Solution of forward kinematics in Stewart platform using six rotary sensors on joints of three legs

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A novel method is proposed for real-time solution of direct kinematics problem of Stewart platform (SP) using six measurements on three legs’ joints consisting of the rotations of three legs in two directions. After the application of the method on a laboratory sample SP, it is observed that the method is preferable to the conventional method that uses the length measurements of all six legs, in the grounds of industrial applicability. It is due to simpler implementation, less expense, easier maintenance, and stress-free assembly. The algorithms of both forward and inverse kinematics are fully derived based on geometric relationships between the platform states and the measurement data. The sensitivity to the measurement errors is analyzed theoretically and is applied through a computer simulation to several configurations of the sample SP which are uniformly distributed in the workspace. The variances of measurement errors for those configurations are compared between the conventional and proposed methods and it is observed that: the proposed method operates more accurate in position measurement especially in lateral movements. Additionally, the proposed method is not too sensitive to direction of movement and geometry of the SP.

Keywords: forward kinematics; direct kinematics; Stewart platform; hexapod

1. Introduction

The Stewart platform (SP) is a six-degree-of-freedom (6DoF) parallel manipulator with a high structural and positional rigidity. It was first introduced by Stewart in 1965 for flight simulator application.[1] Later, it became a widely-used mechanism in robotic applications. It consists of two plates: base plate and platform. The first one is usually fixed and the latter is moving with 6 DoF motion. The plates are connecting to each other by six legs. They are the linear actuators giving 6 DoF motion to the platform (see Figure 4).

A challenging point, often found in the motion control of an SP is the requirement of real-time solution of the highly nonlinear forward kinematics transformation. This transformation is to find the translational and rotational attitudes of platform knowing the length of its legs.

Practically speaking, there are three approaches for solving the forward kinematics. The first approach, which is a straight forward one, is the direct measurement of the platform’s orientation and displacement.[2] The orientation of the platform may be measured by a rate gyro which is relatively an expensive device. The direct measurement of the platform’s position is very complicated in practice. Therefore, the approach is rarely used because of its high cost and the drawback of complicatedness in sensors application especially in measurement of the platform’s position.

The second approach, which is more common, is mainly based on an analytical solution of the forward kinematics from the length measurements of the legs. The details of this approach can be found in references. [2–4] Since the forward kinematics procedure of Stewart platform is mathematically complicated to be dealt with, some methods have been developed based on the preprocessing of the data gathered from the known inverse kinematics for the solution of real-time forward kinematics.[5,6]

Not many found in the literature is the third approach for solution of forward kinematics in practice. It is based on additional length measurements of auxiliary directions other than those of the six legs. It may help to solve the problem of forward kinematics fast and straight forward; this concept is initially utilized by Cheok et al. [7]. They presented two methods for exact solution of the forward kinematics problem using nine measurements of length (instead of six) by means of Linear Variable Displacement Transducers (LVDTs). In the first method, they mounted six LVDTs on six legs and the remaining three on the three additional directions as auxiliary measurements. In the second method, they mounted three LVDTs on three non-adjacent legs and the six others on
six auxiliary directions (Mounting of sensors in these two methods is demonstrated in Figure 1). In both methods, they found the position and orientation of the platform by some algebraic and trigonometric relations.

Although the two aforementioned methods are simple and yet more practical from the computational viewpoint, there are some drawbacks regarding their application. The auxiliary directions, where the LVDTs or string potentiometers are to be assembled, should be chosen among the six legs somewhere beneath the platform and over the base. This will cause their collision with legs in some attitudes of the SP’s platform.[7] Thus, these attitudes have to be avoided in practice and this yields losing some portion of the workspace. In addition, some sophisticated software is needed to analyze the remaining workspace which makes the application of the method quite expensive and time consuming. The probability of collision between the legs and the auxiliary LVDTs can be reduced by mounting the LVDTs quite close to the legs. In addition, the output error will be finally magnified since the auxiliary measurements are very close to the main length measurements of the legs. Later, Hus and Fong [8], Bonev and Ryu [9] and Etemadi and Angeles [10] developed this method by making some improvements on the method presented by Cheok et al. [7].

This paper proposes a novel practical method for an online solution of forward kinematics, which can be categorized into the second approach among the three approaches discussed earlier. Since Stewart platform is indeed a six DoF mechanism, six proper measurements will be sufficient to solve the direct kinematics problem. Conventionally, these six measurements have been the leg’s length which is usually performed by mounting six digital rulers on the legs. The presented method is based on six geometric measurements except the leg’s length which are the rotations of three legs in two directions. After the application of the method on a sample SP, it is observed that while the method has reasonable accuracy, it is simpler in implementation, more practical, less expensive, and easier in maintenance in comparison with the conventional method.

The rest of the paper is organized as follows. In Section 2, the algorithms for solution of both forward and inverse kinematics are fully described followed by some practical considerations. Section 3 deals with the sensitivity analysis of the algorithms and compares them with those of conventional method. Finally, Section 4 concludes the paper.

2. Position and orientation of the platform

2.1. Coordinate frames

Figure 2 shows the ith actuator with a general configuration possessing a ball joint at the top and a universal joint at the bottom. The orientation of the universal joint is quite arbitrary. Without losing the generality, the fixed pivot of the universal joint is considered to be horizontal. The primary coordinate frame used in derivation of formulation is the inertial coordinate frame $X-Y-Z$ which is considered to be fixed at the center of the SP’s base plate (See Figure 4). Then consider an auxiliary right-handed coordinate frame $x_i$-$y_i$-$z_i$ originated at the center of the inertial coordinate frame whose axes are oriented in such a way that the $x_i$-axis is parallel to the fixed pivot of the universal joint and $y_i$-axis lies on the horizontal surface and the $z_i$-axis points downward (see Figure 2). A body coordinate frame represented with unit vectors $\hat{\xi}_{3i}$, $\hat{\xi}_{2i}$, and $\hat{\xi}_{3i}$ is considered to be attached to the center of the universal joint so that $\hat{\xi}_{3i}$ is along the leg. $\hat{\xi}_{2i}$ is perpendicular to both the $\hat{\xi}_{3i}$ and the fixed pivot of the universal joint. Therefore, $\hat{\xi}_{2i}$ remains in the vertical plate while the joint is rotating $\hat{\xi}_{ij}$ makes a right-handed frame with $\hat{\xi}_{2i}$ and $\hat{\xi}_{3i}$ in $\hat{\xi}_{1i}$-$\hat{\xi}_{2i}$-$\hat{\xi}_{3i}$ order. $\hat{\xi}_{2i}$ makes the angle $\Theta_i$ with the horizontal plane and $\hat{\xi}_{1i}$ makes the angle $\Phi_i$ with the direction $x_i$ in a plane perpendicular to $\hat{\xi}_{2i}$ (see Figure 2). In other words, $\Theta_i$ and $\Phi_i$ are rotation angles of the universal joint measured from their zero value corresponding to the vertical position of the leg.

Referring to the figure, $\hat{\xi}_{1i}$, $\hat{\xi}_{2i}$, and $\hat{\xi}_{3i}$ can be transformed to unit vectors $\hat{i}_i$, $\hat{j}_i$, and $\hat{k}_i$ associated with the axes $x_i$, $y_i$, and $z_i$, respectively. It is done by involving rotation angles of the universal joint, $\Theta_i$ and $\Phi_i$, as:

![Figure 1. Nine length measurement for exact method for solution of forward kinematics (a): $M = [k_1 m_1 n_1 k_2 m_2 n_2 k_3 m_3 n_3]$.

![Figure 2.](image-url)
The body and the auxiliary coordinate frames of the ith actuator.

\[
\begin{align*}
\hat{\xi}_{1i} &= (\cos \Phi_i)\hat{j}_i + (\sin \Theta_i \sin \Phi_i)\hat{k}_i + (\cos \Theta_i \sin \Phi_i)\hat{l}_i \\
\hat{\xi}_{2i} &= (-\cos \Phi_i)\hat{j}_i + (\sin \Theta_i)\hat{k}_i \\
\hat{\xi}_{3i} &= \hat{\xi}_{1i} \times \hat{\xi}_{2i}.
\end{align*}
\]

(1)

\[
\begin{bmatrix}
\hat{\xi}_{1i} \\
\hat{\xi}_{2i} \\
\hat{\xi}_{3i}
\end{bmatrix} =
\begin{bmatrix}
\cos \Phi_i & \sin \Theta_i \sin \Phi_i & \cos \Theta_i \sin \Phi_i \\
0 & -\cos \Phi_i & \sin \Theta_i \\
\sin \Phi_i & \sin \Theta_i \cos \Phi_i & -\cos \Theta_i \cos \Phi_i
\end{bmatrix}
\begin{bmatrix}
\hat{i}_i \\
\hat{j}_i \\
\hat{k}_i
\end{bmatrix}
\]

(2)

There is a transformation matrix for each leg with constant elements converting the unit vectors of the corresponding auxiliary coordinate frame \(\hat{i}_i, \hat{j}_i, \text{ and } \hat{k}_i\) to that of the inertial coordinate frame i.e. \(\hat{i}, \hat{j}, \text{ and } \hat{k}\) as:

\[
\begin{bmatrix}
\hat{i}_i \\
\hat{j}_i \\
\hat{k}_i
\end{bmatrix} =
\begin{bmatrix}
e_{11,i} & e_{12,i} & e_{13,i} \\
e_{21,i} & e_{22,i} & e_{23,i} \\
e_{31,i} & e_{32,i} & e_{33,i}
\end{bmatrix}
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix} = E_i
\]

(3)

The matrix \(E_i\) depends on the configuration of the universal joint of the ith leg. Note that the method developed in this paper is based on measurements of attitudes of the three non-adjacent legs. Thus, the index 'i' in Equations (1)–(3) varies over the set \{1, 3, 5\}.

2.2. Mounting of the sensors and the related practical considerations

The mounting of the sensors in the proposed method is rather easy and yet practical. To do that, a laboratory sample SP is so made that its geometry becomes variable by locating the six legs in different radial and peripheral positions (See the figure in Appendix 1). To do that, the base and platform circles possess many peripheral holes and the bottoms and top joints can slide in radial direction and fasten by bolt after they are adjusted.

Potentiometers or incremental encoders are mounted on the pivots of the three non-adjacent universal joints to measure three sets of \(\Theta_i\) and \(\Phi_i\) (see Figure 3).

For calibration of the sensors, one can decide each of the two methods suggested below. First, before assembling the platform, the vertical position of non-adjacent legs is precisely found for example with a digital plummet and then the output signals of the potentiometer in those positions are set to zero value in the interface hardware connecting to central computer. Second, after assembling the platform, the neutral configuration of the Stewart platform is precisely found for example with exact measurements of several distances between the platform and base circle and then the relative orientation of the legs are calculated through solution of inverse kinematics and the corresponding \(\Theta_i\) and \(\Phi_i\) angles are obtained. The obtained values of \(\Theta_i\) and \(\Phi_i\) are set to be as the bias values of those angles in software calculations.

2.3. Development of algorithm for direct kinematics

The perfect attitudes of three legs in SP would be good enough to mathematically obtain the position and orientation of the platform. For increasing the accuracy of the method, it is preferred that the three legs are non-neighboring. The algorithm is well explained below taking into consideration the Figure 4. Depicted in the figure are the directions of three non-neighboring legs. Since the platform is considered to be rigid, the distances between the upper ball joints are fixed and hence a fixed triangle connects the centers of the three non-neighboring joints on the platform. The problem is to locating the corners of this triangle on the three direction lines and
getting the position and orientation of the platform. It converges to unique solution in vicinity of updated attitude of the platform.

Combine Equations (2) and (3) to write \( \hat{\xi}_{3i} \) in the inertial coordinate frame \( X^\prime-Y^\prime-Z^\prime \):

\[
\hat{\xi}_{3i} = (e_{11,i} \sin \Theta_i + e_{21,i} \sin \Theta_i \cos \Phi_i - e_{31,i} \cos \Theta_i \cos \Phi_i) \hat{i} \\
+ (e_{12,i} \sin \Theta_i + e_{22,i} \sin \Theta_i \cos \Phi_i - e_{32,i} \cos \Theta_i \cos \Phi_i) \hat{j} \\
+ (e_{13,i} \sin \Theta_i + e_{23,i} \sin \Theta_i \cos \Phi_i - e_{33,i} \cos \Theta_i \cos \Phi_i) \hat{k}
\]

\( i = 1, 3, 5 \) (4)

The parametric equations describing the direction lines are written as:

\[
X_i = P_{bi,X} + \xi_{3i,X} \cdot t_i \\
Y_i = P_{bi,Y} + \xi_{3i,Y} \cdot t_i \\
Z_i = P_{bi,Z} + \xi_{3i,Z} \cdot t_i, \quad i = 1, 3, 5
\] (5)

In which, \( P_{bi} \) is the vector connecting the origin of inertial coordinate frame to the center of lower joint of the \( i \)th leg. The plane of the platform which is the plane crossing the centers of the three non-adjacent upper joints is considered to have the general form as:

\[ aX + bY + cZ + 1 = 0 \] (6)

and suppose that \( T_1, T_3 \) and \( T_5 \) are, respectively, the parameters \( t_1, t_3 \) and \( t_5 \) in Equations (5) at which the direction lines of those equations coincide with the plane of platform. Hence the following set of equations is valid for the location of upper joints:

\[
a(P_{bi,X} + \xi_{3i,X} \cdot T_i) + b(P_{bi,Y} + \xi_{3i,Y} \cdot T_i) + c(P_{bi,Z} + \xi_{3i,Z} \cdot T_i) + 1 = 0, \quad i = 1, 3, 5
\] (7)

\( T_1, T_3 \) and \( T_5 \), in other words are the length of the three non-adjacent legs. Consider that the triangle connecting the centers of the three non-adjacent upper joints has the side lengths \( L_1, L_3 \) and \( L_5 \) as demonstrated in Figure 4. Then the following equations are held for the distances between the upper joints:

\[
\left[ (P_{bi,X} + \xi_{3i,X} \cdot T_i) - (P_{bj,X} + \xi_{3j,X} \cdot T_j) \right]^2 \\
+ \left[ (P_{bi,Y} + \xi_{3i,Y} \cdot T_i) - (P_{bj,Y} + \xi_{3j,Y} \cdot T_j) \right]^2 \\
+ \left[ (P_{bi,Z} + \xi_{3i,Z} \cdot T_i) - (P_{bj,Z} + \xi_{3j,Z} \cdot T_j) \right]^2
\]

\[ = [L_{i+j-3}]^2 \quad i,j = 1, 3, 5 \text{ and } i \neq j \] (8)

The set of Equations in (8) results in three nonlinear set of equations with the following form:

\[
f(T_1^2, T_2^2, T_1, T_2, T_1 T_2) = 0 \\
g(T_1^2, T_5^2, T_1, T_2, T_1 T_2) = 0 \\
h(T_2^2, T_3^2, T_2, T_3, T_2 T_3) = 0
\] (9)

which should be solved for \( T_1, T_3 \) and \( T_5 \). Since the problem is of forward type, several solutions are possible for it. The appropriate solution is obtained in vicinity of the updated configuration which may be found by Newton-Raphson algorithm. It can be organized as follows:

The Jacobian matrix can be derived as:

\[
[Jac] = \begin{bmatrix}
\frac{\partial f}{\partial T_1} & \frac{\partial f}{\partial T_2} & \frac{\partial f}{\partial T_3} \\
\frac{\partial g}{\partial T_1} & \frac{\partial g}{\partial T_2} & \frac{\partial g}{\partial T_3} \\
\frac{\partial h}{\partial T_1} & \frac{\partial h}{\partial T_2} & \frac{\partial h}{\partial T_3}
\end{bmatrix}
\] (10)

which its elements are not too difficult to derive and \( \{T_1, T_3, T_5\}^T \) is iteratively obtained as follows until it converges to a desired accuracy.

\[
\begin{bmatrix}
T_1 \\
T_3 \\
T_5
\end{bmatrix}_k = \begin{bmatrix}
T_1 \\
T_3 \\
T_5
\end{bmatrix}_{k-1} - [Jac]^{-1}_k \begin{bmatrix}
T_1 \\
T_3 \\
T_5
\end{bmatrix}_k
\] (11)

When proper values for \( T_1, T_3 \) and \( T_5 \) are obtained, they are put in Equations (7) to solve the set of three linear equations for \( \{a, b, c\} \).]

\[
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}_k = \begin{bmatrix}
P_{bi,X} + \xi_{3i,X} T_1 \\
P_{bi,Y} + \xi_{3i,Y} T_1 \\
P_{bi,Z} + \xi_{3i,Z} T_1
\end{bmatrix}_k ^{-1} \begin{bmatrix}
P_{bi,X} + \xi_{3i,X} T_3 \\
P_{bi,Y} + \xi_{3i,Y} T_3 \\
P_{bi,Z} + \xi_{3i,Z} T_3
\end{bmatrix}_k ^{-1} \begin{bmatrix}
P_{bi,X} + \xi_{3i,X} T_5 \\
P_{bi,Y} + \xi_{3i,Y} T_5 \\
P_{bi,Z} + \xi_{3i,Z} T_5
\end{bmatrix}_k ^{-1} \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\] (12)

since in majority of the Stewart platforms, the platform does not possess any singular configuration in its work space.[2] the square matrix in Equation (12) has inverse and its solution is guaranteed. Next, the unit normal vector of the plane will be achieved as:

\[
\hat{n} = \frac{a \hat{i} + b \hat{j} + c \hat{k}}{\sqrt{a^2 + b^2 + c^2}}
\] (13)

Considering the geometry of \( i \)th leg shown in Figure 4, we may restate the Equations (5) as:

\[
P_{bi} = P_{bi,i} + l_i \hat{\xi}_{3i} \\ i = 1, 3, 5
\] (14)

In which, \( P_{bi,i} \) is the vector connecting the origin of inertial coordinate frame to the center of the upper joint of the \( i \)th leg.

If the reference point for platform’s position is selected to be its geometric center, it can be obtained by averaging three upper joint’s coordinates.

\[
P_p = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \frac{1}{3} \sum_{i=1}^{3} P_{ui,(2i-1)}
\] (15)
For obtaining platform’s attitude, it is sufficient to find the transformation between the final and initial configurations. (The initial configuration is often set to be the neutral configuration which is defined in the footnote of page 4). To do so, the rotation of normal vector of the platform together with a reference line on the platform is detected. Although the \( \hat{n} \) in Equation (13) shows the orientation of the platform when compared with its initial value \( \hat{n}_0 \), the Euler angles cannot be obtained from the \( \hat{n} \) itself. It is due to the fact that two rotations are enough to map the initial normal vector \( \hat{n}_0 \) to the final one \( \hat{n} \), while the Euler angles consist of three rotations \( \psi \), \( \theta \), and \( \phi \) which are rotations about the \( x \), \( y \), and \( z \) axes, respectively. Here, we utilize another unit vector as an auxiliary vector and implementation of rotation formula [11] to find the third unknown rotation angle and attain all components of Euler angles. By the agreement, the \( x \) axis of body coordinate frame crosses the middle of the joints of first and second legs (See Figure 4). Note that this agreement does not lose the generality of the method and it can be similarly developed for any other agreement of this type. Hence, if the position vector of the first joint ‘\( P_{11} \)’ rotates about the normal vector \( \hat{n} \) in right-handed manner by \( \frac{\pi}{2} \) (which \( \alpha_n \) is the central angle between two adjacent joints on top plane) it will overlap with the \( x \) axis (See Figure 4). The normalized form of the vector ‘\( P_{11} \)’ is shown by ‘\( \hat{P}_{11} \)’ and is obtained by:

\[
\hat{P}_{11} = \left( \frac{P_{u1} - P_p}{\|P_{u1} - P_p\|} \right)
\]

And the unit vector of the platform ‘\( x \)’ axis ‘\( \hat{i} \)’ can be obtained through the rotation formula presented by Goldstein [11].

\[
\hat{i} = (1 - \cos \theta)\hat{n}(\hat{n} \cdot \hat{P}_{11}) + (\cos \theta)\hat{n} - (\sin \theta)(\hat{n} \times \hat{P}_{11})
\]

In which: \( \theta = \frac{\pi}{2} \) according to the agreement on the choice of direction of the \( x \) axis.

The \( z \) axis coincides with the normal vector \( \hat{n} \). Hence its unit vector \( \hat{k} \) is equal to \( \hat{n} \).

\[
\hat{k} = \hat{n}
\]

And the direction of ‘\( y \)’ axis can be found by cross production of \( \hat{k} \) into \( \hat{i} \).

\[
\hat{j} = \hat{k} \times \hat{i}
\]

The rotation matrix ‘\( C \)’ which describes the orientation of top plane relative to fixed coordinate frame ‘\( X’-Y’-Z’’ is constructed by including from Equations (17)–(19) the \( \hat{i}, \hat{j}, \) and \( \hat{k}, \) respectively, within its columns.

\[
C = [ijk]^T = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}
\]

The Euler angles \( \psi \), \( \theta \), and \( \phi \) of the platform (which are rotations about the \( x \), \( y \), and \( z \) axes, respectively) can be achieved by [12]:

\[
\begin{align*}
\theta &= -\sin^{-1}(c_{13}) \\
\varphi &= a \tan(2c_{23}, c_{33}) \\
\psi &= a \tan(2c_{12}, c_{11})
\end{align*}
\]

where ‘atan2’ is the four-quadrant inverse tangent function, available in most programming languages. These equations automatically put the Euler angles into the proper ranges.

Now, the forward kinematics algorithm is completed and the platform’s position and orientation can be obtained from Equations (15) and (21).

### 2.4. The Inverse Kinematics Algorithm

Referring to Figure 4, the vector along the \( i \)th leg connecting the lower joint to the upper one, can be stated in the auxiliary coordinate frame \( x’-y’-z’_i \) shown in Figure 2 as:

\[
S_i = E_i \times (P_{u,i} - P_{b,i}) \quad i = 1, 3, 5
\]

in which \( E_i \) is the transformation matrix between the fixed coordinate frame \( X’-Y’-Z’ \) to the auxiliary coordinate frame \( x’_i-y’_i-z’_i \) (see Figures 2 and 4). In the \( x’_i-y’_i-z’_i \) coordinate frame, the angles \( \Theta_i \) and \( \Phi_i \) can be easily determined by:

\[
\begin{align*}
\Theta_i &= \arctan\left( \frac{-S_{12}}{S_{13}} \right) \\
\Phi_i &= \arctan\left( \frac{S_{22}}{S_{23}} \right)
\end{align*}
\]

For the sample SP, which is shown in Figure 3, the joints are assembled so that \( E_i’s \) can be described as:
The sensitivity matrix in Jacobian form can then be defined as:

\[
E_i = \begin{bmatrix}
-\cos(\xi) & -\sin(\xi) & 0 \\
\sin(\xi) & -\cos(\xi) & 0 \\
0 & 0 & 1 \\
\cos(\eta) & -\sin(\eta) & 0 \\
\sin(\eta) & \cos(\eta) & 0 \\
0 & 0 & 1
\end{bmatrix}, \\
E_3 = \begin{bmatrix}
-\sin(\xi) & \cos(\xi) & 0 \\
-\cos(\eta) & \sin(\eta) & 0 \\
0 & 0 & 1
\end{bmatrix}, \\
E_5 = \begin{bmatrix}
-\sin(\xi) & \cos(\xi) & 0 \\
-\cos(\eta) & \sin(\eta) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(24)

3. Sensitivity analysis

3.1. Derivation of sensitivity relationships

One of the main advantages of the method presented in this paper is its low sensitivity to the measurement errors, especially in measurement of the platform's position. As discussed earlier, six angle measurements are indeed consisted of three sets of two rotations of lower universal joints (\(\Theta\) and \(\Phi\)) on the three legs (see Figures 2 and 4). They can be represented in a vector form as:

\[
v = \{\Theta_1, \Theta_3, \Phi_1, \Phi_3, \Phi_5\}^T
\]

(25)

Note that the measurements whether being accomplished on non-adjacent legs or not, have no effects on the proposed algorithm. However, the simulations based upon the sensitivity analysis proposed in this section; show that it is preferable to have these measurements on non-adjacent legs for obtaining more accuracy.

Remember that the algorithm outputs are the platform’s Euler angles \(\psi, \theta, \phi\) (Equation (21)) and the displacements \(X, Y, Z\) (Equation (15)) which are wholly referred to as \(\text{platform's states}\); these states can be defined in a vector form as:

\[
\Xi = \{\psi, \theta, \phi, X, Y, Z\}
\]

(26)

The sensitivity matrix in Jacobian form can then be defined as:

\[
\nabla_\Xi = \begin{bmatrix}
\frac{\partial \psi}{\partial \Theta_1} & \frac{\partial \psi}{\partial \Theta_3} & \cdots & \frac{\partial \psi}{\partial \Phi_5} \\
\frac{\partial \theta}{\partial \Theta_1} & \frac{\partial \theta}{\partial \Theta_3} & \cdots & \frac{\partial \theta}{\partial \Phi_5} \\
\frac{\partial \phi}{\partial \Theta_1} & \frac{\partial \phi}{\partial \Theta_3} & \cdots & \frac{\partial \phi}{\partial \Phi_5}
\end{bmatrix}^T
\]

(27)

Note that the partial derivatives in Equation (27) and consequently the errors in the calculation of platform’s states are highly dependent to the configuration of the Stewart platform. The error observed in platform’s states can be described by the following equation:

\[
d\Xi = \nabla_\Xi \cdot d\Xi
\]

(28)

in which \(d\Xi\) is the vector of measurements errors.

Since analytical solutions for obtaining partial derivatives in Equation (27) is almost impossible due to the complexity of the forward kinematics equations, a procedure is utilized based on numerical solution of the inverse kinematics (described in part 2.4) by Newton-Raphson method. The inverse kinematics is solved on various configurations and correspondingly various amounts for \(\nabla_\Xi \) (described in Equation (25)) are obtained.

To derive a proper procedure to perform the sensitivity analysis, first suppose an augmented matrix of measurements as:

\[
A = \begin{bmatrix}
\Theta_{1,1} & \Theta_{3,1} & \Phi_{1,1} & \Phi_{3,1} & \Phi_{5,1} \\
\Theta_{1,2} & \Theta_{3,2} & \Phi_{1,2} & \Phi_{3,2} & \Phi_{5,2} \\
\Theta_{1,3} & \Theta_{3,3} & \Phi_{1,3} & \Phi_{3,3} & \Phi_{5,3} \\
\Theta_{1,5} & \Theta_{3,5} & \Phi_{1,5} & \Phi_{3,5} & \Phi_{5,5}
\end{bmatrix}
\]

\[
\Xi_{\text{perturbed}} = \text{forward}(A_{\text{perturbed}})
\]

(29)

Next, the direct kinematics procedure described in Section 2.3 is applied to the perturbed matrix \(A_{\text{perturbed}}\) as:

\[
\Xi_{\text{perturbed}} = \text{forward}(A_{\text{perturbed}})
\]

(30)

The deviations of platform states (or the errors in platform’s state) are obtained as:

\[
d\Xi = \Xi_{\text{perturbed}} - \Xi
\]

(32)

the Jacobian matrix \(\nabla_\Xi\) in Equation (27) can also be obtained as:

\[
\nabla_\Xi = \left(\frac{d\Xi}{\partial \Theta_1}, \frac{d\Xi}{\partial \Theta_3}, \frac{d\Xi}{\partial \Phi_1}, \frac{d\Xi}{\partial \Phi_3}, \frac{d\Xi}{\partial \Phi_5}\right)^T
\]

(33)

A similar procedure can be derived for the conventional method in which the measurement vector consists of six lengths of the legs.

3.2. Different configurations

The partial derivatives in Equation (27) and consequently the errors in the calculation of platform’s states are highly dependent to the configuration of the Stewart platform. Hence, a full spectrum of the error estimation in
platform’s states is achieved once a full pattern of platform configurations is available. If such a pattern is attained, a statistical investigation can be accomplished to determine the overall sensitivity of the proposed method to approve its effectiveness over the conventional method. Attaining such a pattern is still a challenging task due to the obligation of considering quite a lot of possible configurations within the six-dimensional workspace, which is also a tedious task. The best way to achieve this goal is to take into the account several configurations uniformly distributed in the entire workspace. Hence, the sensitivity analysis proposed in this section, should be performed in 6D task-space on several configurations uniformly distributed in three levels: 1- on boundary of the workspace, 2- in two-thirds of the workspace, and 3- in one-third of the workspace. These levels are respectively associated with large, moderate, and small movements of the platform. To obtain the so-called uniformly distributed configurations, a novel method is proposed based on the solution of the inverse kinematics in 6D task-space by taking into the account the physical constrains of the SP.

3.2.1. Uniformly distributed configurations in 6D task-space

The workspace of parallel mechanisms is restricted by physical limitations such as maximum and minimum limits of legs’ lengths. Two forms of kinematic constrains are often considered for determining the workspace of the SP type hexapods.[14,15] These constraints are: length limits of legs and range of angular movements of joints.

The method, which is used here to obtain the boundary points of the workspace, is mainly based on the solution of the inverse kinematics. To do so, an iterative procedure is developed that solves the inverse kinematics on numerous points on the entire workspace so that the physical constrains at each point can be easily checked out. Without losing the generality, the method is first explained for 3D case because it becomes more feasible to be understood due to the availability of graphical facilities. Consider some vectors starting at the origin and pointing away toward directions which make equal angular divisions with each other. The vectors resemble to a dandelion or fire blast of exploding devices in firework (see Figure 5). Hence, the vectors are referred to as ‘Regular Radial Vectors’ abbreviated as ‘RRV.’ Then, locate several points on each vector in equal small distances from the origin as far as it surely exceeds the border of the workspace. Then in computer simulation, place the platform (or the end effector of the parallel mechanism) on the origin and start to move from point to point in the outward direction on each vector. Meanwhile, check the criteria of exceeding physical limits on each point until the platform goes beyond at least one of the aforementioned kinematic constrains. Once it happens, the point is saved as a boundary point in the corresponding direction. Then restart the same procedure on a different vector and continue this on. After doing so for all of RRV, several points are collected and saved as the border points of the workspace. The border points will have almost a uniform distribution on the border of the workspace due to the choice of the so-called RRV. The precision of this method surely depends on the density of the points as well as the compactness of RRV.

For the space of more than three dimensions, the procedure can still be utilized though no graphical demonstration is accessible but attention must be paid to the difference between the units of different axes correlated to different degrees of freedom. For example, in the six-degrees-of-freedom SP, three axes denoted by x, y, and z are related to position with unit of meter (m) and the three others denoted by ψ, θ, and ϕ are related to rotation with unit of radians (rad). To eliminate the effect of these differences, dimensionless axes are used by normalization of the platform states in the region of [−1, 1].

For the sample SP, 15,624 RRV is considered. Consequently, it yields 15,625 boundary points in 6D-space as the points of the so called ‘level 1.’ At the same time, the same number of points is selected on both two-thirds and one-third of the length of RRV to have the levels 2 and 3 discussed in Section 3.2. Hence, 3 × 15,624 points are considered in the overall workspace which are almost distributed uniformly.

3.3. Simulation results

The sensitivity analysis is applied to both the proposed and conventional methods with respect to the geometric parameters of the prototype SP listed in the Appendix 1. The measurements of the proposed method (i.e. the vector Y in Equation (25)) are accomplished in practice by means of six incremental encoders assembled on the
universal joints with an accuracy of 6000 pulses per revolution. This kind of encoders is accessible as well as inexpensive. It is further assumed that the measurements of the conventional method (i.e. the legs’ lengths) are brought about by means of six digital rulers with an accuracy of 0.005 mm.

For instance, the results for the level 1 (mentioned in the previous section) are depicted in Figure 6. Those are the measurement errors of the six independent axes: x, y, z, ψ, θ, and φ calculated on the 15,625 points of this level. Comparing the diagrams given in Figure 6, of the conventional and the proposed methods, it is seen that while the conventional method in average possesses lower errors in attitude measurements, the proposed method holds lower errors in position measurements. The variances of the errors are a good index for comparison of the results of the two methods as listed in Table 1. The average improvements of variances for the proposed method over the conventional method are shown in the three last rows of Table 1. It can be seen that the proposed method is less efficient in the measurement of orientation especially the one achieved in the roll direction (φ), but it operates better in measurement of position especially the one achieved in the sway direction (y).

Another outstanding feature can be notified in the data illustrated in Table 1. While the conventional method possesses a significant jump in the variance of errors in some directions, the proposed method keeps monotonic variance of errors in different directions. For instance, the conventional method in position measurements in the y direction possesses variances about 8–15 times larger than those hold in the x and z directions or in orientation measurements in the φ direction it possesses variances about 8–16 times lesser than those hold in the ψ and θ directions.

On the other hand, the errors of both the position and orientation measurements obtained from the proposed method are not much sensitive to direction since it shows almost a uniform error variance in all directions.

The results demonstrated in Figure 6 and Table 1 are dependent to both the accuracy of the sensors and the geometric dimensions of the mechanism. Hence, various simulations with different geometric dimensions are required to generalize the aforementioned conclusive results. The sample SP is so made that it can possess different geometries (See Appendix 1). Based on that, 10 simulations with different geometries are performed together with the laboratory tests on the sample SP. The simulation results regarding to the sensitivity analysis

Table 1. Comparison of variances of errors between the presented and conventional methods.

<table>
<thead>
<tr>
<th>Variance × (10⁻⁷) (rad²)</th>
<th>σ²(ψ) (m²)</th>
<th>σ²(θ) (rad²)</th>
<th>σ²(φ) (m²)</th>
<th>σ²(X) (m²)</th>
<th>σ²(Y) (m²)</th>
<th>σ²(Z) (m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The proposed method (6 rotary sensor)</td>
<td>Level 1</td>
<td>1.2311</td>
<td>0.8875</td>
<td>0.8840</td>
<td>0.0169</td>
<td>0.0168</td>
</tr>
<tr>
<td>Level 2</td>
<td>0.5384</td>
<td>0.3918</td>
<td>0.3916</td>
<td>0.0080</td>
<td>0.0079</td>
<td>0.0139</td>
</tr>
<tr>
<td>Level 3</td>
<td>0.1327</td>
<td>0.0971</td>
<td>0.0974</td>
<td>0.0022</td>
<td>0.0021</td>
<td>0.0035</td>
</tr>
<tr>
<td>The conventional method (6 linear sensor)</td>
<td>Level 1</td>
<td>0.2058</td>
<td>0.1506</td>
<td>0.0192</td>
<td>0.0628</td>
<td>0.0452</td>
</tr>
<tr>
<td>Level 2</td>
<td>0.0867</td>
<td>0.0624</td>
<td>0.0048</td>
<td>0.0272</td>
<td>0.0209</td>
<td>0.0158</td>
</tr>
<tr>
<td>Level 3</td>
<td>0.0211</td>
<td>0.0155</td>
<td>0.0009</td>
<td>0.0066</td>
<td>0.0051</td>
<td>0.0032</td>
</tr>
<tr>
<td>Improvement of the proposed method over the conventional method</td>
<td>Level 1</td>
<td>0.17</td>
<td>0.17</td>
<td>0.02</td>
<td>3.7</td>
<td>28.5</td>
</tr>
<tr>
<td>Level 2</td>
<td>0.16</td>
<td>0.16</td>
<td>0.01</td>
<td>3.4</td>
<td>26.6</td>
<td>1.1</td>
</tr>
<tr>
<td>Level 3</td>
<td>0.16</td>
<td>0.16</td>
<td>0.01</td>
<td>3</td>
<td>24.5</td>
<td>0.9</td>
</tr>
</tbody>
</table>
were very promising since it shows that in the proposed method the variances of the errors is slightly dependent to the geometry while in the conventional method the error variances is highly affected by the geometry.

Generally speaking, the proposed method works better than the conventional method in the ground of the position accuracy in various geometries and configurations of the SP. However, it is rather weaker in accurateness of the orientation measurement. And also, it is much less sensitive to geometric diversity than the conventional method.

The abovementioned comparison for sensitivity in position measurement highlights the benefits of the proposed method over the conventional method. The higher accuracy of the proposed method in position measurement is a remarkable outcome since in overall the measurement of position in parallel manipulators of hexapod’s family is more important and still much more complicated than the measurement of the orientation. Other than the proposed method as well as all the methods offered for the indirect determination of orientation in the Stewart platform [3–5,7–9], there are some alternative ways which directly measures the orientation of the platform. For instance, is the one done by the assembling the rate gyros on the platform [7,8]. Yet in the related publications, one cannot find any alternative way to directly measure the position of the platform because it is too challenging to deal with in practice. Hence, the capability of the proposed method in precise measurement of the position in the SP highlights the benefits of the proposed method over the conventional method.

4. Practical considerations in real-time application

The main objective of the proposed method is its application in the real-time solution of forward kinematics in SP which is essential in its motion control. It is mentioned in the previous section that although the proposed method works better than the conventional method in the ground of position measurement, it possesses a weaker performance, in orientation measurement which can be a drawback when high accuracies are demanded. This drawback can be removed by utilization of hybrid methods i.e. either a combination of both the proposed and the conventional methods or a combination of the proposed and a direct method of orientation measurement like the one done by rate gyros. The first hybrid method is accomplished on the sample SP and yields promising outcomes. It is demonstrated in Figure 7.

The length of legs can be measured by use of one of these devices: LVDTs, digital rulers, multi-turn potentiometers, or incremental encoders. Demonstrated in the figure, is the application of the conventional method together with the proposed method. In the sample laboratory SP, since the legs are driven by electrical servo motors their rotations are converted to the prismatic movement by utilization of ball-screw mechanism, it is suitable for this application to measure the legs’ length by assembling incremental encoders on servomotors’ shaft. Since a reductive ratio is needed to transmit the shaft’s rotation to the potentiometer, a worm-gear mechanism is mounted on the output shaft. Six encoders are assembled on the six servomotors to enable the conventional method to be run parallel to the proposed method.

5. Conclusion

A new practical method was proposed for real-time solution of the forward kinematics problem of the Stewart platform with six legs having universal joints at one end. The proposed concept can be generalized to other types of the parallel manipulators especially those belong to the hexapods’ family. It uses six measurements on universal joints of three legs’ which consist of the rotations of the three legs in two directions. It is preferred that the legs are non-adjacent for obtaining higher accuracy. The method was experimentally applied to a laboratory sample SP and it is observed that it offers a powerful tool in real-time solution of forward kinematics problem of the Stewart platform especially in the ground of position measurements.
The algorithms of forward kinematics as well as the inverse kinematics were derived based on geometric relationships. The sensitivity analysis was performed with respect to the measurement errors. This analysis was also performed on the conventional method. Both of these analyses were applied to the sample SP in various configurations which are regularly distributed in the workspace. The results show the following advantages over the conventional method:

1. The method is simpler in implementation, more practical, less expensive, and easier in maintenance.
2. Although it works weaker than the conventional method in orientation measurement, it operates more accurate than the conventional method in position measurement especially in lateral movements.
3. Because of absence of alternative ways, the measurement of position is more challenging than those of orientation. Hence the benefit of higher accuracy of the proposed method in position measurement over the conventional method is greatly highlighted.
4. On the contrary of the conventional method in which the accuracy of the both position and orientation measurements are strongly dependent to direction of movement and geometry of the SP, the proposed method is not too sensitive to direction of movement and geometry of the SP.
5. The disadvantage of lesser accuracy of the proposed method in orientation measurement in compared with the conventional method can be easily removed by combining the proposed method with either the conventional method or a direct method of orientation measurement such as use of rate gyros mounted on the platform.

Note
1. For synergistic configuration, the neutral position is the position in which all legs are in half stroke.

Notes on contributors

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References

Appendix 1. Sample 6-DoF Stewart platform and its characteristics

The sample SP is fabricated for the purpose of laboratory test of the proposed method. It consists of two rings as the base and platform circles, connected to each other by means of six legs (or linear actuators). The movements of the legs are controlled by servo electrical DC actuators. The rotation of servomotors is converted to the prismatic movement by utilization of ball-screw mechanism. The legs possess ball-joints on their top and universal-joints at their bottom. The use of universal-joints in Stewart platforms is convenient to prevent the spin of legs. The SP is so made that its structure is variable because of achieving different tests with variants of geometric parameters. It is done by: 1- making several holes on the periphery of the base and platform rings to get the ability of locating the ends of legs in different angular attitudes and 2- placement of joints on bars which can slide in radial direction to give different radii for base and platform circles. After adjustment of positions of the bars, they are fastened by bolts.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top circle’s radius (m)</td>
<td>0.4045</td>
</tr>
<tr>
<td>Low circle’s radius (m)</td>
<td>0.4045</td>
</tr>
<tr>
<td>Central angle between adjacent joints on top (deg)</td>
<td>10°</td>
</tr>
<tr>
<td>Central angle between adjacent joints on bottom (deg)</td>
<td>30°</td>
</tr>
<tr>
<td>diameter of actuator’s cylinder (m)</td>
<td>0.057</td>
</tr>
<tr>
<td>diameter of actuator’s rod (m)</td>
<td>0.02</td>
</tr>
<tr>
<td>Actuator’s minimum length (m)</td>
<td>0.5175</td>
</tr>
<tr>
<td>Actuator’s maximum length (m)</td>
<td>0.8775</td>
</tr>
<tr>
<td>Upper joint’s bending limit (deg)</td>
<td>25°</td>
</tr>
<tr>
<td>Lower joint’s bending limit (deg)</td>
<td>35°</td>
</tr>
</tbody>
</table>