CONTROLLING PREDATOR-PREY DISCRETE DYNAMICS UTILIZING A THRESHOLD POLICY WITH HYSTERESIS

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Abstract—This paper introduces a threshold policy with hysteresis (TPH) for the control of the logistic one-species model. A nonstandard scheme is used for the discretization of the model since it results in preservation of the qualitative characteristics of the continuous-time models. Two theorems that establish the global stability of the logistic discrete model subject to the threshold policy (TP) and the TPH are proved. The proposed policy (TPH) is more realistic than a pure threshold policy (TP) proposed earlier in the literature and changes the dynamics of the system in such a way that a low amplitude bounded oscillation, far from the extinction region, is achieved. Furthermore, it can be designed by a suitable choice of so called virtual equilibrium points in a simple and intuitive manner.

Keywords—Discrete models, Nonstandard scheme discretization, Threshold policy with hysteresis.

Resumo—Este artigo introduz uma política de limiar com histerese (TPH) para o controle do modelo logístico de uma espécie. Uma discretização não convencional é usada na discretização do modelo logístico, uma vez que preserva as características qualitativas do modelo contínuo. Prova-se dois teoremas estabelecendo a estabilidade global do modelo logístico sujeito às políticas de limiar e de limiar com histerese. A política TPH proposta, que é mais realista que a política de limiar (TP) proposta anteriormente na literatura, faz com que o sistema dinâmico fique confinado a uma região limitada (e pequena) do espaço de estados longe da região de extinção. Adicionalmente, essa política pode ser projetada por uma escolha adequada dos equilíbrios virtuais de maneira simples e intuitiva.

Palavras-chave—Modelos discretos, Discretização não convencional, Política de Limiar com Histerese.

1 Introduction

In population dynamics, there are two kinds of mathematical models: continuous-time models described by differential equations or dynamical systems, and the discrete-time models described by difference equations, discrete dynamical systems or iterative maps (Liu and Xiao, 2007).

Investigations have shown that many economic, physical, and biological phenomena are best represented via difference equations. However, there are many situations for which continuous models, i.e., differential equations, are the best fit. One may then invoke some numerical scheme to transform the given differential equation into a difference equation. The resulting difference equation is said to be dynamically consistent with its continuous counterpart if both exhibit the same qualitative behavior, such as stability, bifurcation, and chaos (Liu and Elaydi, 2001).

A major difficulty in the numerical integration of ordinary differential equations (ODE) is the existence of numerical instabilities (Mickens, 1994). These are solutions to the discrete equations that do not correspond to any of the solutions to the original ODEs. Conventional numerical techniques, such as the forward Euler and higher-order Runge-Kutta schemes, are extensively used in solving systems of nonlinear differential equations. However, these numerical procedures may lead to chaotic or spurious solutions which are strongly scheme dependent. A nonstandard scheme was introduced by Mickens (1994) to alleviate these problems and lead to difference equations that preserve the qualitative behavior of their continuous counterparts.

The effects of (proportional) harvesting on dynamic behavior in discrete-time population models have received increased attention (Basson and Fogarty, 1997; Tang and Chen, 2004, and references therein). This paper studies the response of the logistic discrete-time model to a sustained perturbation (removal of species through harvesting). The contribution of this paper is to show that a continuous-time harvesting or control policy (Meza et al., 2006) called a threshold policy with hysteresis (TPH) can be used successfully in the discrete-time case, provided that nonstandard discretization schemes referred to above are used, instead of the standard discretization schemes.

Preliminaries

Definitions:

The definition of the concepts of real and virtual equilibria, as well as the definition of the threshold policy (TP) and the TPH are briefly summarized in this section for the following discrete time system

$$z_{k+1} = f(z_k, t) - u,$$

where $z_k = [x_k \ y_k]^T$ is the state vector, $x_k$ denotes the prey population density, $y_k$ denotes the predator population density, $f(\cdot)$ represents the interaction function of the species, and the control...
policy \( u \) can be defined as the TP \( u_{TP} = \varepsilon z_k \psi(\tau) \)
or as the TPH \( u_{TPH} = \varepsilon z_k \psi_{hys}(\tau) \), where \( \varepsilon \) is the effort (harvesting effort), \( \psi(\tau) \) is defined as follows

\[
\psi(\tau) = \begin{cases} 
1 & \text{if } \tau > 0 \\
0 & \text{if } \tau < 0,
\end{cases}
\]

where \( \tau \) is the threshold that should be chosen according to the problem to be solved (Figure 1.(a)). A graphical representation of the \( u_{TP} \) is shown in Figure 1.(b).

The threshold policy with hysteresis (TPH) is defined as follows

\[
\psi_{hys}(\tau) = \begin{cases} 
\emptyset & \text{if } \tau < -\sigma, \text{ or } \tau \leq \sigma \text{ and } \\
1 & \text{if } \tau > \sigma, \text{ or } \tau \geq -\sigma \text{ and } \tau(k + 1) > \tau(k).
\end{cases}
\]

where \( \tau \) is the threshold that should be chosen adequately, depending on the problem to be solved, \( \sigma \) is the hysteresis parameter, and \( \tau(k + 1) \), \( \tau(k) \) are the values of \( \tau \) at instant \( k + 1 \) and \( k \) (Figure 2.(a)). A graphical representation of the TPH is shown in Figure 2.(b).

**Definition 1** Let \( x^{G_i} \) be the stable equilibrium point of the dynamics of region \( G_i \) for \( i = \{1, 2\} \). Then \( x^{G_i} \) is called a real equilibrium if it belongs to \( G_i \) and a virtual equilibrium if it belongs to \( G^j, j \neq i \), for \( i = \{1, 2\} \).

Note that a TPH models a TP in which errors in estimation and implementation occur.

**Nonstandard discretization:**

It is well known that the discrete logistic model with a standard discretization, e.g. forward Euler scheme, has solutions with various periods, as well as chaotic solutions (Mickens, 1994; Murray, 2002), see Figure 3. Numerical instabilities are an indication that the discrete equations are not able to model the correct mathematical properties of the solutions to the differential equations of interest. The nonstandard scheme that was introduced by Mickens (1994) to alleviate these problems and lead to difference equations that preserve the qualitative behavior of their continuous counterparts.

![Graphical representation of the TPH in a phase line.](image)

**Figure 2:** (a) Graphical representation of \( \psi_{hys}(\tau) \). (b) Graphical representation of the TPH in a phase line. The interval \( G^3 \) is the hysteresis region, \( x^{G_1} \) and \( x^{G_2} \) are the stable equilibrium points of the dynamics in interval \( G^1 \) and \( G^2 \), respectively, and both are virtual. \( x_{LL} = x_{1h} - \sigma \) and \( x_{RR} = x_{1h} + \sigma \) are the threshold levels.

![Simulation of the discrete logistic model for a nonstandard, \( x_{k+1} = (1 + r \phi)x_k \), and standard scheme discretization, \( x_{k+1} = x_k \left(1 + \frac{r}{K}(K - x_k)\right)\). Parameter values for both simulations are the same. \( r = 3, K = 1, \phi = 1 \), and initial condition 0.1.](image)

**Figure 3:** Simulation of the discrete logistic model for a nonstandard, \( x_{k+1} = (1 + r \phi)x_k \), and standard scheme discretization, \( x_{k+1} = x_k \left(1 + \frac{r}{K}(K - x_k)\right)\). Parameter values for both simulations are the same. \( r = 3, K = 1, \phi = 1 \), and initial condition 0.1.

**2 Logistic discrete model subject to TP and TPH**

In the traditional form of single population model, in which endogenous consumption is considered
together with the control, changes in prey abundance are described by

$$\dot{x} = f(x) - c_{end}(x) - u, \quad (4)$$

where the continuous function $f(x)$ describes prey growth as a function of prey density, the endogenous function $c_{end}(x)$ is the loss rate due to consumption either by herbivores or harvesting (the predator density is assumed constant), and $u$ is the control policy. We consider the following forms for $f(x)$ and $c_{end}(x)$,

$$LG - No \ EC : \ r x \left(1 \ - \ \frac{x}{K}\right) \quad (5)$$
$$LG - Holling Type II EC : \ r x \left(1 \ - \ \frac{x}{K}\right) - \frac{c x}{x + d} \quad (6)$$
$$LG - Holling Type III EC : \ r x \left(1 \ - \ \frac{x}{K}\right) - \frac{c x^2}{x^2 + d} \quad (7)$$

where $LG$ means Logistic Growth, EC means Endogenous Consumption, $r$ is the intrinsic growth rate, $K$ is the carrying capacity, $c$ is the endogenous consumption rate, and $d$ relates to the prey density at which predator satiation occurs.

The one-species discrete models are obtained according to nonstandard discretization method (Mickens, 1994), by making the following replacements: (i) $\frac{dx}{dt} \rightarrow \frac{x_{k+1} - x_k}{\Delta t}$ where $\phi$ is such that $\phi(h) = h + \phi(h^2)$ and $h = \Delta t$, e.g., $\phi(h) = e^h - 1$, (ii) $x \rightarrow x_k$, (iii) $x^2 \rightarrow x_{k+1} x_k$. The resulting discrete models, from its continuous counterpart (5), (6) and (7), respectively, are as follows

$$x_{k+1} = \frac{(1 + r \phi) x_k}{1 + \frac{r}{K} \phi x_k} \quad (8)$$
$$x_{k+1} = \frac{(1 + r \phi - \frac{c x}{x + d}) x_k}{1 + \frac{r}{K} \phi x_k} \quad (9)$$
$$x_{k+1} = \frac{(1 + r \phi) x_k}{1 + \left(\frac{r}{K} + \frac{c^2}{x_k^2 + d}\right) \phi x_k} \quad (10)$$

The logistic discrete model (8) has been studied in Mickens (1994), the logistic discrete model with endogenous consumption Holling Type II (9) and Holling Type III (10) are introduced in this paper. As an application example we take the continuous logistic model, which although it assumes a drastic simplification of the real-world complexities inherent in predator-prey systems, can still provide useful insight into the underlying behavior of the system. This type of model has been studied by Kot (2001) in the continuous-time case with an emphasis on the productive aspects of renewable-resource management.

In the simplest form of the model, changes in the species abundance are described by (5), where $x$ denotes the population density. The logistic model (5), under nonstandard discretization (Mickens, 1994), becomes (8).

### 2.1 Logistic discrete model subject to TP

The model (8) under TP is as follows,

$$x_{k+1} = \frac{(1 + r \phi) x_k}{1 + \frac{r}{K} \phi x_k} - u_{TP} \quad (11)$$

where the control is given by

$$u_{TP} = \varepsilon x_k \psi(\tau)$$

and $\psi(\tau)$ is defined as in (2), where $\tau$ is the threshold and $\varepsilon$ is an effort parameter. The simplest choice is:

$$\tau = x_k - x_{th}$$

where $x_{th}$ is the threshold level of population density.

#### Equilibrium points under TP

The equilibrium point of the system without harvesting (without control) is $x_{VNC} = K$ and the equilibrium point of the system with harvesting (proportional control) is $x_{VC} = K - \frac{r \phi - \varepsilon}{r \phi(1 + \varepsilon)}$. If the effort parameter satisfies the restriction $\varepsilon < r \phi$ then the equilibrium point $x_{VC}$ is positive. If $\varepsilon$ does not satisfy this restriction, then the system without control goes to extinction. In this paper, we consider values of $\varepsilon$ slightly greater than $r \phi$, i.e., $\varepsilon \geq r \phi$ (see Remark 1). We chose the threshold level such that the equilibria $x_{VNC}$ and $x_{VC}$ are virtual, i.e.,

$$x_{VC} < x_{th} < x_{VNC}.$$

![Figure 4: Simulation of the logistic discrete model subject to a TP](image-url)
2.2 Logistic discrete model subject to TPH

Using nonstandard discretization method (Mickens, 1994) the logistic model (5) under TPH is as follows:

\[ x_{k+1} = \frac{(1 + r \phi) x_k}{1 + \frac{r}{K} \phi} - u_{TPH} \]  

where the control is given by:

\[ u_{TPH} = \varepsilon x_k \psi_{k+1}(\tau) \]

and \( \psi_{k+1}(\tau) \) is defined as in (3), where \( \tau \) is the threshold that should be chosen according to the problem to be solved, and \( \varepsilon \) is an effort parameter. The simplest choice of \( \tau \) is:

\[ \tau = x_k - x_{th} \]

where \( x_{th} \) is the threshold level of population density. We choose the threshold level such that:

\[ x_{VC} < x_{th} - \sigma < x_{th} < x_{th} + \sigma < x_{VNC}. \]

\[ \text{Figure 5: Simulation of the logistic discrete model subject to a TPH. Parameter values are: } r = 1, \ K = 90, \ h = 0.01, \ \phi = 0.0101, \ x_{th} = 45, \ \varepsilon = 0.0201, \ \sigma = 5, \ \text{initial conditions 15 and 75.} \]

**Theorem 1** Consider a system of the following type:

\[ \begin{cases} x_{k+1} = f(x_k, \phi) - u_{TP} \\ u_{TP} = \varepsilon x_k \psi(\tau) \end{cases} \]  

where \( \psi(\tau) \) is defined as in equation (2). Assume that:

1. the function \( f(x_k, \phi) - x_k \) is nonnegative on the interval \([0, x_{VNC}]\), in which \( x_{VNC} \) is the equilibrium point of the system without control, and

2. the effort parameter \( \varepsilon \) satisfies

\[ \varepsilon < \frac{1}{1 + r \phi}. \]  

Under these assumptions, the system stabilizes in a neighborhood of the threshold level \( x_{th} \), i.e., in an invariant interval, if the threshold level is chosen such that the equilibrium points of the system with control and without control are virtual.

**Proof:** Trajectories initiating in the intervals \( I_1 = [0, x_L] \), \( I_2 = [x_R, \infty) \) (bold lines, Figure 6) will be shown to converge to the interval \( I_2 = (x_L, x_R) \) which contains the point \( x_{th} \), which determines the threshold, and can be thought as a “desired equilibrium” in the sense that convergence occurs to a neighborhood \( (I_2) \) of this point. \( I_2 \) is shown to be an invariant interval. While convergence to the invariant interval \( I_2 \) is in progress, the system is said to be in the reaching phase and the proof of the convergence is carried out using a Liapunov function referred to a reaching phase Liapunov function.

**Figure 6:** Graphical representation of the phase line. \( x_{VC} \) and \( x_{VNC} \) are the equilibria of the system with control and without control, respectively. \( x_L \) and \( x_R \) the lower and upper bounds of the interval \( I_2 \), respectively. \( x_{th} \) is the threshold level.

**Determining the bounds of the globally attractive invariant interval**

We first calculate an interval that contains \( x_{th} \) (the threshold value) by reasoning as follows. Suppose that the state \( x_k = x_{th} \) (i.e. infinitesimally to the left of \( x_{th} \)), then it is in the region without control and the next state, which will denote \( x_R \), can be written as:

\[ x_R = \frac{(1 + r \phi) x_{th}}{1 + \frac{r}{K} \phi} \]  

Similarly if \( x_k = x_{th}^+ \) (infinitesimally to the right of \( x_{th} \)) and therefore in the region with control, the next state, denoted \( x_L \), can be calculated as

\[ x_L = \frac{(1 + r \phi) x_{th}}{1 + \frac{r}{K} \phi} - \varepsilon x_{th}. \]

We will show that \( x_L \) and \( x_R \) are the left and right endpoints of an interval, denoted \( I_2 \), that will be proved to be globally attractive and invariant under the switching threshold policy.

Equations (15) and (16) define the endpoints of the interval \( I_2 \).

**Proof of the global attractivity and invariance of the interval \( I_2 \)**

A suitable reaching phase Liapunov function, adapted from Goh (1980), is a generalized distance
from the threshold given by:

$$V(x_k) = x_k - 2x_{th} + \frac{x_{th}^2}{x_k} = \frac{(x_k - x_{th})^2}{x_k}$$ \hspace{1cm} (17)$$
for which

$$\Delta V(x_k) = V(x_{k+1}) - V(x_k)$$
$$= (x_{k+1} - x_k) \left( 1 - \frac{x_{th}^2}{x_{k+1} x_k} \right) \hspace{1cm} (18)$$

This function is used to prove:

Global attractivity of interval $I_2$

We now use $V, \Delta V$ to carry out an interval analysis of the system (13).

Interval $I_1$ : Assume $x_k \in I_1$. If $x_{k+1} \in I_1$, then $x_{k+1} \leq x_{th}$, and from the dynamics in this interval, we have

$$x_{k+1} = \frac{(1 + r \phi) x_k}{1 + \frac{r}{K} \phi x_k}.$$ \hspace{1cm} (19)

Since $x_k < x_{VNC} = K, \forall k$, the factor $(1 + r \phi) / (1 + \frac{r}{K} \phi x_k)$ is greater than 1, thus from (19), $x_{k+1} > x_k$. Therefore, from (18), $\Delta V < 0$ and all trajectories with initial condition in the interval $I_1$ converge to the interval $I_2$.

Interval $I_3$ : Assume $x_k \in I_3$. If $x_{k+1} \in I_3$, then $x_{k+1} \geq x_{th}$, and from the dynamics in this interval, we have

$$x_{k+1} = \frac{(1 + r \phi) x_k}{1 + \frac{r}{K} \phi x_k} - \varepsilon x_k.$$ \hspace{1cm} (20)

The factor $(1 + r \phi) / (1 + \frac{r}{K} \phi x_k) - \varepsilon$ is less than 1 for $x_k > \frac{K}{r} \left( \frac{\phi}{1 + \frac{r}{K} \phi} \right)$. The latter holds by (14) since $x_k$ is a nonnegative variable.

Thus $x_{k+1} < x_k, \forall x_k \in I_3$. Therefore, $\Delta V < 0$ and all trajectories with initial condition in the interval $I_3$ converge to the interval $I_2$.

Therefore, trajectories initiating in the intervals $I_1$ and $I_3$ converge to the interval $I_2$, proving that it is globally attractive.

Invariance of interval $I_2$

Inside interval $I_2$, the plan of the proof is to show that points in $I_2$ to the left of $x_{th}$ cannot jump, under the dynamics (13), to the right of $x_R$ and, similarly, points to the right of $x_{th}$ cannot jump to the left of $x_L$. This completes the proof of invariance of interval $I_2$.

1. Consider an initial condition $x_k$ in the region without control (Figure 7) that belongs to the interval $I_2^R = (x_L, x_{th})$ and suppose that the next state is on the right of $x_R$, i.e., $x_{k+1} \in I_3$, and $x_{k+1} > x_R$,

$$\frac{(1 + r \phi) x_k}{1 + \frac{r}{K} \phi x_k} > \frac{(1 + r \phi) x_{th}}{1 + \frac{r}{K} \phi x_{th}}.$$ \hspace{1cm} (20)

Inequality (20) is satisfied if $x_k > x_{th}$, which contradicts $x_k \in I_2^R$. In other words, if $x_k \in I_2^R$, then $x_{k+1}$ is smaller than $x_R$, i.e., the trajectory remains inside the interval $I_2$.

2. Consider an initial condition $x_k$ in the region with control (Figure 7) that belongs to the interval $I_2^R = (x_{th}, x_R)$ and suppose that the next state is on the left of $x_L$, i.e., $x_{k+1} \in I_1$.

Let $a = 1 + r \phi, \ b = r \phi / K$ for notational simplicity. Then $x_{k+1} < x_L$ can be written as:

$$\frac{a x_k}{1 + b x_k} - \varepsilon x_k < \frac{a x_{th}}{1 + b x_{th}} - \varepsilon x_{th},$$

which implies

$$(x_k - x_{th})(a - \varepsilon (1 + b x_{th})) > 0.$$ \hspace{1cm} (21)

We will prove that the second term of (21) is positive. In fact,

$$a - \varepsilon (1 + b x_{th}) > 0 \iff \varepsilon < \frac{1 + r \phi}{(1 + \frac{r}{K} \phi x_{th})(1 + \frac{r}{K} \phi x_{th})}.$$ \hspace{1cm} (20)

By assumption 2 and for values of $x_k$ and $x_{th}$ in the interval $[0, x_{VNC})$, we have

$$\varepsilon < \frac{1 + r \phi}{1 + \frac{r}{K} \phi x_{th}} < \frac{1 + r \phi}{1 + \frac{r}{K} \phi x_{th}} < 1 + r \phi.$$ \hspace{1cm} (20)

Thus, the expression (21) can only be satisfied for $x_k < x_{th}$, which contradicts $x_k \in I_2^R$. In other words, if $x_k \in I_2^R$, then $x_{k+1}$ is greater than $x_L$, i.e., the trajectory remains inside the interval $I_2$.

Remark 1 In order to ensure overexploitation, the additional condition $\varepsilon \geq r \phi$ must be imposed and can be satisfied in conjunction with assumption 2 iff $r \phi \in \left[0, \frac{\sqrt{5} - 1}{2}\right]$.
Theorem 2 Consider a system of the following type
\[
\begin{align*}
x_{k+1} &= f(x_k, \phi) - u_{TPH} \\
u_{TPH} &= \varepsilon x_k \psi_{hys}(\tau)
\end{align*}
\]
(22)
where \(\psi_{hys}(\tau)\) is defined as in equation (3). Assume that:
1. the function \(f(x_k, \phi) - x_k\) is nonpositive on the interval \([0, x^{VNC}]\), in which \(x^{VNC}\) is the equilibrium point of the system without control, and
2. the effort parameter \(\varepsilon\) satisfies
\[
\varepsilon < \frac{1}{1 + r \phi}.
\]
Under these assumptions, the system stabilizes in the invariant interval \((x_L, x_R)\), if the threshold levels \(x_{LL} = x_{th} - \sigma\) and \(x_{RR} = x_{th} + \sigma\) are chosen such that the equilibrium points of the system with control and without control are virtual.

Proof: Trajectories initiating in the intervals \(I_1 = [0, x_L]\), \(I_3 = [x_R, \infty)\) (bold lines, Figure 8) will be shown to converge to the interval \(I_2 = (x_L, x_R)\) which contains the points \(x_{LL} = x_{th} - \sigma\) and \(x_{RR} = x_{th} + \sigma\) which determine the thresholds, in the sense that convergence occurs to a neighborhood of the interval \((x_{LL}, x_{RR})\). \(I_2\) is shown to be an invariant interval.

The proof of Theorem 2 is similar to the proof of Theorem 1. The difference is that there are two different switching points (thresholds), one switching point for trajectories which initiate in the interval \([0, x_{LL}]\) and switch when crossing the point \(x_{RR} = x_{th} + \sigma\), and another switching point for trajectories which initiate in the interval \([x_{RR}, \infty)\) and switch when crossing the point \(x_{LL} = x_{th} - \sigma\). All trajectories remain inside the interval \(I_2\).

Figure 8: Graphical representation of the phase line. \(x_{VNC}\) and \(x_{VNC}\) are the equilibrium point of the system with control and without control, respectively. \(x_L\) and \(x_R\) are the lower and the right upper bounds of the interval \(I_2\), respectively. \(x_{LL} = x_{th} - \sigma\) and \(x_{RR} = x_{th} + \sigma\) are the threshold levels.

To economize space, this demonstration is omitted, but it follows the idea of the proof of Theorem 1.

The proof of the analogs of Theorems 1 and 2 for the logistic discrete model with endogenous consumption of Holling Type II (9) and Holling Type III (10) models can be carried out similarly, and this is reported on elsewhere.