Conditions for \( L \)-th band filters of order \( 2N \) as cascades of identical linear-phase FIR spectral factors of order \( N \)

Amir Eghbali\textsuperscript{a,}\textsuperscript{*}, Tapio Saramäki\textsuperscript{b}, Håkan Johansson\textsuperscript{a}

\textsuperscript{a} Division of Electronics Systems, Department of Electrical Engineering, Linköping University, Sweden
\textsuperscript{b} Department of Signal Processing, Tampere University of Technology, Finland

\textbf{A R T I C L E  I N F O}

Article history:
Received 22 May 2013
Received in revised form 13 September 2013
Accepted 18 October 2013
Available online 25 October 2013

Keywords:
Pulse shaping
\( L \)-th band filters
Spectral factors
FIR filters
Linear-phase

\textbf{A B S T R A C T}

This paper presents formulas for the number of optimization parameters (degrees of freedom) when designing Type I linear-phase finite-length impulse response (FIR) \( L \)-th band filters of order \( 2N \) as cascades of identical linear-phase FIR spectral factors of order \( N \). We deal with two types of degrees of freedom referred to as (i) the total degrees of freedom \( D_T \) and (ii) the remaining degrees of freedom \( D_R \). Due to the symmetries or antisymmetries in the impulse responses of the spectral factors, \( D_T \) roughly equals \( N/2 \). Some of these parameters are specifically needed to meet the \( L \)-th band conditions because, in an \( L \)-th band filter, every \( L \)th coefficient is zero and the center tap equals \( 1/L \). The remaining \( D_R \) parameters can then be used to improve the stopband characteristics of the overall \( L \)-th band filter. We derive general formulas for \( D_R \) with given pairs of \( L \) and \( N \). It is shown that for a fixed \( L \), the choices of \( N \), in a close neighborhood, may even decrease \( D_R \) despite increasing the arithmetic complexity, order, and the delay.

\( \textcopyright \) 2013 Elsevier B.V. All rights reserved.

\textbf{1. Introduction}

Nyquist or \( L \)-th band filters find applications in, e.g., filter banks (FBs), transmultiplexers (TMUXs), pulse shaping, and interpolation/decimation [1,2]. For pulse shaping, these filters are used as matched pairs of \( H(z) \) and \( G(z) \) where \( P(z) = H(z)G(z) \) is an \( L \)-th band filter such that, in the causal case, we have [1]

\[
V(z) = \sum_{k=0}^{L-1} P(zW_L^k) = z^{-N}, \quad W_L = e^{-j2\pi/L}.
\] (1)

As the order of an \( L \)-th band filter must be even [1], this paper assumes \( P(z) \) to be a Type I linear-phase finite-length impulse response (FIR) filter of order \( 2N \) where \( N \) can be either odd or even. It is generally expressed as

\[
N = LM - k, \quad k \in \{0, 1, ..., L-1\}, \quad M \text{ integer}.
\] (2)

If \( P(z) \) is a causal Type I linear-phase FIR \( L \)-th band filter of order \( 2N \), we have [1]

\[
p(2N - n) = p(n), \quad n = 0, 1, ..., 2N,
\] (3)

where

\[
p(N) = \frac{1}{L},
\] (4)

and

\[
p(N - mL) = 0, \quad m = 1, 2, ..., \left[ \frac{N}{L} \right].
\] (5)

Here, \( [\cdot] \) represents the floor operation. Ideally, \( p(n) \) has an infinite length and a zero transition band. Then, \( p(n) = \text{sinc}(n) = \sin(\pi n)/(\pi n) \) is the only solution [2]. In practice, one often prefers FIR approximations for \( p(n) \) and filter optimization results in more efficient solutions than the truncated (or windowed) versions of \( \text{sinc}(n) \) [3].
1.1. Contribution of the paper and relation to previous work

The literature on the design of Lth-band filters can be classified into two groups. The first group, e.g., [3–5], derives the spectral factors of an Lth-band filter whereas the second group, e.g., [6–8], designs Lth-band filters without deriving the spectral factors. Almost all of these methods use nonlinear-phase spectral factors. Then, the phase response of $G(z)$ compensates for that of $H(z)$ so that $P(z)$ has a linear phase. However, we can obtain Lth-band filters with linear-phase FIR spectral factors as well [9]. As opposed to the present paper, [9] (or any other literature to the best of our knowledge) does not treat the derivations for the degrees of freedom.

This paper assumes linear-phase FIR spectral factors. In a practical communication system, $h(n)$ is usually matched to $g(n)$ as $g(n) = h^*(N–n)$ where the superscript * denotes complex conjugation. With Type I and II linear-phase FIR filters (as mostly considered in this paper) having real symmetric coefficients, we thus automatically get $g(n) = h(N–n)$ giving $H(z) = G(z)$. This choice has two advantages as follows:

- The implementation complexity is reduced due to the linear-phase property\(^1\) and thus the coefficient symmetry or antisymmetry.
- The choice $H(z) = G(z)$ reduces the design complexity due to the reduced number of unknowns.

This paper focuses on deriving formulas for the degrees of freedom in the filter design. We deal with two types of degrees of freedom referred to as (i) the total degrees of freedom $D_T$ and (ii) the remaining degrees of freedom $D_R$. With our derived formulas, we can determine whether two different orders, say $N = 104$ and $N = 105$, as seen later in Table 8, give improvements in $D_R$ or not. This helps decide which of these orders must be selected so that we minimize the arithmetic complexity while maximizing $D_R$. We will later see that for a given specification, as in Table 8, it is better to select a filter of order $N = 104$ because such a filter has a lower arithmetic complexity, as compared to a filter of order $N = 105$. However, it has a larger $D_R$.

With linear-phase FIR spectral factors, $D_T$ is roughly equal to half of the order of each spectral factor, i.e., $N/2$. However, in the design of an Lth-band filter, we cannot utilize all of these $D_T$ parameters. The reason is that these $D_T$ parameters are further constrained because $p(n)$ has some predefined coefficients, according to (4) and (5). Consequently, the optimization routine can only determine $D_R < D_T$ parameters. Therefore, it is important for a designer to choose a suitable $N$ so that the value of $D_R$ is reasonable while keeping the arithmetic complexity as low as possible. This paper derives explicit formulas for $D_R$. It will be shown that the value of $D_R$ can be either $D_R = -1$, $D_R = 0$, or $D_R > 0$.

With $D_R = -1$, we cannot obtain Lth-band filters because we do not have enough degrees of freedom even to meet (4) and (5). If $D_R = 0$, we will show that each spectral factor should be composed of a maximum of two nonzero coefficients. Although this results in filters which meet (4) and (5), the stopband characteristics are not useful in practice. With $D_R > 0$, we can obtain arbitrarily good Lth-band filters by a proper selection of $N$. For some consecutive values of $N$, we can however not improve the stopband characteristics of $P(z)$ because $D_R$ does not increase. Specifically, the arithmetic complexity, order, and the delay associated with an order $N + 1$ can be larger than those related to an order $N$. However, the corresponding $D_R$ can be the same or even smaller than that of an order $N$.

1.2. Paper outline

Following this introduction, some prerequisites are outlined in Section 2 whereas Sections 3 and 4 derive the formulas for the degrees of freedom. The design examples are illustrated in Section 5 with the concluding remarks given in Section 6.

2. Prerequisites

Consider the transfer function, assuming $H(z) = G(z)$,

$$P(z) = \sum_{n=0}^{2N} p(n)z^{-n} = G(z)G(z) \quad (6)$$

with

$$G(z) = \sum_{n=0}^{N} g(n)z^{-n}. \quad (7)$$

From (3), (6), and (7), one has for $0 \leq n \leq N$,

$$p(n) = \left\{ \begin{array}{ll}
g(0)g(0) & n = 0 \\
\left( g\left(\frac{n}{2}\right) g\left(\frac{n}{2}\right) \right) + 2 \sum_{m=0}^{n/2-1} g(m)g(n-m) & n \text{ even} \\
2 \sum_{m=0}^{(n-1)/2} g(m)g(n-m) & n \text{ odd}. \end{array} \right. \quad (8)$$

The number of constraints, bound by the Lth-band conditions in (4) and (5), is

$$D_R = \left\lfloor \frac{N}{L} \right\rfloor + 1 = \left\lfloor \frac{N + L}{L} \right\rfloor. \quad (9)$$

The reason for (9) is that every Lth coefficient, of $p(n)$, is zero and its center tap equals $1/L$. As we will see in Section 3.2.1, these constraints restrict some of the coefficients in $g(n)$. The remaining coefficients, of $g(n)$, are then used to improve the characteristics of $P(z)$.

3. Type I and II linear-phase FIR spectral factors

If $G(z)$ is a Type I or II linear-phase FIR filter, we have

$$g(N-n) = g(n), \quad n = 0, 1, ..., N. \quad (10)$$

Without the conditions in (4) and (5), the total degrees of freedom, i.e., the total number of distinct coefficients

\(^1\) Although nonlinear-phase FIR filters require lower orders than the linear-phase FIR filters, their arithmetic complexity, in terms of the number of multipliers, is larger than that of the linear-phase FIR filters [10].
of \( g(n) \), is
\[
D_T = \left\lfloor \frac{N}{2} \right\rfloor + 1 = \left\lfloor \frac{N + 2}{2} \right\rfloor.
\] (11)

From (9) and (11), the remaining degrees of freedom thus becomes
\[
D_R = D_T - D_B = \left\lfloor \frac{N + 2}{2} \right\rfloor - \frac{N + L}{L} = \left\lfloor \frac{N}{2} \right\rfloor - \frac{N}{L}.
\] (12)

We can generally increase \( D_R \) by increasing \( N \) but this increase is not monotonic. For example, consider \( L=5 \) and \( M=21 \). Here, the choices \( k=0 \) and \( k=3 \), in (2), lead to \( N=105 \) and \( N=102 \), respectively. Then, (12) gives \( D_R=31 \) in both cases. Further discussion on this will be given in Section 3.2.

3.1. Degrees of freedom

Inserting (2) in (12), it is easy to verify some special cases, i.e., \( D_R=0 \), with some values of \( L, M, \) and \( k \). These special cases are summarized in Table 1. For \( L=2 \), we always have \( D_R=0 \) and there is also a resemblance between the conventional alias-free two-channel maximally decimated FBs which we will discuss in Section 3.1.1. If \( D_R > 0 \), we can use the formulas derived later in Section 3.1.2.

3.1.1. \( L=2 \)

In this case, (2) and (12) give
\[
D_R = \left\lfloor \frac{2M+2}{2} \right\rfloor - \left\lfloor \frac{2M+2}{2} \right\rfloor = 0, \quad N=2M
\] (13)

and
\[
D_R = \left\lfloor \frac{2M-1+2}{2} \right\rfloor - \left\lfloor \frac{2M-1+2}{2} \right\rfloor = 0, \quad N=2M-1.
\] (14)

Therefore, all of the degrees of freedom are used to satisfy (4) and (5). With linear-phase FIR filters in conventional alias-free two-channel maximally decimated FBs, the only option is to have the analysis filters as \( H(z) = az^{-A} + \beta z^{-B} \) and \( G(z) = \gamma(az^{-A} - \beta z^{-B}) \) which meet the power complementary property defined as, for \( z = e^{j\omega T} \), [11]
\[
|H(e^{j\omega T})|^2 + |G(e^{j\omega T})|^2 = 1.
\] (15)

Note that the symmetry, in an \( N \)-th order Type II linear-phase FIR filter \( H(z) = az^{-A} + \beta z^{-B} \), necessitates that \( A+B = N \). To meet (15), appropriate values of \( \alpha, \beta, \) and \( \gamma \) should also be chosen. Hence, the filters must have a maximum of two nonzero coefficients [1]. In such a case, the stopband will not have reasonable characteristics and the resulting filters are not useful in practice.

In the context of this paper, the choice \( G(z) = H(z) = (1+z^{-1}) \) meets (4) and (5) with \( L=2 \) but it does not meet (15). This shows a fundamental difference from the conventional alias-free two-channel maximally decimated FBs. In this paper, the only choice with identical \( N \)-th order Type I linear-phase FIR spectral factors is
\[
H(z) = G(z) = az^{-N/2}.
\] (16)

For a proof, see Appendix A. Then, (6) gives \( P(z) = a^2 z^{-N} \) and (4) results in \( a^2 = \frac{1}{2} \) leading to \( \alpha = \pm \sqrt{2} \). On the other hand, with identical Type II linear-phase FIR spectral factors of order \( N \), we have
\[
H(z) = G(z) = az^{-A} + az^{-B}.
\] (17)

For a proof, see Appendix B. As \( N = A + B \) must be odd for a Type II linear-phase FIR filter, we should then have \( A = 2m_1 \) and \( B = 2m_1 + 1 \) for some integers \( m_1 \) and \( m_2 \) where \( m_1 \leq m_2 \). If \( g(0) \neq 0 \), we must have \( m_1 = 0 \), as shown in Appendix B. With (6) and (17), we get \( P(z) = a^2 z^{-2A} + 2a^2 z^{-(A+B)} + a^2 z^{-2B} \) where (4), (6), and \( A+B = N \), lead to \( 2a^2 = \frac{1}{2} \) giving \( \alpha = \pm \frac{1}{\sqrt{2}} \).

3.1.2. General \( L \)

\[
N = LM,
\]
\[N = LM - 1,
\]
\[N = LM - 2,
\] (18)

Assuming general orders as in (2), the expression for \( D_R \) becomes as in (18). For \( N=LM \), we have
\[
D_R = \left\lfloor \frac{(L-2)M}{2} \right\rfloor,
\] (19)

whereas for \( N = LM-k \), \( k \in \{1,2,\ldots,L-1\} \), we get
\[
D_R = \left\lfloor \frac{(L-2)M-(k-2)}{2} \right\rfloor.
\] (20)

Note that these formulas apply to any \( L, M, \) and \( k \) thereby allowing us to even verify the special cases of Table 1.

<table>
<thead>
<tr>
<th>( M )</th>
<th>( L )</th>
<th>( k )</th>
<th>( D_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>( k \in {0,1} )</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>( k \in {0,2} )</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( L &gt; 3 )</td>
<td>( k = L-1 )</td>
<td>0</td>
</tr>
<tr>
<td>( M &gt; 1 )</td>
<td>2</td>
<td>( k \in {0,1} )</td>
<td>0</td>
</tr>
</tbody>
</table>
3.2. Discussion

The value of $D_R$ generally increases (although in steps) when increasing $N$. This is shown in Fig. 1 for two pairs of $L$ and $M$. Therefore, the designer must consider the amount of (possible) increase, in $D_R$, when moving from an order, say $N$, to another order, say $N+1$. This helps avoid cases where one cannot increase $D_R$ despite increasing $N$. The values of $L$ and $M$, in (2), could generally be odd or even leading to a total of four cases. Fig. 2 shows the trend of $D_R$ for some pairs of $L$ and $M$ where we have covered all of these four cases. With Case 4, i.e., when both $L$ and $M$ are odd, the choices of $k \in 0, 2, 3$ lead to the same $D_R$. With choices other than those in Case 4, the values of $k \in 0, 1, 2$ lead to the same $D_R$. The above discussion requires $L \geq 4$ so as to allow $k \in 0, 1, 2, 3$ according to (2).

3.2.1. Simple illustrative examples

The above-mentioned facts can also be understood by the simple examples in Tables 2 and 3. Consider, for example, $k=0$ and $k=1$ in Table 2. With $k=1$, we get

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$g(n)$</th>
<th>$D_T$</th>
<th>$D_B$</th>
<th>$D_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>$a, b, c, b, a$</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$a, b, c, b, a$</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>$a, b, b, a$</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>$a, b, a$</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>$a, a$</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2
Illustration of the values of $D_T$, $D_B$, and $D_R$ for some trivial filters with $L=5$ and $M=1$, as in Case 4, using (9), (11), and (12).

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$g(n)$</th>
<th>$D_T$</th>
<th>$D_B$</th>
<th>$D_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>$a, b, c, b, a$</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$a, b, b, a$</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$a, b, a$</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$a, a$</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3
Illustration of the values of $D_T$, $D_B$, and $D_R$ for some trivial filters with $L=4$ and $M=1$, as in Cases 1–3, using (9), (11), and (12).
As can be seen, despite an increased arithmetic complexity or delay, it is the values of \( N \) and \( L \) which determine whether \( D_R \) increases, does not change, or even decreases. In these tables, we have chosen \( M = 1 \) so as to simplify the illustration.

With \( L = 5, N = 5 \), and \( g(n) = a, b, c, b, a \), we can use (8) so as to obtain \( p(n) = a^2, 2ab, b^2 + 2ac, 2ab + 2bc, 2a^2 + 2b^2 + 2c^2 \) for \( 0 \leq n \leq N \). According to (4) and (5), we will then have \( p(0) = 0 \) giving \( a = 0 \) and \( p(5) = 2a^2 + 2b^2 + 2c^2 = 1/L \). In this case, and with \( a = 0 \), the coefficients \( b \) and \( c \) are related as \( b = \sqrt{1/2L - c^2} \). Therefore, the only free optimization parameter is \( c \) which is in accordance with \( D_R = 1 \) as outlined in Table 2.

On the other hand, with \( L = 5, N = 4, \) and \( g(n) = a, b, c, b, a \), (8) gives \( p(n) = a^2, 2ab, b^2 + 2ac, 2ab + 2bc, 2a^2 + 2b^2 c^2 \) for \( 0 \leq n \leq N \). According to (4), we will then have \( p(4) = c^2 + 2a^2 + 2b^2 = 1/L \). For this trivial example, (5) does not force any of the coefficients of \( p(n) \) to be zero because the order \( N \) is too low. Consequently, we can obtain \( a \) in terms of the other coefficients as \( a = \sqrt{1/2L - c^2/2 - b^2} \). Therefore, the only free optimization parameters are \( b \) and \( c \). This is also in accordance with \( D_R = 2 \) as outlined in Table 2.

## 4. Type III and IV linear-phase FIR spectral factors

If the spectral factors are not lowpass filters, we could choose them to be Type III or IV linear-phase FIR filters of order \( N \) where

\[
g(N-n) = -g(n), \quad n = 0, 1, ..., N. \tag{21}
\]

This finds applications in, e.g., FBs and TMUXs, which utilize bandpass or highpass filters in different branches. Irrespective of the type of the spectral factors, the filter \( P(z) \) will always be a Type I linear-phase FIR filter of order \( 2N \). Consequently, (9) does not change. However, (11) will be different for Type III linear-phase FIR spectral factors because, in such a case, we have \( g(N/2) = 0 \). With Type III linear-phase FIR spectral factors, we hence get

\[
D_T = \left\lfloor \frac{N}{2} \right\rfloor \tag{22}
\]

and

\[
D_R = D_T - D_B = \left\lfloor \frac{N}{2} \right\rfloor - \left\lfloor \frac{N+L}{2} \right\rfloor = \left\lfloor \frac{N}{2} \right\rfloor - \left\lfloor \frac{N}{L} \right\rfloor - 1. \tag{23}
\]

### Table 4

<table>
<thead>
<tr>
<th>( M )</th>
<th>( L )</th>
<th>( k )</th>
<th>( D_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M \geq 1 )</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>( k = 0 )</td>
<td>2</td>
</tr>
<tr>
<td>( L &gt; 4 )</td>
<td>( k = L-2 )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>( k = 0 )</td>
<td>0</td>
</tr>
</tbody>
</table>

For Type IV linear-phase FIR spectral factors, (11) still holds and the discussion, in Section 3, also applies to the case with Type IV linear-phase FIR spectral factors.

### 4.1. Degrees of freedom

For Type III linear-phase FIR spectral factors, and like in Section 3, there are some special cases, i.e., \( D_B = 0 \) or \( D_R = -1 \), for some values of \( L, M, \) and \( k \). These are summarized in Table 4. From (9), (22), and (23), \( L = 2, M = 2, \) and \( k = 0 \), we get \( D_R = -1 \) which means that we cannot even meet (4) and (5). For example, consider \( L = 2 \) and a trivial Type III linear-phase FIR filter \( G(z) = \frac{1}{2} (1 - z^{-4}) \) ignoring the sign, the filter \( P(z) = \frac{1}{4} (1 - 2z^{-4} + z^{-8}) \) then meets (4). However, it does not meet (5).

With \( M = 2, L = 3, \) and \( k = 0 \), we get \( D_R = 0 \). As shown in Appendix C, the only option is hence to choose

\[
G(z) = a z^{-A} - a z^{-B}, \quad A + B = N, \quad A \in (1, 2]. \tag{24}
\]

Then, \( P(z) = a^2 z^{-2A} - 2a^2 z^{-A+B} + a^2 z^{-2B} \) and (4) can be met if \( 2a^2 = \frac{1}{4} \) giving \( a = \pm \frac{1}{\sqrt{6}} \). If \( D_R > 0 \), we can use (23) to derive formulas similar to (19) and (20). With \( N = LM \), we then have

\[
D_R = \left[ \frac{(L-2)M}{2} \right] - 1, \tag{25}
\]

whereas for \( N = LM - k, k \in (1, 2, ..., L-1) \), we get

\[
D_R = \left[ \frac{(L-2)M - (k-2)}{2} \right] - 1. \tag{26}
\]

These formulas also apply to any \( L, M, \) and \( k \) so that we can easily verify the special cases of Table 4.

## 5. Design examples

In this section, we will give the relations between the ripples of \( H(z) \) and \( G(z) \). Then, we will present some design examples assuming Type I and II linear-phase FIR spectral factors.

### 5.1. Overall filter ripples

In an \( L \)th-band filter, the passband/stopband edges are

\[
\omega_c T = \frac{\pi(1-\rho)}{L}, \quad \omega_s T = \frac{\pi(1+\rho)}{L}, \tag{27}
\]

where \( \rho \) is the roll-off factor. Also, the passband ripple \( \delta_c \) and the stopband ripple \( \delta_s \) are related as \( \delta_c \leq (L-1)\delta_s \) [12].
With (6), we have [9]
\[ |G(e^{j\omega T})| \leq \sqrt{\delta_i}, \quad \omega T \in [\omega_i T, \pi], \]
\[ 1 - \delta^c_i \leq |G(e^{j\omega T})| \leq 1 + \delta^c_i, \quad \omega T \in [0, \omega_i T], \quad \delta^c_i \leq \frac{1 - \sqrt{\delta_i}}{2}. \]
(29)

5.2. Filter design

With noncausal ideal (impractical) filters, we have
\[ p(z) = \begin{cases} 1 & \text{in the passband} \\ 0 & \text{in the stopband}. \end{cases} \]
(30)

In practice, such ideal responses must be approximated by finding suitable filter coefficients. This section treats a minimax design problem as
\[ \min \delta \text{ subject to } \]
\[ p(N) = \frac{1}{T}, \]
\[ p(N - mL) = 0, \quad m = 1, 2, \ldots, \left\lfloor \frac{N}{T} \right\rfloor, \]
\[ |P(e^{j\omega T})| \leq \delta, \quad \omega T \in [0, \omega_i T]. \]
(31)

Here, \( \omega_i T \) is given by (27) where we have discretized it into 2001 points. In the examples of this paper, we have \( M = 5, 6, 20, 21, L = 5, \rho = 0.125, 0.25, \) and \( k = 0, 1, \ldots, L - 1. \) We also use the \texttt{firpm} algorithm in MATLAB along with (27)–(29) to obtain the initial solutions to the nonlinear optimization problem of (31). To solve (31), we use the \texttt{fminimax} algorithm in MATLAB.

5.3. Quality measures

For pulse shaping filters, the peak inter-symbol interference (ISI) is defined as [3]
\[ P_{\text{ISI}} = \sum_{k = -\infty}^{\infty} |h(kL + K)|, \quad K = \frac{2N}{\pi} \text{ mod } L. \]
(32)

with \((2N/\pi) \text{ mod } L\) being the remainder of \((2N/\pi)/L\). A Type I or II linear-phase FIR filter \( G(z) \), of order \( N \), requires \( C_{\text{Mult.}} = \left\lfloor \frac{N}{2} \right\rfloor + 1 \) even \( N \)
\[ C_{\text{Mult.}} = \begin{cases} \left\lfloor \frac{N}{2} \right\rfloor + 1 & \text{even } N \\ N + 1 & \text{odd } N \end{cases} \]
(33)
multipliers and \( C_{\text{Add}} = N + 1 \) adders. The corresponding delay is
\[ \Delta = \frac{N}{2}. \]
(35)

5.4. Results and discussion

Tables 5–12 show the design parameters for some Nyquist filters obtained from (31). Here, \( \delta \) is the objective value in (31). Also, Fig. 3 plots the frequency responses of \( P(e^{j\omega T}), G(e^{j\omega T}), \) and \( |V(e^{j\omega T}) - e^{-j\rho 2\pi T}| \) for the design with \( k = 1 \) in Table 8. As can be seen, if \( \Delta > 0 \), we can obtain \( L \)-th band filters and we can also improve the stopband attenuation. However, for different values of \( N, D_R, C_{\text{Mult.}}, C_{\text{Add.}}, \) and \( \Delta \), the values of \( \delta \) become different. In other words, and to choose suitable parameters, the designer
must consider (i) the remaining degrees of freedom $D_k$, (ii) the number of multipliers $C_{\text{Mult}}$, (iii) the number of adders $C_{\text{Add}}$, and (iv) the delay $\Delta$. For example, in Table 5, $k=3$ will be a suitable choice whereas in Table 6, $k=1$ seems to be suitable. Furthermore, in Table 9, it is better to select $k=1$ but Table 12 shows that $k=2$ is a better option.

6. Conclusion

Formulas were derived for the degrees of freedom when designing $L$th-band filters as cascades of identical linear-phase FIR spectral factors. These derivations can be used to determine the number of degrees of freedom as well as the feasibility of useful $L$th-band filters. For some filter orders, it does not help to increase the filter order by a small number, because $D_k$ does not increase. Then, no improvements are obtained in the frequency response despite the increased arithmetic complexity, order, and delay.

Appendix A. Proof of (16)

Here, we consider $N$ to be even with $p(N) = \frac{1}{2}$ and $p(n) = 0$ for $n \in \{0, 2, ..., N-2\}$. According to (8),

$$p(0) = g^2(0) = 0 \Rightarrow g(0) = 0.$$  \hspace{1cm} (A.1)

We then automatically get, from (8), that $p(1) = 2g(0) \quad g(1) = 0$ for an arbitrary $g(1)$. Also,

$$p(2) = g^2(1)+2g(0)g(2) = 0 \Rightarrow g(1) = 0.$$  \hspace{1cm} (A.2)

and, consequently, we automatically have $p(3) = 2g(0)\quad g(3) + 2g(1)g(2) = 0$ for arbitrary $g(2)$ and $g(3)$. However, as $p(4) = g^2(2)+2g(0)g(4)+2g(1)g(3) = 0 \Rightarrow g(2) = 0$.  

we then, automatically, get $p(5) = 2g(0)g(5)+2g(1)g(4)+2g(2)g(3) = 0$ with arbitrary $g(4)$ and $g(5)$. Continuing in this way until $n = N-1$, we can conclude that $g(n) = 0, \quad 0 \leq n \leq \frac{N}{2} - 1$. \hspace{1cm} (A.4)

Then, (8) gives (A.5) for $n=N$.

$$p(N) = g^2(\frac{N}{2})+2g(0)g(N)+2g(1)g(N-1)+2g(2)g(N-2) + \cdots + g\left(\frac{N}{2}-1\right)g\left(N-\frac{N}{2}+1\right).$$  \hspace{1cm} (A.5)

In (A.5), all of the terms, except the first term, are zero. This is a result of (A.4). We should hence have $g(N/2) = \pm \sqrt{\frac{2}{\pi}} \pi$ which would then lead to the only choice as $G(z) = \pm \sqrt{\frac{2}{\pi}} \pi$ thereby meeting (16). \(\square\)

Appendix B. Proof of (17)

In this case, $N$ is assumed to be odd with $p(N) = \frac{1}{2}$ and $p(n) = 0$ for $n \in \{1, 3, ..., N-2\}$. Here,

$$p(0) = g^2(0) \neq 0 \Rightarrow g(0) \neq 0,$$  \hspace{1cm} (B.1)

and

$$p(1) = 2g(0)g(1) = 0 \Rightarrow g(1) = 0.$$  \hspace{1cm} (B.2)

From $p(2) = g^2(1)+2g(0)g(2)$, we can have an arbitrary value for $p(2)$ if $g(2)$ has an arbitrary value. Further, $p(3) = 2g(0)g(3)+2g(1)g(2) = 0 \Rightarrow g(3) = 0$.  \hspace{1cm} (B.3)

As $p(4) = g^2(2)+2g(0)g(4)+2g(1)g(3)$ could also have an

### Table 10

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$C_{\text{Mult}}$</th>
<th>$C_{\text{Add}}$</th>
<th>$\Delta$</th>
<th>$D_k$</th>
<th>$\delta$ (dB)</th>
<th>$\rho_{\text{SI}}$ (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>30</td>
<td>16</td>
<td>31</td>
<td>15</td>
<td>9</td>
<td>-40.9151</td>
<td>-336.5294</td>
</tr>
<tr>
<td>1</td>
<td>29</td>
<td>15</td>
<td>30</td>
<td>14.5</td>
<td>9</td>
<td>-42.6833</td>
<td>-345.673</td>
</tr>
<tr>
<td>2</td>
<td>28</td>
<td>15</td>
<td>29</td>
<td>14</td>
<td>9</td>
<td>-40.9151</td>
<td>-341.4559</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
<td>14</td>
<td>28</td>
<td>13.5</td>
<td>8</td>
<td>-37.6308</td>
<td>-342.3958</td>
</tr>
<tr>
<td>4</td>
<td>26</td>
<td>14</td>
<td>27</td>
<td>13</td>
<td>8</td>
<td>-35.0628</td>
<td>-337.2904</td>
</tr>
</tbody>
</table>

### Table 11

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$C_{\text{Mult}}$</th>
<th>$C_{\text{Add}}$</th>
<th>$\Delta$</th>
<th>$D_k$</th>
<th>$\delta$ (dB)</th>
<th>$\rho_{\text{SI}}$ (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
<td>51</td>
<td>101</td>
<td>50</td>
<td>30</td>
<td>-96.7273</td>
<td>-337.3329</td>
</tr>
<tr>
<td>1</td>
<td>98</td>
<td>50</td>
<td>99</td>
<td>49</td>
<td>30</td>
<td>-96.7345</td>
<td>-337.2863</td>
</tr>
<tr>
<td>2</td>
<td>97</td>
<td>49</td>
<td>98</td>
<td>48.5</td>
<td>29</td>
<td>-93.7384</td>
<td>-337.5787</td>
</tr>
<tr>
<td>3</td>
<td>96</td>
<td>49</td>
<td>97</td>
<td>48</td>
<td>29</td>
<td>-88.0725</td>
<td>-337.1671</td>
</tr>
</tbody>
</table>

### Table 12

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$C_{\text{Mult}}$</th>
<th>$C_{\text{Add}}$</th>
<th>$\Delta$</th>
<th>$D_k$</th>
<th>$\delta$ (dB)</th>
<th>$\rho_{\text{SI}}$ (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>105</td>
<td>53</td>
<td>106</td>
<td>52.5</td>
<td>31</td>
<td>-96.1375</td>
<td>-336.757</td>
</tr>
<tr>
<td>1</td>
<td>104</td>
<td>53</td>
<td>105</td>
<td>52</td>
<td>32</td>
<td>-97.0223</td>
<td>-340.6256</td>
</tr>
<tr>
<td>2</td>
<td>103</td>
<td>52</td>
<td>104</td>
<td>51.5</td>
<td>31</td>
<td>-97.534</td>
<td>-333.3645</td>
</tr>
<tr>
<td>3</td>
<td>102</td>
<td>52</td>
<td>103</td>
<td>51</td>
<td>31</td>
<td>-95.9889</td>
<td>-334.3385</td>
</tr>
<tr>
<td>4</td>
<td>101</td>
<td>51</td>
<td>102</td>
<td>50.5</td>
<td>30</td>
<td>-96.5959</td>
<td>-335.7096</td>
</tr>
</tbody>
</table>

Fig. 3. The frequency responses of $\Phi(e^{j\omega T})$, $G(e^{j\omega T})$, and $|V(e^{j\omega T})-e^{-j\omega T}|$ for the case with $k=1$ in Table 8. In the upper figure, the solid line represents $\Phi(e^{j\omega T})$ whereas the dotted line represents $G(e^{j\omega T})$. 

Please note that the text contains a variety of mathematical expressions and formulas, which are not part of the original image.
arbitrary value, we can then have an arbitrary value for \( g(4) \). In addition,

\[
p(5) = 2g(0)g(5) + 2g(1)g(4) + 2g(2)g(3) = 0 \Rightarrow g(5) = 0.
\]  

(B.4)

Continuing in this way until \( n = N - 1 \) and ignoring the symmetry property in (10), it is easy to conclude that

\[
g(n) = \begin{cases} 
0, & n \text{ odd, } n \leq N - 2 \\
\text{arbitrary, } & n \text{ even, } n \leq N - 1.
\end{cases}
\]  

(B.5)

Setting \( n=N \), in (8), thus leads to (B.6).

\[
p(N) = 2g(0)g(N) + 2g(1)g(N - 1) + 2g(2)g(N - 2) + \cdots + g \left( \frac{N - 1}{2} \right) g \left( N - \frac{N - 1}{2} \right).
\]  

(B.6)

In (B.6), all of the summation terms, except the first term, will be zero due to (B.5). The reason is that for any value \( n \in \{0, 1, \ldots, (N - 1)/2\} \), either \( n \) or \( (N - n) \) is odd and the corresponding coefficient, and hence the term \( g(n)g(N - n) \), becomes zero.

Now consider a trivial example with \( N=3 \) as \( g(n) = a, b, c, d \) where we have not considered any symmetry or antisymmetry in \( g(n) \). With \( g(0) \neq 0 \) as in (B.1), we have \( b=0 \) and the values of \( a, c, \) and \( d \) can be arbitrary. If \( g(z) \) is a Type II linear-phase FIR filter, then (10) holds. This hence necessitates that \( a=d \) and \( b=c=0 \) giving \( g(n) = a, 0, 0, a \) with \( a \neq 0 \). This discussion can easily be extended to larger values of \( N \) but the conclusion is the same. In other words, if \( g(n) \) is zero, the value of \( g(N-n) \) is also zero due to (10). Consequently, with (10) and \( g(0) \neq 0 \), the only option is \( G(z) = \pm \frac{1}{2} (1 + z^{-N}) \).

Note that if some (say \( C \)) coefficients at the beginning and at the end of the impulse response \( g(n) \), of order \( N \), are zero, we can treat such an impulse response as a new impulse response \( \tilde{g}(n) \) having an odd order of \( N = 2C \). Then, the same argument, as above, holds for \( \tilde{g}(n) \) so that \( G(z) = \pm \frac{1}{2} z^{-C} (1 + z^{-2C}) \) where \( D + 2C = N \) for some integers \( D \) and \( C \). To exemplify, with \( N=7 \), all of the filters

\[
G(z) = \pm \frac{1}{2} (z^{-3} + z^{-7}) = \pm \frac{1}{2} z^{-3} (1 + z^{-1}),
\]  

(B.7)

\[
G(z) = \pm \frac{1}{2} (z^{-2} + z^{-5}) = \pm \frac{1}{2} z^{-2} (1 + z^{-3}),
\]  

(B.8)

\[
G(z) = \pm \frac{1}{2} (z^{-1} + z^{-6}) = \pm \frac{1}{2} z^{-1} (1 + z^{-5}),
\]  

(B.9)

\[
G(z) = \pm \frac{1}{2} (z^{-0} + z^{-7}) = \pm \frac{1}{2} z^{-0} (1 + z^{-7})
\]  

(B.10)

are subclasses of (17). □

Appendix C. Proof of (24)

Here, we consider a special case with \( M=2, L=3 \), and \( k \in \{0, 2\} \), in (2). With \( k=0 \), we get \( N=6, p(6) = \frac{1}{2} \), and \( p(n) = 0 \) for \( n \in \{3\} \). Then, \( p(1) = 2g(0)g(1) = 0 \) requires that \( g(1) \) is arbitrary although \( p(1) \) will anyhow be zero. Also, \( p(2) = g^2(1) + 2g(0)g(2) \) can result in an arbitrary value for \( p(2) \) if \( g(2) \) has an arbitrary value. However, from \( p(3) = 2g(0)g(3) + 2g(1)g(2) = 0 \), we must have either \( g(1) = 0 \) or \( g(2) = 0 \). In other words, assuming \( g(1) = a \neq 0 \), we get \( g(n) = 0, a, 0, 0, a, 0 \) whereas assuming \( g(2) = a \neq 0 \) gives \( g(n) = 0, a, 0, a, 0 \) for \( k=2 \), we have \( N=4, p(4) = \frac{1}{2} \), and \( p(1) = 0 \). Then, a similar proof can be derived leading to \( g(0) = 0 \) and \( g(1) = 0 \) thereby meeting (24). □

References