>
</tr>
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<td>0.119</td>
<td>0.117</td>
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<tr>
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<td>0.441</td>
<td>0.003</td>
<td>0.688</td>
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<td>0.096</td>
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<td>0.122</td>
<td>0.114</td>
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<tr>
<td></td>
<td>0.002</td>
<td>0.544</td>
<td>0.460</td>
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<td>0.992</td>
</tr>
<tr>
<td>LW1F</td>
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<td>0.105</td>
<td>0.147</td>
<td>0.230</td>
<td>0.109</td>
</tr>
<tr>
<td></td>
<td>0.319</td>
<td>0.945</td>
<td>0.174</td>
<td>0.315</td>
<td>0.699</td>
</tr>
</tbody>
</table>
Table 5: Turnover of the portfolio strategies

The table reports the average monthly turnover of the portfolio strategies with an estimation period of $\tau=120$ months. Turnover is measured by the percentage of total portfolio wealth traded in each period over the sample, normalized by the number of trading periods.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>6 FF</th>
<th>8+1 MSCI</th>
<th>10 IND</th>
<th>25FF</th>
<th>30 IND</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min-L</td>
<td>0.208</td>
<td>0.164</td>
<td>0.155</td>
<td>0.625</td>
<td>0.433</td>
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<td>Min-Up</td>
<td>0.177</td>
<td>0.151</td>
<td>0.154</td>
<td>0.721</td>
<td>0.473</td>
</tr>
<tr>
<td>Min-B</td>
<td>0.235</td>
<td>0.219</td>
<td>0.169</td>
<td>0.973</td>
<td>0.450</td>
</tr>
<tr>
<td>$1/N$</td>
<td>0.014</td>
<td>0.031</td>
<td>0.020</td>
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<td>0.025</td>
</tr>
<tr>
<td>VW</td>
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<td>0.019</td>
<td>0.005</td>
<td>0.017</td>
<td>0.007</td>
</tr>
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<td>Min-U</td>
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<td>0.241</td>
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<td>0.061</td>
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<td>0.561</td>
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References


