BIVARIATE DELTA-EVOLUTION EQUATIONS AND CONVOLUTION POLYNOMIALS: COMPUTING POLYNOMIAL EXPANSIONS OF SOLUTIONS.

ANA LUZÓN* AND MANUEL A. MORÓN**

*Departamento de Matemática Aplicada a los Recursos Naturales. E.T. Superior de Ingenieros de Montes. Universidad Politécnica de Madrid. 28040-Madrid, SPAIN
e-mail: anamaria.luzon@upm.es

**Departamento de Geometría y Topología. Facultad de Matemáticas. Universidad Complutense de Madrid. 28040- Madrid, SPAIN.
Fax: 34 91 394 45 64. Tel: 34 91 3944511
e-mail: ma.moron@mat.ucm.es

Abstract. This paper describes an application of Rota and collaborator’s ideas on the foundations on combinatorial theory to the computing of solutions of some linear functional partial differential equations. We give a dynamical interpretation of the convolution families of polynomials as the entries, in the matrix representation, of the exponentials of certain contractive linear operators in the ring of formal power series. This is the starting point to get symbolic solutions to some functional-partial differential equations. We introduce the bivariate convolution product of convolution families to obtain symbolic solutions of a natural extension of a functional-evolution equation related to delta-operators. We put some examples to show how these symbolic methods allow us to get closed formulas for solutions of genuine partial differential equations and of some functional-evolution equations. We create an adequate framework to base, theoretically, some of the constructions and to get some existence and uniqueness results.

This paper is dedicated to Jose Maria Montesinos Amilibia with admiration and on occasion of his 65th birthday.

Key words and phrases. Ultrametric, Convolution family, delta-operator, Riordan group, delta-evolution equation, bivariate convolution product.

1. Introduction

D. E. Knuth in [1] introduced the concept of convolution family as a sequence

\[ F_0(t), F_1(t), \cdots, F_n(t), \cdots \]

such that \( F_n(t) \) is a polynomial with \( \deg(F_n(t)) \leq n \), for every \( n \in \mathbb{N} \), and satisfying the following convolution condition:

\[ F_n(s+t) = F_n(s)F_n(t) + F_{n-1}(s)F_1(t) + \cdots + F_1(s)F_{n-1}(t) + F_0(s)F_n(t) \quad \forall s, t \text{ and } n \geq 0 \]

As described there in [1] many such families are known and they appear frequently in applications. Earlier in [2], [3] appeared many families of this kind, all of them associated to some delta-operator. Special families of this type were introduced and studied in [4] and [5] for different purposes, more related to combinatorics, and with additional conditions related to the evaluation at the integers. The real beginning of the use of this families of polynomials can be dated much earlier as one can see in [2], [3] and in the corresponding references.

Knuth pointed out that a simple rule that characterizes all non-null convolution families is the following: \( (F_n(t))_{n \in \mathbb{N}} \) is a non-null convolution family if and only if there is a formal power series \( F(x) = 1 + F_1x + \cdots + F_nx^n + \cdots \) such that \( F_n(t) = [x^n](F(x))^t \), where the notation \([x^n]\) stands for the \( n \)-th ”coefficient” in the expression.

\[ (F(x))^t = \sum_{n \geq 0} F_n(t)x^n \]

Earlier in [2] and [3] the same was proved for families associated to delta-operators.

This paper begins, in Section 2, with a trivial dynamical interpretation of the convolution families of polynomials. Our main aim, contained in Section 3, is to show how convolution families can be used to find symbolic solutions of some functional equations related to partial linear differential equations. We finish the paper, with the Section 4, stating an adequate theoretical framework related to ultrametrics, to base some of the constructions used before and to get some results on the uniqueness of solutions. Our main tool in the last section of this paper is the well-known Banach’s fixed point Theorem, see [6] for example.
With the help of a natural complete ultrametric $d$ in $\mathbb{K}[[x]]$, and by means of the use of contractive maps, we characterize the convolution families as the entries of an infinite lower triangular Toeplitz matrix with coefficients in the ring of polynomials $\mathbb{K}[t]$. In fact this matrix is a special matrix representation of the exponential $e^{tL}$ where $L$ is a contractive $\mathbb{K}[[x]]$-module homomorphism in $(\mathbb{K}[[x]], d)$. The exterior product that we consider in this module is the usual Cauchy product of series. It is easily seen that the action of any such operator $L$ is univocally represented by the Cauchy product by a fixed series $g = \sum_{n \geq 1} g_n x^n$, note that $g(0) = 0$. The family $\{e^{tL}\}_{t \in \mathbb{K}}$ is a one-parameter subgroup (the parameter varies through $\mathbb{K}$) of the group of $\mathbb{K}$-linear onto isometries in $(\mathbb{K}[[x]], d)$. After that we treat what we call \textit{delta-evolution equations}, which are obtained by substituting the partial derivative respect to a variable by a natural action of a delta-operator in an adequate framework.

We also use an interesting paper due to Roland Bacher [7] to give the symbolic solutions to some evolution initial value problems in terms of one-parameter subgroups of the Riordan group, see, [8], [9], [10], [11] and [12] for basic references on this group. In our previous papers [13, 14, 15, 16] we introduced a different approach, and a different notation, for this group that we will follow herein.

We usually impose, for many series appearing herein, that the independent term is null, this is because many operators related to the series are automatically contractive and the complete ultrametric $d$ allows us to get easily many results. Using again some results in [7] we should avoid this condition in some cases. In fact in the last section we also show how to interpret, among other things, the more general case treated by Bacher in [7] related to the Lie Algebra of the Riordan group.

We also introduce herein a bivariate convolution product of convolution families of polynomials to find symbolic solutions to some delta-evolution equations where another delta-operator is involved. To point out that this symbolic results allows us to find solutions of initial value problems for some real partial linear differential equations, we put some examples. We compute, at hand, the solutions. Following the Rodrigues’s formula relating consecutive terms in a polynomial sequence of binomial type, see again
[2] and [3], we introduce and compute solutions for what we call the Rodrigues-evolution equation induced by a delta-operator.

We always move in the symbolic world but we put real examples. To do that we choose, in many examples, polynomial initial conditions that help us in the computation too. In fact our natural frameworks are related to the rings of formal power series with some natural ultrametrics. The advantage of these ultrametrics is that we can recognize easily many contractive functions and then the Banach's fixed point Theorem and the iterative procedure described in it are in order. We have to say that in the realm of analytic functions many of the results we give could be applied to obtain solutions for some partial differential equations when \( K \) is the real or the complex field. We also think that some results can be reached from here, for functions, if one follows the line described in the classical textbook [17] about polynomial expansions of analytic functions because most of the convolution sequences we use are Sheffer sequences of polynomials in the sense defined in [17]. In fact they are very closely related to polynomial sequences of binomial type as defined in [2] and [3]. This paper can be summarized as an application of polynomial sequences of binomial type, or of the Rota and collaborators's Finite Operator Calculus, to compute polynomial expansions of solutions of some families of functional linear partial differential equations associated to delta-operators. See [2] and [3] for all the related basic definitions.

All along this paper \( K \) represents a field of characteristic zero, \( \mathbb{N} \) represents the natural numbers ( in this field) including zero and \( D \) denotes the usual derivative operator on power series.

We recommend [18] for information on ultrametrics among many other things. We also recommend [2] and [3], in particular, for information about delta-operators that will be used all along this paper. In particular both papers describe some ways to construct the convolution family of polynomials associated to a delta-operator. Also in [16] are described recurrence relations to obtain all of them.

We used the classical textbook of Weinberger [19] to get, and to refresh, some general knowledge on partial differential equations. We do not use any previous result on this subject. We also used the survey [20] for a general view, and historical notes, on partial
differential equations. We hope that this paper can help along the line on the last paragraph in the introduction in [20]. Literally,

"⋯ computations of solution of PDE’s is the major concern in scientific computing."

2. Convolution sequences as exponentials of contractive module-homomorphisms

Let $\mathbb{K}$ be a field of characteristic zero. Let $\mathbb{K}[[x]]$ be the ring of formal power series with coefficients in $\mathbb{K}$.

Consider the complete ultrametric $d$ on $\mathbb{K}[[x]]$ given by

$$d(f, g) = \frac{1}{2^{\omega(f-g)}}, \text{ for } f, g \in \mathbb{K}[[x]]$$

where $\omega(h)$ means the order of the power series $h = \sum_{n \geq 0} h_n x^n$, see [18], [14], defined by

$$\omega(h) = \min\{n \in \mathbb{N}, | h_n \neq 0\}$$

Consider also the Cauchy product of series denoted by $\cdot$. So $(\mathbb{K}[[x]], +, \cdot)$ has a natural structure of $\mathbb{K}[[x]]$-module over the ring $(\mathbb{K}[[x]], +, \cdot)$. That is, consider the abelian group $(\mathbb{K}[[x]], +)$ with the "external" product given by the Cauchy product. Note also that $\mathbb{K}[[x]]$ has a natural structure of vector space over the field $\mathbb{K}$. The first proposition we want to point out is the following

Proposition 1. Let $L : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x]], d)$ be a $\mathbb{K}[[x]]$-module homomorphism. Then there is a unique $g \in \mathbb{K}[[x]]$ such that $L(f) = g \cdot f$ for every $f \in \mathbb{K}[[x]]$. Consequently $L$ is either contractive or an isometry respect to the ultrametric $d$.

Proof. The first part is obvious. In fact $g = L(1)$ where $1 = 1 + 0x + 0x^2 + \cdots + 0x^n + \cdots$. Moreover if $g = \sum_{n \geq 0} g_n x^n$ then $L$ is at least $\frac{1}{2}$-contractive if and only if $g_0 = 0$ and $L$ is an isometry if and only if $g_0 \neq 0$. \qed

From now on we denote by $L_g$ to the corresponding $\mathbb{K}[[x]]$-homomorphism given by $L_g(f) = g \cdot f$. 5
Obviously any $\mathbb{K}[[x]]$-homomorphism is a continuous $\mathbb{K}$-linear map in $\mathbb{K}[[x]]$ and so, it has an associated matrix as in Definition 16 in [14]. In this case if $g = \sum_{n \geq 0} g_n x^n$, then the associated matrix to $L_g$ is:

$$L_g \equiv \begin{pmatrix} g_0 & g_1 & g_2 & \cdots & g_0 \\
 & g_1 & g_0 & \cdots & \cdot \\
 & & g_2 & g_1 & g_0 \\
& & & \vdots & \vdots \\
& & & & g_n \\
& & & & & \cdot \cdot \cdot 
\end{pmatrix}$$

which is a lower triangular Toeplitz matrix. The ultrametric $d$ allows us to give the following result on the action of a power series on a contractive operator:

**Proposition 2.** Let $f = \sum_{n=0}^{\infty} f_n x^n$ be a power series and $L : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x]], d)$ a contractive linear operator. Then $f(L) : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x]], d)$,

$$f(L) = \sum_{n=0}^{\infty} f_n L^n = f_0 I + f_1 L + f_2 L^2 + \cdots$$

defines a non-expansive linear operator on $\mathbb{K}[[x]]$. Moreover

a) $f(L)$ is an isometry if and only if $f(0) \neq 0$.

b) $f(L)$ is contractive if and only if $f(0) = 0$.

**Proof.** Let $g \in \mathbb{K}[[x]]$ and consider the partial sums $S_n(g) = (\sum_{k=0}^{n} f_k L^k)(g)$. Then $d(S_n(g), S_{n+1}(g)) = d(f_{n+1} L^{n+1}(g), 0)$. Since $L$ is at least $\frac{1}{2}$-contractive we get $d(f_{n+1} L^{n+1}(g), 0) \leq \frac{1}{2^{n+1}} d(g, 0) \leq \frac{1}{2^n}$. Consequently $\{S_n(g)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathbb{K}[[x]], d)$ because $d$ is an ultrametric. From the completeness of $(\mathbb{K}[[x]], d)$ we obtain that $f(L)(g)$ is well defined as $\lim_{n \rightarrow \infty} S_n(g)$.

Suppose that $n_0, k_0 \in \{0, 1, 2, 3, \cdots \}$ are such that $d(f, 0) = \frac{1}{2^{n_0}}$ and $d(L(g), L(h)) \leq \frac{1}{2^{k_0}} d(g, h)$, for every $g, h \in \mathbb{K}[[x]]$. Then

$$d(f(L)(g), f(L)(h)) = d(L^{n_0}(g), L^{n_0}(h)) \leq \frac{1}{2^{n_0 k_0}} d(g, h)$$

(1)

So $f(L)$ is non-expansive.
To obtain an equality in (1) we need $n_0 = 0$ and we get (a), in a similar way if $n_0 > 0$ and using the contractivity of $L$ we have (b). □

As a consequence we obtain

**Corollary 3.** Let $g = \sum_{n \geq 0} g_n x^n$ be a power series with $g_0 = 0$ and $t \in \mathbb{K}$. Then

$$e^{tL_g} = I + tL_g + \frac{t^2}{2!} L_g^2 + \cdots + \frac{t^n}{n!} L_g^n + \cdots$$

being $L_g^k$ the $k$-fold composition of $L_g$, defines a $\mathbb{K}[[x]]$-isomorphism which is also an isometry respect to the ultrametric $d$. Moreover $e^{tL_g}(h) = L_{e^{t\theta}}(h) = he^{t\theta}$ for any $h \in \mathbb{K}[[x]]$.

To get our main result in this section we need two explanations on notations:

1. Given a formal polynomial $P(t) \in \mathbb{K}[[t]]$, then it induces a natural polynomial map $P : \mathbb{K} \rightarrow \mathbb{K}$. So when we write $P(t_0)$, we are talking about the evaluation of such map at the point $t_0 \in \mathbb{K}$. Remember that $\mathbb{K}$ is always a field of characteristic zero and then the polynomial is completely determined by the induced polynomial map and viceversa.

2. If $u(x, t) = \sum_{k \geq 0} P_k(t)x^k \in \mathbb{K}[t][[x]]$ then we denote by $\frac{\partial u(x, t)}{\partial t}$ to the element of $\mathbb{K}[t][[x]]$ defined by $\sum_{k \geq 0} D(P_k(t)) x^k$ where $D$ is the corresponding derivative in $\mathbb{K}[t]$.

So we are now ready to prove the following

**Theorem 4.** Let $(F_n(t))_{n \in \mathbb{N}}$ be a non-null sequence of polynomials in $\mathbb{K}[t]$ with degree$(F_n) \leq n$. Then $(F_n(t))$ is a convolution family if and only if there is $g \in \mathbb{K}[[x]]$ with $g(0) = 0$ such that for any $t_0 \in \mathbb{K}$

$$e^{t_0L_g} \equiv \begin{pmatrix} F_0(t_0) \\ F_1(t_0) & F_0(t_0) \\ F_2(t_0) & F_1(t_0) & F_0(t_0) \\ \vdots & \vdots & \vdots & \ddots \\ F_n(t_0) & F_{n-1}(t_0) & F_{n-2}(t_0) & \cdots & F_0(t_0) \\ \vdots & \vdots & \vdots & \cdots & \ddots \end{pmatrix}$$
Moreover if \( f = \sum_{n \geq 0} f_n x^n \) is any power series, then

\[
\begin{align*}
  u(x, t) &= \sum_{n \geq 0} \left( \sum_{k=0}^{n} f_k F_{n-k}(t) \right) x^n \\
  &\in \mathbb{K}[t][[x]]
\end{align*}
\]

is the unique solution of the partial differential equation (in \( \mathbb{K}[t][[x]] \))

\[
\begin{align*}
  \frac{\partial u(x, t)}{\partial t} &= g(x) u(x, t), \\
  u(x, 0) &= f(x)
\end{align*}
\]

Proof. Suppose that \((F_n(t))_{n \in \mathbb{N}}\) is a convolution family in the sense of Knuth [1], then

\[
\sum_{n \geq 0} F_n(t)x^n = e^{t g(x)} \quad \text{for some } g \in \mathbb{K}[[x]] \text{ with } g(0) = 0 \quad \text{which obviously implies that}
\]

\[
e^{t_0 T_g} \equiv \begin{pmatrix}
  F_0(t_0) \\
  F_1(t_0) & F_0(t_0) \\
  F_2(t_0) & F_1(t_0) & F_0(t_0) \\
  \vdots & \vdots & \vdots & \ddots \\
  F_n(t_0) & F_{n-1}(t_0) & F_{n-2}(t_0) & \cdots & F_0(t_0) \\
  \vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}
\]

for any \( t_0 \in \mathbb{K} \).

On the contrary suppose that

\[
e^{t_0 L_g} \equiv \begin{pmatrix}
  F_0(t_0) \\
  F_1(t_0) & F_0(t_0) \\
  F_2(t_0) & F_1(t_0) & F_0(t_0) \\
  \vdots & \vdots & \vdots & \ddots \\
  F_n(t_0) & F_{n-1}(t_0) & F_{n-2}(t_0) & \cdots & F_0(t_0) \\
  \vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}
\]

for any \( t_0 \in \mathbb{K} \). Consider the formal polynomials \( F_k(t) \in \mathbb{K}[t] \). Since \( e^{0L_g} = I \) and degree\( F_0 \leq 0 \), this means that \( F_0(t) = 1 \). Moreover since \( (e^{tL_g} \circ e^{sL_g})(f) = e^{tg}e^{sg}f = \)
\[ e^{(t+s)} f = e^{(t+s)T_s}(f). \] This translates to the corresponding product of matrices as

\[
\begin{pmatrix}
F_0(t) \\
F_1(t) & F_0(t) \\
F_2(t) & F_1(t) & F_0(t) \\
\vdots & \vdots & \vdots & \ddots \\
F_n(t) & F_{n-1}(t) & F_{n-2}(t) & \cdots & F_0(t) \\
\vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}
\begin{pmatrix}
F_0(s) \\
F_1(s) & F_0(s) \\
F_2(s) & F_1(s) & F_0(s) \\
\vdots & \vdots & \vdots & \ddots \\
F_n(s) & F_{n-1}(s) & F_{n-2}(s) & \cdots & F_0(s) \\
\vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}
= 
\begin{pmatrix}
F_0(t+s) \\
F_1(t+s) & F_0(t+s) \\
F_2(t+s) & F_1(t+s) & F_0(t+s) \\
\vdots & \vdots & \vdots & \ddots \\
F_n(t+s) & F_{n-1}(t+s) & F_{n-2}(t+s) & \cdots & F_0(t+s) \\
\vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}
\begin{pmatrix}
f_0 \\
f_1 \\
f_2 \\
\vdots \\
f_n \\
\vdots
\end{pmatrix}
\]

and consequently

\[
\sum_{k=0}^{n} F_{n-k}(t)F_k(s) = F_n(s + t)
\]

which is the convolution condition. Consider the bivariate power series

\[
u_1(x, t) = \sum_{k \geq 0} F_k(t)x^k = e^{t g(x)}
\]

see [1]. If \( f = \sum_{n \geq 0} f_n x^n \) then

\[
u(x, t) = f(x)e^{t g(x)} = f(x)u_1(x, t)
\]

converts, in matrix form, to

\[
\begin{pmatrix}
F_0(t) \\
F_1(t) & F_0(t) \\
F_2(t) & F_1(t) & F_0(t) \\
\vdots & \vdots & \vdots & \ddots \\
F_n(t) & F_{n-1}(t) & F_{n-2}(t) & \cdots & F_0(t) \\
\vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}
\begin{pmatrix}
F_0(s) \\
F_1(s) & F_0(s) \\
F_2(s) & F_1(s) & F_0(s) \\
\vdots & \vdots & \vdots & \ddots \\
F_n(s) & F_{n-1}(s) & F_{n-2}(s) & \cdots & F_0(s) \\
\vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}
= 
\begin{pmatrix}
f_0 \\
f_1 \\
f_2 \\
\vdots \\
f_n \\
\vdots
\end{pmatrix}
\]
So \( u(x, 0) = f(x) \) and \( \frac{\partial u(x, t)}{\partial t} = g(x)u(x, t) \). The uniqueness of the solution in \( \mathbb{K}[t][[x]] \) is obvious and will be reproved, using Banach fixed point theorem, in the last section of this paper. \( \square \)

3. Delta-evolution equations and convolution polynomials

Some of the main classes of convolution families of polynomials, due to its relations with delta-operators and umbral calculus, are those obtained directly from polynomial sequences of binomial type or, equivalently, from basic sequences of polynomials for delta-operators, see [2] and [3] for all these definitions and relations.

One of the consequences of the first expansion Theorem in [2] or [3] is that for any delta-operator \( Q \) there is a power series \( q \), invertible for the composition operation, such that \( Q = q(D) \) where \( D : \mathbb{K}[x] \to \mathbb{K}[x] \) is the usual derivative on polynomials. So \( q(x) = \sum_{n \geq 1} q_n x^n \) with \( q_1 \neq 0 \).

Given a delta-operator \( Q : \mathbb{K}[t] \to \mathbb{K}[t] \) consider the sequence of basic polynomials \( (p_n(t))_{n \in \mathbb{N}} \) of \( Q \). So \( p_0(t) = 1, p_n(0) = 0 \) if \( n \geq 1 \) and \( Q(p_n(t)) = np_{n-1}(t) \). Since \( (p_n(t))_{n \in \mathbb{N}} \) is a polynomial sequence of binomial type we have:

**Proposition 5.** If \( Q \) and \( (p_n(t))_{n \in \mathbb{N}} \) are as described above and \( F_n(t) = \frac{p_n(t)}{n!} \). Then \( (F_n(t))_{n \in \mathbb{N}} \) is a convolution sequence with \( Q(F_n(t)) = F_{n-1}(t) \) and \( \sum_{n \geq 0} F_n(t)x^n = e^{q^{-1}(x)} \) where \( q^{-1} \) is the compositional inverse, in \( \mathbb{K}[[x]] \), of the series \( q \) such that \( q(D) = Q \).

The proof of this proposition is contained in [2] and [3]. In the situation described in the previous proposition we say that the convolution family, or sequence, \( (F_n(t))_{n \in \mathbb{N}} \) is associated, or related, to the delta-operator \( Q \).

**Definition 6.** Let \( u(x, t) \in \mathbb{K}[t][[x]] \). That is \( u(x, t) = \sum_{n \geq 0} r_n(t)x^n \) with \( r_n(t) \in \mathbb{K}[t] \). Let \( Q \) be any delta-operator. We define \( Q_t(u(x, t)) \) as:

\[
Q_t(u(x, t)) = \sum_{n \geq 0} Q(r_n(t))x^n \in \mathbb{K}[t][[x]]
\]

For example if \( Q = D \) the derivative then \( D_t(u(x, t)) = \sum_{n \geq 0} D(r_n(t))x^n \). So \( D_t : \mathbb{K}[t][[x]] \to \mathbb{K}[t][[x]] \) defines a linear operator. Usually we denote as \( \frac{\partial u}{\partial t} \) to \( D_t(u(x, t)) \).
In general if \( Q = q(D) \) with \( q(t) = \sum_{n \geq 1} q_n t^n \) with \( q_1 \neq 0 \) then \( Q_t = q(D_t) \) and we denote it by \( Q_t = \sum_{n \geq 1} q_n \frac{\partial^n}{\partial t^n} A(x^n) \).

By the same way if we have a linear operator \( A : \mathbb{K}[[x]] \to \mathbb{K}[[x]] \) which is continuous for the ultrametric \( d \). We define \( A_x : \mathbb{K}[[x,t]] \to \mathbb{K}[[x,t]] \) by the following way if \( u(x,t) = \sum_{n \geq 0} u_n(t)x^n \) with \( u_n(t) \in \mathbb{K}[[t]] \), then \( A_x(u(x,t)) = \sum_{n \geq 0} u_n(t)A(x^n) \).

The fact that \( A_x(u(x,t)) \in \mathbb{K}[[x,t]] \) is due to the continuity of \( A \) respect to \( d \). So given \( m \in \mathbb{N} \) there is an \( n_m \in \mathbb{N} \) such that \( d(A(x^p),0) \leq \frac{1}{2^m m!} \) for \( p \geq n_m \). Consequently \( A(x^p) = x^{m+1}(A_p(x)) \) with \( A_p \in \mathbb{K}[[x]] \) and the power \( x^m \) does not appear in \( A(x^p) \). So

\[
A_x(u(x,t)) = \sum_{k \geq 0} a_k(t)x^k
\]

where the coefficients \( a_k(t) \) are finite linear combinations of the coefficients \( \{u_n(t)\}_{n \in \mathbb{N}} \).

Moreover if in addition \( u(x, t) \in \mathbb{K}[t][[x]] \) then \( A_x(u(x,t)) \in \mathbb{K}[t][[x]] \).

With this ingredients we give our first result

**Theorem 7.** Let \( Q \) be a delta-operator on \( \mathbb{K}[t] \) such that \( Q = q(D) \) with \( q \in \mathbb{K}[[t]] \) invertible for composition, i.e. \( q(0) = 0, q'(0) \neq 0 \). Let \( A : \mathbb{K}[[x]] \to \mathbb{K}[[x]] \) be any linear contractive operator (respect to the ultrametric \( d \) in \( \mathbb{K}[[x]] \)). Then, the initial value problem in \( \mathbb{K}[t][[x]] \):

\[
\begin{cases}
Q_x(u(x,t)) = A_x(u(x,t)), \\
u(x,0) = f(x)
\end{cases}
\]

has the bivariate power series given by \( u(x,t) = e^{t\nu^{-1}(A)}(f) \) as a solution. Moreover if \( (F_n(t))_{n \in \mathbb{N}} \) is the convolution sequence associated to the delta-operator \( Q \), then

\[
e^{t\nu^{-1}(A)}(f) = \sum_{k \geq 0} F_k(t)A^k(f)
\]

where \( A^k \) is the \( k \)-fold composition.

**Proof.** Consider the bivariate power series \( u(x,t) = \sum_{k \geq 0} F_k(t)A^k(f) \). In fact \( u(x,t) \in \mathbb{K}[t][[x]] \) because \( A \) is contractive. \( F_k(t) = \frac{p_k(t)}{k!} \) where \( (p_k(t))_{k \in \mathbb{N}} \) is the basic sequence associated to the delta-operator \( Q \) so \( F_k(0) = 0 \) if \( k \geq 1 \) and \( F_0(t) \equiv 1 \), so \( u(x,0) = A^0(f) = 0 \) so \( u(0,t) = 0 \).
Moreover \( A_x(u(x,t)) = \sum_{k \geq 0} F_k(t) A^{k+1}\) and \( Q_t(u(x,t)) = \sum_{l \geq 0} Q(F_l(t)) A^l(f) \). Since \( Q(F_l(t)) = F_{l-1}(t) \) for \( l \geq 1 \) and \( Q(F_0(t)) = 0 \). So
\[
Q_t(u(x,t)) = \sum_{l \geq 1} F_{l-1}(t) A^l(f) = \sum_{k \geq 0} F_k(t) A^{k+1}(f)
\]
and the equality is proved.

Using Corollary 3 in page 15 of [3] we get that
\[
\sum_{k \geq 0} F_k(t)x^k = e^{tq^{-1}(x)}
\]
So, since \( q^{-1}(A) \) is a contractive linear operator, we have easily the following chain of equalities:
\[
e^{tq^{-1}(x)}(A) = e^{tq^{-1}(A)} = \sum_{k \geq 0} F_k(t)A^k
\]
Consequently
\[
e^{tq^{-1}(A)}(f) = \sum_{k \geq 0} F_k(t)A^k(f)
\]
and then this solution is given by
\[
u(x,t) = e^{tq^{-1}(A)}(f)
\]
In general the solution given above is not unique in a sufficiently large framework as we will point out in the last section.

**Example 8.** Find a solution of
\[
\begin{cases}
\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = g(x)u(x,t) \\
u(x,0) = f(x)
\end{cases}
\]
with \( g(0) = 0 \). In this case \( Q = D + D^2 = q(D) \) for \( q(x) = x + x^2 \). So \( q^{-1}(x) = \frac{-1 + \sqrt{1 - 4x}}{2} \).

We also have, for this case, \( A_x(f) = L_g(f) = g(x)f(x) \). Consequently
\[
e^{tq^{-1}(A)}(f) = f(x)e^{t\left(\frac{-1 + \sqrt{1 - 4g(x)}}{2}\right)}
\]
and hence our solution is
\[
u(x,t) = f(x)e^{t\left(\frac{-1 + \sqrt{1 - 4g(x)}}{2}\right)}
\]
Example 9. Find a solution of
\[
\begin{cases}
\left( \sum_{k \geq 1} \frac{1}{k!} \frac{\partial^k}{\partial t^k} \right) (u(x, t)) = g(x)u(x, t) \\
u(x, 0) = f(x)
\end{cases}
\]
with \(g(0) = 0\). So the equation is \(\sum_{k \geq 1} \frac{1}{k!} \frac{\partial^k}{\partial t^k} = g(x)u(x, t)\). In this case \(Q = q(D)\) where 
\(q(x) = e^x - 1\). So, \(q^{-1}(x) = \log(1 + x)\). Again \(A_x(f) = L_g(f) = g(x)f(x)\). According to
our theorem we have that a solution is given by
\[u(x, t) = f(x)(1 + g(x))t\]
or, see [1]:
\[u(x, t) = f(x) \sum_{n \geq 0} \left( \sum_{n \geq 0} \left\{ \begin{array}{c} n \\ k \end{array} \right\} t^k \frac{n!}{n!} g^n(x) + \cdots \right)\]
where the lower factorial polynomials appear. Note also that, in this case \(Q = q(D) = \Delta\),
where \(\Delta\) is the difference operator, see [2], so our example is the same as
\[
\begin{cases}
u(x, t + 1) - u(x, t) = g(x)u(x, t) \\
u(x, 0) = f(x)
\end{cases}
\]

Example 10. Consider the initial value problem
\[
\begin{cases}
\sum_{k \geq 0} \frac{(-1)^k}{k+1} \frac{\partial^{k+1}}{\partial x^{k+1}} = g(x)u(x, t) \\
u(x, 0) = f(x)
\end{cases}
\]
with \(g(0) = 0\). In this case the delta-operator is \(Q = q(D)\) where \(q(x) = \log(1 + x)\).
Consequently \(q^{-1}(x) = e^x - 1\). In this case our solution is
\[u(x, t) = f(x)e^{t(e^x-1)}\]
Following [1] we obtain
\[u(x, t) = f(x) \left( \sum_{n \geq 0} \frac{n!}{n!} g^n(x) \right)\]
where the Stirling number \(\left\{ \begin{array}{c} n \\ k \end{array} \right\}\) is the number of partitions of a set with \(n\)-elements into exactly \(k\) subsets.
3.1. **A special example: Using the Riordan group to solve certain evolution equations.** We shall start with the following example that can be solved using our last theorem.

**Example 11.** Solve

\[
\begin{cases}
\frac{\partial u}{\partial t} = x^2 \frac{\partial u}{\partial x} + xu(x, t) \\
u(x, 0) = f(x)
\end{cases}
\]

In this case \( Q = D, q(x) = x \) and \( A(f) = x^2 f' + xf \). It is easy to prove that the linear operator \( A \) is \( \frac{1}{2} \)-contractive. So the solution we found before is given by \( u(x, t) = e^{tA}(f) \).

The matrix associated to \( A \), in the sense of [14], is just

\[
A = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 2 & \cdots & 0 \\
0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & n-1 \\
0 & 0 & 0 & \cdots & 0 & n \\
\vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}
\]

the creation matrix, see [21], then

\[
e^{tA} = \begin{pmatrix}
1 \\
t \\
t^2 & 2t & 1 \\
t^3 & 3t^2 & 3t & 1 \\
\vdots & \vdots & \vdots & \ddots \\
t^n & {n \choose 1}t^{n-1} & \cdots & {n-1 \choose 1}t & 1 \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]

which is a Riordan array for any \( t \in \mathbb{K} \). In fact \( e^{tA} = T(1 \mid 1 - xt) \) in our notation used in [13, 14, 15, 16]. Consequently

\[
(e^{tA})(f) = T(1 \mid 1 - xt)(f) = \frac{1}{1 - xt} f \left( \frac{x}{1 - xt} \right)
\]
So our solution is

\[ u(x, t) = \frac{1}{1 - xt} f \left( \frac{x}{1 - xt} \right) \]

This last example is a particular one of a family of examples where the Riordan group is involved to give a closed symbolic formula for the solution. In this case the main tool we will use is an interesting preprint of Roland Bacher [7]. There, the author recognized the Lie algebra of the Riordan group. In the sequel we are going to recall some results in [7] but we will change the notation. Given two series \( \alpha = \sum_{n \geq 0} \alpha_n x^n \), \( \beta = \sum_{n \geq 0} \beta_n x^n \), with \( \alpha_0 = \beta_0 = 0 \), we denote by \( L_{\alpha, \beta} \) to the linear operator in \( \mathbb{K}[[x]] \) defined by

\[ L_{\alpha, \beta}(h) = \alpha h + x \beta h' \]

For any \( h \in \mathbb{K}[[x]] \) and where \( h' \) represents the derivative of the series \( h \). Note that if \( \beta = 0 \) we are in the case studied in the previous section. It is clear that \( L_{\alpha, \beta} \) is \( \frac{1}{2} \)-contractive, for the ultrametric \( d \), because \( \alpha_0 = \beta_0 = 0 \). The \( k \)-column, beginning at \( k = 0 \), of the matrix associated to \( L_{\alpha, \beta} \) is formed by the coefficients of the series \( x^k(\alpha + k\beta) \), i.e. the coefficients of the consecutive terms of the arithmetic-geometric progression, in \( \mathbb{K}[[x]] \), with first term \( \alpha \), rate \( x \) and difference \( \beta \). Following now Proposition 2 we have that

\[ e^{tL_{\alpha, \beta}} : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x]], d) \]

defines a linear isometry. Of special interest for us will be the series of the form \( e^{tL_{\alpha, \beta}}(1) \), i.e. the image of the series constantly 1.

Using our previous theorem and the results of Bacher in [7], essentially those contained in the sections 5 and 6, we have

**Proposition 12.** Let \( \alpha = \sum_{n \geq 0} \alpha_n x^n \), \( \beta = \sum_{n \geq 0} \beta_n x^n \) be two power series with \( \alpha_0 = \beta_0 = 0 \). Then the initial value problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= x \beta(x) \frac{\partial u}{\partial x} + \alpha(x) u(x, t) \\
u(x, 0) &= f(x)
\end{aligned}
\]

has a unique solution given by

\[ u(x, t) = e^{tL_{\alpha, \beta}} f(x e^{tL_{\beta, \beta}}(1)) \]
Remark 13. (1) Bacher in [7] describes some procedures to calculate $e^{tL_{\alpha,\beta}(1)}$ and $e^{tL_{\beta,\beta}(1)}$.

(2) Note that using our notation on Riordan arrays the solution to the problem above can be written as $u(x,t) = T(e^{tL_{\alpha,\beta}(1)}|e^{tL_{\beta,\beta}(1)})f(t)$.

(3) Note also that the previous example corresponds to the case $\alpha = \beta = x$.

(4) The uniqueness, and the corresponding result without the conditions $\alpha_0 = \beta_0 = 0$ will be obtained, in the last section at once.

(5) A more general result than that given in the previous proposition can be given. In fact:

The initial value problem in $\mathbb{K}[t][[x]]$

$$\begin{cases}
Q_t(u(x,t)) = x\beta(x)\frac{\partial}{\partial x} + \alpha(x)u(x,t) \\
u(x,0) = f(x)
\end{cases}$$

where $Q = q(D)$ is a delta-operator, has a solution in $\mathbb{K}[t][[x]]$ given by

$$u(x,t) = e^{tq^{-1}(L_{\alpha,\beta})}(f) = \sum_{k \geq 0} F_k(t)L_{\alpha,\beta}^k(f)$$

being $F_k(t)$ the convolution sequence associated to the delta-operator $Q$.

3.2. The bivariate convolution product and the symbolic solutions for some delta-evolution equations. In this final part of this section we concentrate ourselves in the discussion of the equations

$$Q_t(u(x,t)) = R_x(u(x,t))$$

where $Q$ and $R$ are both delta-operators.

What is the framework where the above equation makes sense? Of course if we have a power series $u(x,t) \in \mathbb{K}[[x,t]]$ such that, doing algebraic operations, one can describe $u(x,t)$ as an element in both $\mathbb{K}[t][[x]]$ and $\mathbb{K}[x][[t]]$, then the sense of the above equation is to apply $Q_t$ to the $\mathbb{K}[t][[x]]$ expression of $u(x,t)$ and to apply $R_x$ to the $\mathbb{K}[x][[t]]$ expression of $u(x,t)$.

Suppose that, as delta-operators, $Q = q(D)$ and $R = r(D)$ with $q$ and $r$ power series such that $q(0) = r(0) = 0$ and $q'(0) \neq 0$, $r'(0) \neq 0$ and $D$ is the derivative
operator. If $q$ and $r$ are polynomials then $Q_t(u(x,t))$ and $R_x(u(x,t))$ make sense for every $u(x,t) \in \mathbb{K}[x,t]$.

Let us describe a general symbolic framework to treat our problem:

Suppose that $Q : \mathbb{K}[x] \rightarrow \mathbb{K}[x]$ is a delta-operator and denote by $(F_k(x))_{k \in \mathbb{N}}$ the corresponding convolution family associated to $Q$. We denote by $\mathbb{K}[[F_k(x)]]$ to the set of all formal expressions $\sum_{k \geq 0} a_k F_k(x)$ with the equality relation $\sum_{k \geq 0} a_k F_k(x) = \sum_{k \geq 0} b_k F_k(x)$ if and only if $a_k = b_k$ for every $k \in \mathbb{N}$. We can define a natural structure of $\mathbb{K}$-vectorial space in $\mathbb{K}[[F_k(x)]]$. The delta-operator $Q$ acts on $\mathbb{K}[[F_k(x)]]$ as a derivative by means of the formula $Q(\sum_{k \geq 0} a_k F_k(x)) = \sum_{k \geq 1} a_k F_{k-1}(x)$. When $Q = D$ then $F_n(x) = \frac{x^n}{n!}$ and there is a natural identification between $\mathbb{K}[[x]]$ and $\mathbb{K}[[F_k(x)]]$ by the equality $\sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} n! a_n \frac{x^n}{n!}$.

Using analogous arguments one can also define the bivariate framework $\mathbb{K}[[F_k(x)), (G_k(t))]]$ for two convolution families $(F_k(x))$ and $(G_k(t))$. How can be described the elements in $\mathbb{K}[[F_k(x)), (G_k(t))]]$? Suppose that $\nu = (\nu_1, \nu_2) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $u(x,t) \in \mathbb{K}[[F_k(x)), (G_k(t))]]$ if and only if it is a formal expression of the form

$$u(x,t) = \sum_{\nu \in \mathbb{N}^2} a_{\nu} F_{\nu_1}(x) G_{\nu_2}(t)$$

where $a_{\nu} \in \mathbb{K}$ for every $\nu \in \mathbb{N}^2$.

When, in the above expression, $a_{\nu} = 0$ except for $\nu$ in a finite subset of $\mathbb{N}^2$, then we simply say that $u(x,t)$ is a polynomial and we represent the set of all this kind of elements as $\mathbb{K}[(F_k(x)), (G_k(t))]]$. In the case that both convolution families are associated to delta-operators, we create no confusion with this notation because since $\{F_k(x)\}_{k \in \mathbb{N}}$ and $\{G_k(t)\}_{k \in \mathbb{N}}$ are vectorial bases for $\mathbb{K}[x]$ and $\mathbb{K}[t]$, respectively, then there is a natural identification that allows as to say that $\mathbb{K}[(F_k(x)), (G_k(t))]] = \mathbb{K}[x,t]$.

If one want to treat more deeply these new frameworks, we are not going to do it now, one should follow the line described by L. Ferrari in [22].

One of the operations between convolution families described by Knuth in [1] is the following: given two convolution families $(F_n(x))$, $(G_n(x))$ if $H_n(x) = \sum_{k=0}^{n} F_k(x) G_{n-k}(x)$
then \((H_n(x))\) is a convolution family. In order to obtain our main result in this section we give a bivariate version of the above relation:

Suppose that \(F(x) = (F_k(x))_{k \in \mathbb{N}}\) and \(G(t) = (G_k(t))_{k \in \mathbb{N}}\) are convolutions families in independent variables \(x\) and \(t\) respectively, we define the bivariate convolution product as

\[
H(x, t) = F(x) \ast G(t) = (H_n(x, t))_{n \in \mathbb{N}} \quad \text{where} \quad H_n(x, t) = \sum_{k=0}^{n} F_k(x) G_{n-k}(t).
\]

Note that \(H_n(x, t)\) is a polynomial of degree \(n\) in two variables when \(F(x)\) and \(G(t)\) are convolution families associated to delta-operators.

Note also that when \(F = G\) then \(H(x, t) = F(x) \ast F(t) = (F_n(x + t))_{n \in \mathbb{N}}\) by the convolution condition. In this case given \(f(x) = \sum_{k \geq 0} f_k F_k(x) \in \mathbb{K}[[F_k(x)]]\), we write

\[
f(x + t) = \sum_{k \geq 0} f_k (F(x) \ast F(t))_k = \sum_{k \geq 0} f_k F_k(x + t) \in \mathbb{K}[[F_k(x), (F_k(t))]].
\]

As before, suppose that \(f(x) = \sum_{k \geq 0} f_k F_k(x) \in \mathbb{K}[[F_k(x)]]\). What should be the meaning of \(f(x - t)\)? Symbolically we should have \(f(x - t) = \sum_{k \geq 0} f_k F_k(x - t)\). Using now the convolution equality we have \(F_k(x - t) = F_k(x + (-t)) = \sum_{k \geq 0} f_k (F(x) \ast F(-t))_k\).

Call now \(J_k(t) = F_k(-t)\). Is \((J_k(t))\) a convolution sequence related to any delta-operator \(S\) supposing that \((F_k(t))\) is a convolution sequences related to a delta-operator \(Q^*\)? We answer this question in the following way

**Proposition 14.** Let \(Q : \mathbb{K}[t] \rightarrow \mathbb{K}[t]\) be a delta-operator being \((F_k(t))_{k \in \mathbb{N}}\) its associated convolution sequence. Suppose also that \(Q = q(D)\) where \(q(t) = \sum_{k \geq 1} q_k t^k\) with \(q_1 \neq 0\). Consider the sequence \((J_k(t) = F_k(-t))_{k \in \mathbb{N}}\). Then \((J_k(t))_{k \in \mathbb{N}}\) is a convolution sequence associated to the delta-operator \(Q^* = q(-D)\)

**Proof.** Using Corollary 3 in [3] we have that

\[
\sum_{k \geq 0} F_k(t) x^k = e^{t q^{-1}(x)}.
\]

Then \(\sum_{k \geq 0} J_k(t) x^k = e^{t(-q^{-1}(x))}\). It is obvious that \(h(x) = q(-x)\) is the compositional inverse of \(-q^{-1}(x)\).

Consequently, by Corollary 3 in [3] again, we get the result. \(\square\)

Note that \(D^* = -D\) and that \((Q^*)^* = Q\)

Now we define \(f(x-t) = \sum_{k \geq 0} f_k (F(x) \ast F(-t))_k = \sum_{k \geq 0} f_k F_k(x - t) \in \mathbb{K}[[F_k(x), (F_k(-t))]]\).
Another property that would be of interest in the future is:

**Proposition 15.** Let $Q : \mathbb{K}[t] \to \mathbb{K}[t]$ be a delta-operator being $(F_k(t))_{k \in \mathbb{N}}$ its associated convolution sequence. Then $-Q$ is a delta-operator with associated convolution sequence $((-1)^kF_k(t))_{k \in \mathbb{N}}$.

To understand the next result we have to explain something more:

Let $Q$ be a delta-operator in $\mathbb{K}[t]$ and suppose that $(F_n(t))_{n \in \mathbb{N}}$ is the convolution family associated to $Q$. Suppose also that $R$ is a delta-operator in $\mathbb{K}[x]$ being $(G_n(x))_{n \in \mathbb{N}}$ the convolution family associated to $R$. Take

$$u(x, t) = \sum_{\nu \in \mathbb{N}^2} a_{\nu} G_{\nu_1}(x) F_{\nu_2}(t) \in \mathbb{K}[[G_k(x)], (F_k(t))].$$

we understand that $Q_t(u(x, t)) = \sum_{\nu \in \mathbb{N}^2} a_{\nu} G_{\nu_1}(x) F_{\nu_2-1}(t)$ with the agreement $F_{-1} = 0$ and that $R_t(u(x, t)) = \sum_{\nu \in \mathbb{N}^2} a_{\nu} G_{\nu_1-1}(x) F_{\nu_2}(t)$ with $G_{-1} = 0$.

**Theorem 16.** Let $Q$ be a delta-operator in $\mathbb{K}[t]$ and suppose that $(F_n(t))_{n \in \mathbb{N}}$ is the convolution family associated to $Q$. Suppose also that $R$ is a delta-operator in $\mathbb{K}[x]$ being $(G_n(x))_{n \in \mathbb{N}}$ the convolution family associated to $R$. Let $g(x) = \sum_{k \geq 0} g_k G_k(x) \in \mathbb{K}[[G_k(x)]]$. Then the initial value problem

$$\begin{cases}
Q_t(u(x, t)) = R_x(u(x, t)), \\
u(x, 0) = g(x)
\end{cases}$$

has the symbol $u(x, t) = \sum_{k \geq 0} g_k H_k(x, t) \in \mathbb{K}[[G_k(x)], (F_k(t))]$ as solution, where $H_k(x, t) = \sum_{j=0}^{k} G_j(x) F_{k-j}(t)$ or, equivalently, $H(x, t) = G(x) F(t)$. In particular if $Q = R$, then the solution to the problem is given by $g(x + t)$. Finally if $Q = R^*$ then the solution is given by $g(x - t)$.

**Proof.** Consider $u(x, t) = \sum_{k \geq 0} g_k \sum_{j=0}^{k} G_j(x) F_{k-j}(t)$. Doing algebraic operations it can be easily proved that $u(x, t) = \sum_{k \geq 0} R^k(g(x)) F_k(t)$ where $R(g(x)) = \sum_{k \geq 1} g_k G_{k-1}(x)$ as described before and $R^k$ is the $k$-fold composition of $R$. Now $Q_t(u(x, t)) = \sum_{k \geq 1} R^k(g(x)) F_{k-1}(t)$. 


and \( R_x(u(x,t)) = \sum_{k \geq 0} R^{k+1}(g(x))F_k(t) \). Obviously \( Q_t(u(x,t)) = R_x(u(x,t)) \). Since \( F_k(0) = 0 \) if \( k \geq 1 \) and \( F_0(0) = 1 \) we have that the initial condition is also satisfied. □

Another symbolic result that can be easily proved is the following:

**Proposition 17.** Let \( A : \mathbb{K}[x] \rightarrow \mathbb{K}[x] \) be a continuous \( \mathbb{K} \)-linear function. Then \( \sum_{k \geq 0} F_k(t)A^k(f) \) is a solution of

\[
\begin{align*}
Q_t(u(x,t)) &= A_x(u(x,t)), \\
u(x,0) &= f(x)
\end{align*}
\]

in \( \mathbb{K}[[x]][[F_k(t)]] \). In particular \( u(x,t) = \sum_{k \geq 0} \frac{A^k(f)}{k!} t^k \in \mathbb{K}[[x,t]] \) is a solution of

\[
\begin{align*}
\frac{\partial u}{\partial t} &= A_x(u(x,t)), \\
u(x,0) &= f(x)
\end{align*}
\]

**Remark 18.** \( u(x,t) \in \mathbb{K}[[x]][[F_k(t)]] \) if and only if it is a symbolic expression of the form \( u(x,t) = \sum_{k \geq 0} u_k(x)F_k(t) \) where \( u_k(x) \in \mathbb{K}[[x]] \) for any \( k \in \mathbb{N} \).

Using different convolution sequences, associated to delta-operators, extracted from [1], [2] or [3], one can put many concrete examples related to the previous theorem:

**Example 19.** Compute a solution of

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial u(x,t)}{\partial x}, \\
u(x,0) &= f(x)
\end{align*}
\]

Using the two results above we have:

\[
u(x,t) = \sum_{k \geq 0} \frac{f^{(k)}(x)}{k!} t^k = f(x + t) \in \mathbb{K}[[x,t]]\]

In fact we can extend the shift operator \( E^t(p(x)) = p(x + t) \) for polynomials to power series \( E^t(f(x)) = f(x + t) \). In this sense, since \( A = D \) in this case, we still have that the solution is given by

\[
u(x,t) = (e^{tD})(f) = f(x + t)\]

because \( e^{tD} = E^t \) on polynomials. Although \( D \) is an expansive \( \mathbb{K} \)-linear map in \( \mathbb{K}[[x]] \).
Remark 20. In future work we will analyze the three basic equation of the Mathematical Physics, heat equation, wave equation and Laplace equation in the general context of delta-operators. In fact we have more possibilities. We propose to analyze the following equations:

\[ Q_t = Q_{xx}, \quad Q^*_t = Q_{xx}, \quad Q_{tt} = Q_{xx}, \quad Q_{tt}^* = -Q_{xx} \]

and so on. Note that when \( Q = D \) all of them reduce to the three classical one because \( D^* = -D \)

Only to point out that the above theorem, which is a symbolic result, can be used efficiently we develop the following examples

Example 21. Find a solution of

\[
\begin{cases}
\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t^3} + \frac{\partial^4 u}{\partial t^4} = \frac{\partial u}{\partial x} + \frac{1}{2!} \frac{\partial^2 u}{\partial x^2} + \frac{1}{3!} \frac{\partial^3 u}{\partial x^3} + \frac{1}{4!} \frac{\partial^4 u}{\partial x^4}, \\
u(x,0) = 1 + x^4
\end{cases}
\]

Of course our choice of this particular example is to avoid as much computations as possible but it is still significant.

We are under the conditions imposed in the above theorem. In fact our equation is of the form \( Q_t = R_x \) for \( Q = q(D), \ R = r(D) \). Where \( q(t) = t - t^2 + t^3 - t^4 \) and \( r(x) = x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 \) and consequently both, \( Q \) and \( R \), are delta-operators. The next steps are to find the corresponding convolution families associated to \( Q \) and \( R \), to put the initial condition in coordinates relative to the convolution family associated to \( R \) and finally to apply the formula for the solution in the above theorem.

In order to avoid some of these computations we will interpret our problem in terms of different delta-operators \( \tilde{Q}, \tilde{R} \) for which the corresponding associated convolution families are either previously known for us or easily computable. The procedure that we will describe can be used for any polynomial initial condition of degree less than or equal to 4, instead of \( 1 + x^4 \).

Following [2] or [3], any delta-operator transforms a polynomial of degree \( n \) into a polynomial of degree \( n - 1 \). So if we consider \( \tilde{q}(x) = \frac{e^x}{1+x^2} \), \( \tilde{r}(x) = e^x - 1 \) and \( \tilde{Q} = \tilde{q}(D) \), \( \tilde{R} = \tilde{r}(D) \), we have \( \tilde{Q}_t(u(x,t)) = Q_t(u(x,t)) \), \( \tilde{R}_x(u(x,t)) = R_x(u(x,t)) \) for any
$u(x, t) \in \mathbb{K}_4[x, t]$, being $\mathbb{K}_4[x, t]$ the bivariate polynomials of degree less than or equal to 4. As we know, see [1], the first five elements in the convolution sequence associated to $\tilde{R}$ are: $G_0(x) = 1$, $G_1(x) = x$, $G_2(x) = \frac{x(x-1)}{2!}$, $G_3(x) = \frac{x(x-1)(x-2)}{3!}$, $G_4(x) = \frac{x(x-1)(x-2)(x-3)}{4!}$. Now

$$1 + x^4 = G_0(x) + G_1(x) + 14G_2(x) + 36G_3(x) + 24G_4(x)$$

We only need to calculate the first five terms $F_0(t)$, $F_1(t)$, $F_2(t)$, $F_3(t)$, $F_4(t)$ of the convolution family associated to $\tilde{Q}$. As a consequence of a general result proved in our paper [16] relating convolution families with Riordan matrices, the family $(F_k(t))$ satisfies

$$T(1 - x | 1 - x)(e^{tx}) = \sum_{k \geq 0} F_k(t)x^k$$

in our notation on Riordan arrays. But

$$T(1 - x | 1 - x) = \begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 1 & 4 & 6 & 4 & 1 \\
& \vdots & : & : & : & : & : & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ t \\ \frac{t^2}{2!} \\ \frac{t^3}{3!} \\ \frac{t^4}{4!} \\
& \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} F_0(t) \\ F_1(t) \\ F_2(t) \\ F_3(t) \\ F_4(t) \end{pmatrix}$$

This means that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 1 & 0 & \cdots \\ 0 & 1 & 3 & 3 & 1 & \cdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ t \\ \frac{t^2}{2!} \\ \frac{t^3}{3!} \\ \frac{t^4}{4!} \\
& \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} F_0(t) \\ F_1(t) \\ F_2(t) \\ F_3(t) \\ F_4(t) \end{pmatrix}$$

Consequently, $F_0(t) = 1$, $F_1(t) = t$, $F_2(t) = t + \frac{t^2}{2!}$, $F_3(t) = t + \frac{2t^2}{2!} + \frac{t^3}{3!}$, $F_4(t) = t + \frac{3t^2}{2!} + \frac{3t^3}{3!} + \frac{t^4}{4!}$. According to our theorem we have that

$$u(x, t) = 1 + x + t + 14 \left[ t + \frac{t^2}{2!} + xt + \frac{x(x-1)}{2!} \right] + 36 \left[ t + t^2 + \frac{t^3}{3!} + x(t + \frac{t^2}{2!}) + \frac{x(x-1)}{2!}t + \frac{x(x-1)(x-2)}{3!} \right] +$$

$$+ 24 \left[ t + \frac{3t^2}{2} + \frac{t^3}{3!} + x(t + t^2 + \frac{t^3}{3!}) + \frac{x(x-1)}{2!}(t + \frac{t^2}{2!}) + \frac{x(x-1)(x-2)}{3!}t + \frac{x(x-1)(x-2)(x-3)}{4!} \right]$$
is a solution of our initial value problem.

Example 22. (Computing solutions for some functional-evolution equations)

In this family of examples we fix our first delta-operator, in the previous theorem, as $Q_t(u(x, t)) = \frac{\partial u(x, t)}{\partial t}$. Then the convolution family associated is always $F_n(t) = \frac{t^n}{n!}$. We also choose some delta-operators in [2], pages 179-180, to play the role of $R$. We always consider as initial condition $f(x) = 1 + x^4$ as before.

(i) Find a solution of

$$
\begin{align*}
\left\{ \frac{\partial u(x, t)}{\partial t} = u(x + 1, t) - u(x, t) \\
u(x, 0) = 1 + x^4
\right. \\
\end{align*}
$$

In this case the delta-operator $R : \mathbb{K}[x] \rightarrow \mathbb{K}[x]$ is the difference operator defined by $R(p(x)) = p(x + 1) - p(x)$, in the notation in [2], whose associated convolution family is given by $G_n(x) = \frac{x(x-1)\cdots(x-n+1)}{n!}$. As in the previous example we have $1 + x^4 = G_0(x) + G_1(x) + 14G_2(x) + 36G_3(x) + 24G_4(x)$.

Using now our theorem in this section we obtain the following solution

$$
u(x, t) = 1 + t + x + 14 \left[ \frac{t^2}{2!} + xt + \frac{x(x-1)}{2!} \right] + 36 \left[ \frac{t^3}{3!} + \frac{t^2}{2!} + \frac{x(x-1)}{3!} t + \frac{x(x-1)(x-2)}{4!} \right] + 24 \left[ \frac{t^4}{4!} + \frac{t^3}{3!} + \frac{x(x-1)}{2!} t^2 + \frac{x(x-1)(x-2)}{3!} t + \frac{x(x-1)(x-2)(x-3)}{4!} t \right]
$$

(ii) Find a solution of

$$
\begin{align*}
\left\{ \frac{\partial u(x, t)}{\partial t} = u(x, t) - u(x-1, t) \\
u(x, 0) = 1 + x^4
\right. \\
\end{align*}
$$

In this case the delta-operator $R : \mathbb{K}[x] \rightarrow \mathbb{K}[x]$ is the backward difference operator defined by $R(p(x)) = p(x) - p(x - 1)$. The associated convolution sequence is given by $G_0(x) = 1$, $G_n(x) = \frac{x(x+1)\cdots(x+n-1)}{n!}$ for $n \geq 1$. All of we need is to find the coordinates $1 + x^4 = \lambda_0 G_0(x) + \lambda_1 G_1(x) + \lambda_2 G_2(x) + \lambda_3 G_3(x) + \lambda_4 G_4(x)$ of the initial condition with respect to the base $(G_n(x))$ and use again our theorem.
Find a solution of
\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial u(x+a,t)}{\partial x} \\
u(x,0) &= 1 + x^4
\end{align*}
\]
where \(a \in \mathbb{K}\).

Here the delta-operator \(R : \mathbb{K}[x] \rightarrow \mathbb{K}[x]\) is the Abel operator defined by
\[
R(p(x)) = p'(x + a).
\]
The associated convolution sequence is given by \(G_n(x) = \frac{x(x - an)^{n-1}}{n!}\) for \(n \geq 1\). Moreover
\[
1 + x^4 = G_0(x) + 52a^3G_1(x) + 48a^2G_2(x) + 72aG_3(x) + 24G_4(x)
\]
So our solution is
\[
u(x,t) = 1 + 52a^3(t + x) + 48a^2 \left[ \frac{t^2}{2!} + xt + \frac{x(x - 2a)}{2!} \right] + 72a \left[ \frac{t^3}{3!} + \frac{t^2}{2!} + \frac{x(x - 2a)}{2!} t + \frac{x(x - 3a)^2}{3!} \right] + 24 \left[ \frac{t^4}{4!} + \frac{t^3}{3!} \frac{x(x - 2a)}{2!} + \frac{x(x - 3a)^2}{3!} t + \frac{x(x - 4a)^3}{4!} \right]\]

Find a solution of
\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= -\int_0^\infty e^{-s} \frac{\partial u(x+s,t)}{\partial x} ds \\
u(x,0) &= 1 + x^4
\end{align*}
\]
Only to say that in this case the delta-operator \(R : \mathbb{K}[x] \rightarrow \mathbb{K}[x]\) is the Laguerre operator defined by \(R(p(x)) = r(D)(p(x)) = -\int_0^\infty e^{-s} p'(x+s) ds\) where \(r(x) = \frac{x}{x-1}\). The associated convolution sequence is given by \(G_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n-1}{k-1} x^k\).
The remaining part follows as in the previous examples.

All unexplained results used in this example can be found in [2]

Now we are going to point out a result, which is a corollary of the theorem in this subsection, giving us solutions for any functional-evolution equations of the form
\[
\frac{\partial u(x,t)}{\partial t} = R_x(u(x,t))
\]
where \(R\) is a delta-operator. This is the particular case of our theorem when \(Q = D\). The easy proof is left to the reader. This corollary can help in the previous example to
get a more compact formula for the solutions and to get quickly symbolic formulas for the solution for any initial condition.

**Corollary 23.** Let \( R: \mathbb{K}[x] \rightarrow \mathbb{K}[x] \) be a delta-operator and suppose that \((G_n(x))_{n \in \mathbb{N}}\) is the convolution family associated to \( R \). Fix any \( f(x) = \sum_{n \geq 0} f_n G_n(x) \in \mathbb{K}[[G_n(x)]] \).

Denote again by \( f(t) \in \mathbb{K}[[t]] \) the series \( f(t) = \sum_{n \geq 0} f_n \frac{t^n}{n!} \). Then, the initial value problem

\[
\frac{\partial u(x,t)}{\partial t} = R_x(u(x,t)) \\
u(x,0) = f(x)
\]

has the series \( u(x,t) = \sum_{n \geq 0} f^{(n)}(t) G_n(x) \in \mathbb{K}[[t]][[G_n(x)]] \) as a solution, where \( f^{(n)}(t) \) represents the usual \( n \)-th derivative of \( f(t) \) in \( \mathbb{K}[[t]] \).

### 3.3. Computing solutions for the Rodrigues-evolution equation induced by a delta-operator.

Let \( Q: \mathbb{K}[x] \rightarrow \mathbb{K}[x] \) be a delta-operator such that \( Q = q(D) \) where \( D : \mathbb{K}[x] \rightarrow \mathbb{K}[x] \) is the usual derivative on polynomials. So, \( q(x) = \sum_{n \geq 1} q_n x^n \) with \( q_1 \neq 0 \).

Suppose that \( F_Q = (F_n(x))_{n \in \mathbb{N}} \) is the convolution family associated to \( Q \).

From [2], page 192, we extract the definition of the Pincherle derivative, \( Q' \), of \( Q \) as a linear operator \( Q' : \mathbb{K}[x] \rightarrow \mathbb{K}[x] \) defined, using the notation in [2] page 179, by:

\[
Q'(p(x)) = Q(xp(x)) - xQ(p(x))
\]

As proved in [2], \( Q' \) can be expanded into a series in the derivative. In fact \( Q'(p(x)) = q'(D)(p(x)) = \left( \sum_{n \geq 1} n q_n D^{n-1} \right)(p(x)) \). \( Q' \) is a linear isomorphism and the inverse \((Q')^{-1}\) is defined by \((Q')^{-1} = \frac{1}{q'}(D)\). Note that \( q' \) is invertible for the Cauchy product because \( q'(0) = q_1 \neq 0 \). In fact

\[
(Q')^{-1} = \frac{1}{q'}(D) = \sum_{n \geq 0} d_n D^n
\]

where

\[
d_n = -\frac{2q_2}{q_1} d_{n-1} - \frac{3q_3}{q_1} d_{n-2} - \cdots - \frac{(n+1)q_{n+1}}{q_1} d_0
\]

for \( n \geq 1 \), \( d_0 = \frac{1}{q_1} \).
From Theorem 4 in [2], pages 193-194 which is in fact the so called Rodrigues-type formula., we get that
\[ x(Q)'^{-1}(F_k(x)) = (k + 1)F_{k+1}(x) \] for every \( k \in \mathbb{N} \). This formula induces a natural linear transformation
\[ x(Q)'^{-1} : \mathbb{K}[[t]][[F_k(x)]] \rightarrow \mathbb{K}[[t]][[F_k(x)]] \]
given by \( x(Q)'^{-1}\left(\sum_{n \geq 0} u_n(t)F_n(x)\right) = \sum_{n \geq 0} (n + 1)u_n(t)F_{n+1}(x) \) for any \( u(x, t) = \sum_{n \geq 0} u_n(t)F_n(x) \in \mathbb{K}[[t]][[F_k(x)]] \).

For any \( f(x) = \sum_{n \geq 0} f_n F_n(x) \in \mathbb{K}[[F_k(x)]] \), we denote by \( \tilde{f}(x) \in \mathbb{K}[[x]] \) to the power series \( \tilde{f}(x) = \sum_{n \geq 0} f_n x^n \). Note that if \( Q = D \) then \( F_n(x) = \frac{x^n}{n!} \). So if \( f(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!} \), then
\[ \tilde{f}(x) = \sum_{n \geq 0} f_n x^n. \]

Another notation we need is the following: Suppose that \( v(x, t) = \sum_{n \geq 0} v_n(t)x^n \in \mathbb{K}[[x, t]] \), then \( F_Q(v(x, t)) = \sum_{n \geq 0} v_n(t)F_n(x) \in \mathbb{K}[[t]][[F_k(x)]] \). With all these notations we have the following result where the Pascal Triangle is involved

Theorem 24. Let \( Q : \mathbb{K}[x] \rightarrow \mathbb{K}[x] \) be any delta-operator and suppose that \( F_Q = (F_n(x))_{n \in \mathbb{N}} \) is the convolution family associated to \( Q \). Then, the initial value problem
\[
\begin{cases}
\frac{\partial u(x, t)}{\partial t} = x(Q)'^{-1}(u(x, t)) \\
u(x, 0) = f(x) = \sum_{n \geq 0} f_n F_n(x)
\end{cases}
\]
in \( K[[t]][[F_k(x)]] \) has \( u(x, t) = F_Q\left(\frac{1}{1-x} \tilde{f}\left(\frac{x}{1-x}\right)\right) \) as a solution. Moreover \( u(x, t) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} f_k t^{n-k}\right) F_n(x) \)

Proof. As we saw at the beginning of the subsection 3.1, \( v(x, t) = \frac{1}{1-x} \tilde{f}\left(\frac{x}{1-x}\right) \) is a solution of
\[
\begin{cases}
\frac{\partial v}{\partial t} = x^2 \frac{\partial v}{\partial x} + xv(x, t) \\
v(x, 0) = \tilde{f}(x)
\end{cases}
\]
in \( \mathbb{K}[[x,t]] \) but now

\[
\frac{\partial}{\partial t} \left( \mathcal{F}_Q \left( \frac{1}{1 - xt} \tilde{f} \left( \frac{x}{1 - xt} \right) \right) \right) = \mathcal{F}_Q \left( \frac{\partial}{\partial t} \left( \frac{1}{1 - xt} \tilde{f} \left( \frac{x}{1 - xt} \right) \right) \right)
\]

So, by linearity

\[
\frac{\partial u(x,t)}{\partial t} = \mathcal{F}_Q(x^2 \frac{\partial v}{\partial x}) + \mathcal{F}_Q(xv(x,t))
\]

If we write \( v(x,t) = \sum_{n \geq 0} v_n(t)x^n \), then \( \mathcal{F}_Q(x^2 \frac{\partial v}{\partial x}) = \sum_{n \geq 1} nv_n(t)F_{n+1}(x) \) and \( \mathcal{F}_Q(xv(x,t)) = \sum_{n \geq 0} v_n(t)F_{n+1}(x) \). Consequently \( u(x,t) \) satisfies the equation and, obviously, the initial condition. The last part follows easily because \( v(x,t) = T(1 | 1 - xt)(\tilde{f}) \) and

\[
T(1 | 1 - xt) = P^t
\]

where \( P \) is the Pascal Triangle.

We refer to the equation

\[
\frac{\partial u(x,t)}{\partial t} = x(Q')^{-1}(u(x,t))
\]

as the \textit{Rodrigues-evolution equation} induced by the delta-operator \( Q \).

For example if \( Q = D \) then the equation converts to \( \frac{\partial u(x,t)}{\partial t} = xu(x,t) \), because \( Q' \) is the identity. Moreover the corresponding convolution sequence is \( \mathcal{F}_D = \left( \frac{x^n}{n!} \right)_{n \in \mathbb{N}} \). It is obvious, and will be reproved in a more general framework in the next section, that the problem

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u(x,t)}{\partial t} = xu(x,t) \\
u(x,0) = f(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}
\end{array} \right.
\end{align*}
\]

has a unique solution. But this solution can be written, using our previous theorem, as

\[
u(x,t) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} f_k t^{n-k} \right) x^n.
\]

Moreover it is obvious, and was also computed in our section 2, that the solution is \( u(x,t) = f(x)e^{tx} \). So, when \( f_0 \neq 0 \), we obtain as a consequence the following

\[
\]

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Corollary 25. A sequence of polynomials \((A_n(t))_{n \in \mathbb{N}}\) is an Appel sequence if and only if there is a sequence \((f_k)_{k \in \mathbb{N}}\) in the field \(K\), with \(f_0 \neq 0\) such that

\[
A_n(t) = \sum_{k=0}^{n} \binom{n}{k} f_k t^{n-k} \frac{n!}{n!}
\]

The definition of Appel sequence that we are using is just that in [20], page 18.

For any power series \(g(x) = \sum_{n \geq 0} g_n x^n\), with \(g_0 \neq 0\) the functional equation

\[
\frac{\partial u}{\partial t} = x \sum_{k \geq 0} g_k \frac{\partial^k u}{\partial x^k}
\]

where \(\frac{\partial^0 u}{\partial x^0}\) is the identity, represents a Rodrigues-evolution equation for a suitable delta-operator. In fact this operator is \(Q = q(D)\) where \(q(x) = \int_{x=0}^{x} \frac{1}{g(x)}\), where

\[
\int_{x=0}^{x} \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} \frac{a_n}{n+1} x^{n+1}.
\]

Example 26. Find a solution of

\[
\begin{array}{l}
\frac{\partial u(x, t)}{\partial t} = -x \left( u(x, t) - 2 \frac{\partial u(x, t)}{\partial x} + \frac{\partial^2 u(x, t)}{\partial x^2} \right) \\
u(x, 0) = 1 + x^2
\end{array}
\]

Note that the right part in the equation can be written as \(x \left( ( -1 + 2x - x^2)(D_x) \right) (u(x, t))\).

So, consider the delta-operator \(Q\) such that \(Q = q(D)\) and \(\frac{1}{q(x)} = -(1 - x)^2\). Consequently \(q(x) = \frac{x}{x-1}\) and \(Q\) is the Laguerre operator described before. The associated convolution sequence is given by \(F_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n-1}{k-1} x^k\). So our equation is the Rodrigues-evolution equation induced by the Laguerre operator. Since

\[
1 + x^2 = F_0(x) - 2F_1(x) + 2F_2(x).
\]

We get that our solution is

\[
u(x, t) = F_Q \left( \frac{1}{1 - xt} - 2 \frac{x}{(1 - xt)^2} + 2 \frac{x^2}{(1 - xt)^3} \right)
\]

or

\[
u(x, t) = 1 - x(t-2) + \sum_{n \geq 2} (t^n - 2nt^{n-1} + n(n-1)t^{n-2}) \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n-1}{k-1} x^k
\]
4. Uniqueness of solutions for the non-expansive linear functional-evolution equation

Although the general things that we are going to describe now can be developed in the general framework of multivariate power series, we focus on the bivariate case which is the framework of this paper.

Let \( K \) be a field (of characteristic zero). Denote by \( \mathbb{N} \) the set of non-negative integers. Recall that a bivariate power series \( f \in K[[x, t]] \) is a formal expression \( f = \sum_{\nu \in \mathbb{N}^2} f_\nu x^{\nu_1} t^{\nu_2} \) where \( \nu = (\nu_1, \nu_2) \) and \( f_\nu \in K \). We denote, as usual, \( |\nu| = \nu_1 + \nu_2 \).

Given \( k \in \mathbb{N} \) and the series \( f \) as above we denote by \( T_k(f) \) the Taylor polynomial of order \( k \), that is \( T_k(f) = \sum_{|\nu| \leq k} f_\nu x^{\nu_1} t^{\nu_2} \).

Recall that the order of a series \( f = \sum_{\nu \in \mathbb{N}^2} f_\nu x^{\nu_1} t^{\nu_2} \), denoted by \( \omega(f) \), is the smallest integer \( p \geq 0 \) such that \( f_\nu \neq 0 \) for some \( \nu \in \mathbb{N}^2 \) with \( |\nu| = p \), provided that some \( f_\nu \) is \( \neq 0 \). Otherwise, that is, if \( f = 0 \), we write \( \omega(f) = +\infty \). It is easy to see that

\[
\omega(f + g) \geq \min\{\omega(f), \omega(g)\} \quad \omega(f \cdot g) = \omega(f) + \omega(g)
\]

The next observation is important for our purposes (\( \mathbb{R}_+ \) represents the non-negative real numbers)

**Proposition 27.** The map \( d : K[[x, t]] \times K[[x, t]] \to \mathbb{R}_+ \) defined by \( d(f, g) = \frac{1}{2^\omega(f - g)} \), is a complete ultrametric on \( K[[x, t]] \). Moreover \( d(f, g) \leq \frac{1}{2^\omega(f - g)} \) if and only if \( T_k(f) = T_k(g) \).

Finally the sum and product of series are continuous if we consider the corresponding product topology in \( K[[x, t]] \times K[[x, t]] \).

**Proof.** The proofs of all facts above are easy consequences of the properties of the order of a series. Let us choose two of them to prove. Firstly the strong triangle inequality, which defines the ultrametric among metrics.
Let $f, g, h \in \mathbb{K}[[x, t]]$, then $\omega(f - h) = \omega((f - g) + (g - h)) \geq \min \{\omega(f - g), \omega(g - h)\}$ consequently $d(f, h) = \frac{1}{2^{v(t/\nu)}} \leq \max \{\frac{1}{2^{v(t/\nu)}}, \frac{1}{2^{v(t/\nu)}}\} = \max \{d(f, g), d(g, h)\}$.

Finally we are going to prove completeness. Suppose that $\{h_n\}_{n \in \mathbb{N}}$ is a Cauchy sequences in $(\mathbb{K}[[x, t]], d)$. For any $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ such that $n_k < n_{k+1}$ and $d(h_n, h_m) < \frac{1}{2^k}$ for $n, m \geq h_k$. Consider the, well defined, series $h$ where $T_l(h) = T_l(h_{n_k})$ for $0 \leq l \leq k$. It is clear that $\{h_n\} \to h$ in $(\mathbb{K}[[x, t]], d)$.

**Remark 28.** Note that if $f \in \mathbb{K}[[x, t]]$ then $\lim_{k \to \infty} T_k(f) = f$ in $(\mathbb{K}[[x, t]], d)$ and then the set of polynomials $\mathbb{K}_t[x, t]$ is, topologically, dense in the space of series. The induced metric $d$ on $\mathbb{K}_t[x, t]$ is uniformly discrete, where the subscript $t$ means ”degree less or equal to $t$“.

At the beginning of the second section of this paper we used an ultrametric, denoted also by $d$, in the univariate framework $\mathbb{K}[[x]]$. It creates no confusion because there is a natural isometrical embedding of $(\mathbb{K}[[x]], d)$ into $(\mathbb{K}[[x, t]], d)$. In fact given a univariate power series $h = \sum_{n \geq 0} h_n x^n$ we can identify this with the bivariate series $\tilde{h} = \sum_{\nu \in \mathbb{N}^2} \tilde{h}_\nu x^{\nu_1} t^{\nu_2}$ where $\tilde{h}_\nu = 0$ if $\nu_2 \neq 0$ and $\tilde{h}_\nu = h_{\nu_1}$ otherwise where $\nu = (\nu_1, \nu_2)$. In this sense we say that $\mathbb{K}[[x]] \subset \mathbb{K}[[x, t]]$.

Suppose also that we have a $\mathbb{K}$-linear continuous function function $L : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x]], d)$, then $L$ has a natural continuous linear extension $L_x : (\mathbb{K}[[x, t]], d) \rightarrow (\mathbb{K}[[x, t]], d)$ given by

$$L_x(u(x, t)) = \sum_{\nu \in \mathbb{N}^2} u_\nu L(x^{\nu_1}) t^{\nu_2}$$

for any series $u(x, t) = \sum_{\nu \in \mathbb{N}^2} u_\nu x^{\nu_1} t^{\nu_2} \in \mathbb{K}[[x, t]]$. In order to see only that $L_x(u(x, t))$ is well defined we have to prove that $\sum_{\nu \in \mathbb{N}^2} u_\nu L(x^{\nu_1}) t^{\nu_2} \in \mathbb{K}[[x, t]]$. We can write

$$\sum_{\nu \in \mathbb{N}^2} u_\nu L(x^{\nu_1}) t^{\nu_2} = \sum_{\nu_2 \geq 0} (\sum_{\nu_1 \geq 0} u_{(\nu_1, \nu_2)} L(x^{\nu_1})) t^{\nu_2}.$$

By the continuity and $\mathbb{K}$-linearity of $L : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x]], d)$, we have that

$$\sum_{\nu_2 \geq 0} (\sum_{\nu_1 \geq 0} u_{(\nu_1, \nu_2)} L(x^{\nu_1})) t^{\nu_2} = \sum_{\nu_2 \geq 0} L(\sum_{\nu_1 \geq 0} u_{(\nu_1, \nu_2)} x^{\nu_1}) t^{\nu_2}$$
and then obviously \( L_x(u(x, t)) \in \mathbb{K}[[x, t]]. \) The continuity follows from analogous arguments. In fact with a little more care one can prove:

**Proposition 29.** \( L : (\mathbb{K}[[x]], d) \longrightarrow (\mathbb{K}[[x]], d) \) be a \( \mathbb{K} \)-linear map such that there is \( \alpha \in \mathbb{R}_+ \) with \( d(L(f), L(g)) \leq \alpha d(f, g) \), i.e. \( \alpha \)-lipschitz. Then

\[
L_x : (\mathbb{K}[[x]], d) \longrightarrow (\mathbb{K}[[x]], d)
\]

is also \( \alpha \)-lipschitz.

Note that only the case when \( \alpha = 2^k \) for \( k \in \mathbb{Z} \) needs to be proved.

We call \( L_x \) as the *partial x-extension* of \( L \). Of course one can define the partial \( t \)-extension by the analogous way (and we represent it by \( L_t \)). As used along this paper, the partial derivatives are the corresponding partial extensions of the usual derivative. By the same way we can define the *partial integration*, or the *partial Volterra operator* in \( \mathbb{K}[[x, t]] \), in the following way: Given \( u(x, t) = \sum_{\nu \in \mathbb{N}^2} u_{\nu} x^{\nu_1} t^{\nu_2} \), then we define

\[
\mathcal{V}_t(u(x, t)) = \int_0^t u(x, t) = \sum_{\nu \in \mathbb{N}^2} \frac{u_{\nu}}{\nu_2 + 1} x^{\nu_1} t^{\nu_2 + 1}
\]

and we have

**Corollary 30.** The *partial Volterra operator*

\[
\mathcal{V}_t : (\mathbb{K}[[x, t]], d) \longrightarrow (\mathbb{K}[[x, t]], d)
\]

is \( \frac{1}{2} \)-contractive for the ultrametric \( d \).

Recall that a map \( M : (\mathbb{K}[[x, t]], d) \longrightarrow (\mathbb{K}[[x, t]], d) \) is said to be *non-expansive* if \( d(M(u), M(v)) \leq d(u, v) \) for every \( u, v \in \mathbb{K}[[x, t]]. \)

Our main result is the following

**Theorem 31.** Let \( M : (\mathbb{K}[[x, t]], d) \longrightarrow (\mathbb{K}[[x, t]], d) \) be a non-expansive map. Suppose that \( f \in \mathbb{K}[[x]] \) is any power series. Then the initial value problem

\[
\begin{cases}
\frac{du}{dt} = M(u(x, t)) \\
u(x, 0) = f(x)
\end{cases}
\]
has a unique solution. Moreover if \( M = M_x \) is the partial \( x \)-extension of a non-expansive \( \mathbb{K} \)-linear map \( M : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x]], d) \), then the solution is given by \( u(x, t) = \sum_{n \geq 0} M^n(f) \frac{t^n}{n!} \).

**Proof.** One gets the proof by a standard use of Banach’s fixed point Theorem in this context. In fact \( u(x, t) \) is a solution of the above problem if and only if \( u(x, t) = f(x) + V_t(M(u(x, t))) \) where \( V_t \) is the partial Volterra operator. If we consider the map \( F : (\mathbb{K}[[x, t]], d) \rightarrow (\mathbb{K}[[x, t]], d) \) given by \( F(u(x, t)) = f(x) + (V_t \circ M)(u(x, t)) \) we have that \( F \) is \( \frac{1}{2} \)-contractive and, consequently, the solution is unique. Suppose now that that \( M = M_x \) is the partial \( x \)-extension of a non-expansive \( \mathbb{K} \)-linear map \( M : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x]], d) \), then \( M \) is non-expansive (using the above results). Consequently \( F \) is contractive. Let us iterate \( F \) beginning at the series 0. So \( F(0) = f(x) \), because \( V_t \circ M \) is linear and \( M(f) \in \mathbb{K}[[x]] \subset \mathbb{K}[[x, t]] \). Then \( M(f) \) does not depend on \( t \). Consequently

\[
F^{n+1}(0) = \sum_{k=0}^{n} M^k(f) \frac{t^k}{k!}
\]

being \( M^0 \) the identity. Banach’s fixed point Theorem says also that the sequences \( \{ F^{n+1}(0) \}_{n \in \mathbb{N}} \) converges to the solution \( u(x, t) \) in \((\mathbb{K}[[x, t]], d)\). This really means that

\[
u(x, t) = \sum_{n \geq 0} M^n(f) \frac{t^n}{n!} \text{ in } (\mathbb{K}[[x, t]], +, ., d)\]

\( \square \)

In view of the above proposition, Given a non-expansive \( \mathbb{K} \)-linear function \( M : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x]], d) \), we can define the **exponential** as the map

\[
e^{tM} : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x, t]], d)
\]

by \( e^{tM}(f) = \sum_{n \geq 0} M^n(f) \frac{t^n}{n!} \). Note that the difference when \( M \) is in fact contractive is that for any \( t_0 \in \mathbb{K} \) \( e^{t_0M} : (\mathbb{K}[[x]], d) \rightarrow (\mathbb{K}[[x]], d) \) is a linear isometry while for non-expansivity the evaluation at \( t_0 \) makes non-sense in general up to \( t_0 = 0 \) which is the identity. So, for contractivity, the formal power series \( e^{tM} = \sum_{n \geq 0} M^n \frac{t^n}{n!} \) (on the variables \( t \) and \( M \)) is convergent, to a linear isometry on \((\mathbb{K}[[x]], d)\), for any \( t \). Note
that with the above interpretation of the exponential one could treat the initial value problem (of the real partial differential equation)

\[
\begin{cases}
\frac{\partial u}{\partial t} = x\beta(x)\frac{\partial u}{\partial x} + \alpha(x)u(x, t) \\
u(x, 0) = f(x)
\end{cases}
\]

without the hypothesis \(\alpha(0) = \beta(0) = 0\). i.e., for arbitrary \(\alpha, \beta \in \mathbb{K}[[x]]\), because in this case the \(\mathbb{K}\)-linear map \(L_{\alpha, \beta} : (\mathbb{K}[[x]], d) \longrightarrow (\mathbb{K}[[x]], d)\) given by

\[
L_{\alpha, \beta}(h) = \alpha h + x\beta h'
\]
is non-expansive and \(x\beta(x)\frac{\partial u}{\partial x} + \alpha(x)u(x, t)\) is just the action of the partial \(x\)-extension of \(L_{\alpha, \beta}\).

The previous result can be used to get uniqueness in all results where we claim it along this paper. Note that, in general, the initial value problems treated in this paper are *ill-posed*. In particular there is no uniqueness in a sufficiently large framework. For example, consider again the problem

\[
\begin{cases}
\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = g(x)u(x, t) \\
u(x, 0) = f(x)
\end{cases}
\]

with \(g(0) = 0\).

This problem makes sense in the whole space \(\mathbb{K}[[x, t]]\). As proved before

\[
u(x, t) = f(x)e^{t\left(-\frac{1 + \sqrt{1 - 4g(x)}}{2}\right)}
\]
is a solution but in \(\mathbb{K}[[x, t]]\):

\[
v(x, t) = f(x)e^{t\left(-\frac{1 + \sqrt{1 - 4g(x)}}{2}\right)} + 1 - e^{-t}
\]
is another different one.

Fix now two power series \(f, h \in \mathbb{K}[[x]]\), then the map \(F : \mathbb{K}[[x, t]] \longrightarrow \mathbb{K}[[x, t]]\) given by

\[
F(u(x, t)) = f(x) + t(f(x) + h(x)) - \int_0^t u(x, t) + \int_0^t \int_0^t g(x)u(x, t)
\]
is $\frac{1}{2}$-contractive and then the problem

$$\begin{cases}
\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = g(x)u(x, t) \\
u(x, 0) = f(x) \\
\frac{\partial u}{\partial t}(x, 0) = h(x)
\end{cases}$$

with $g(0) = 0$ has a unique solution which is the unique fixed point of

$$F(u(x, t)) = f(x) + t(f(x) + h(x)) - \int_0^t u(x, t) + \int_0^t \int_0^t g(x)u(x, t).$$

Using the above ideas one can really prove the following more general uniqueness result for certain functional-partial differential equations in $\mathbb{K}[[x, t]]$:

**Theorem 32.** Fix a positive integer $n$. Let $f_0, f_1, \cdots, f_{n-1} \in \mathbb{K}[[x]]$ and suppose that $Q : \mathbb{K}[t] \rightarrow \mathbb{K}[t]$ is a delta-operator such that $Q = q(D)$ where $q$ is a polynomial of degree $n$. Suppose also that $M : (\mathbb{K}[[x, t]], d) \rightarrow (\mathbb{K}[[x, t]], d)$ is a $2^{n-1}$-lipschitz map, i.e. $d(M(u(x, t)), M(v(x, t))) \leq 2^{n-1}d(u(x, t), v(x, t))$. Then, the problem:

$$\begin{cases}
Q_t(u(x, t)) = M(u(x, t) \\
u(x, 0) = f_0(x), \frac{\partial u}{\partial t}(x, 0) = f_1(x), \cdots, \frac{\partial^{n-1}_t u}{\partial t^{n-1}}(x, 0) = f_{n-1}(x)
\end{cases}$$

has a unique solution in $\mathbb{K}[[x, t]]$.

One can construct many examples of the above situation in particular if $M$ is the $x$-partial extension of a delta-operator $R$ such that $R = r(D)$ with $r$ a polynomial of degree less than $n$.

Following again the algebraic constructions in [22] and adapting the ultrametric $d$ to $\mathbb{K}[[((F_k(x)), (G_k(t)))]$, in the very natural way suggested by the definition of $d$, we can obtain analogous existence and uniqueness results in $\mathbb{K}[[((F_k(x)), (G_k(t)))]$ for the more general framework of delta-evolution equations

**Acknowledgment:** The first author was partially supported by DGES grant MICINN-FIS2008-04921-C02-02. The second author was partially supported by DGES grant MTM-2006-0825.
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