A class of reconfigurable and low-complexity two-stage Nyquist filters

Amir Eghbali *, Håkan Johansson

Division of Electronics Systems, Department of Electrical Engineering, Linköping University, Sweden

**Abstract**

This paper introduces a class of reconfigurable two-stage Nyquist filters where the Farrow structure realizes the polyphase components of linear-phase finite-length impulse response (FIR) filters. By adjusting the variable predetermined multipliers of the Farrow structure, various linear-phase FIR Nyquist filters and integer interpolation/decimation structures are obtained, online. However, the filter design problem is solved only once, offline. Design examples, based on the reweighted \( \ell_1 \)-norm minimization, illustrate the proposed method. Savings in the arithmetic complexity are obtained when compared to the reconfigurable single-stage structures.

© 2013 Elsevier B.V. All rights reserved.

**1. Introduction**

In multicarrier communications, the prototype filter of the corresponding filter bank (FB) is the spectral factor of a Nyquist filter [1]. Matched Nyquist filters allow efficient spectrum sensing [2], pulse shaping [3], and timing/carrier recovery [4]. Interpolation/decimation is composed of anti-imaging/anti-aliasing filters and upsamplers/downsamplers. If the cascade of these filters is a Nyquist filter, we can exactly recover the input signal [5]. Efficient design and implementation of matched Nyquist filters are thus a necessity in communication systems.

Multistandard communications require different bandwidths and sampling rate conversion SRC structures [6–8]. For example, in long term evolution (LTE), the desired bandwidths are 1.4, 3, 5, 10, 15, and 20 MHz. Another example is the nonuniform transmultiplexers (TMUXs) which require filters with different bandwidths and SRC factors [5]. Such scenarios can be handled by using dedicated blocks for each SRC factor but we would then need to either (i) design a large set of filters offline or (ii) design the filters online. These approaches have a high complexity, in design and implementation, if a dynamic system is desired. To dynamically perform SRC at a low cost, we hence need reconfigurable matched filters having a low arithmetic complexity.

For matched filtering, and using FIR filters, we need the filters \( H(z) \) and \( G(z) \) where

\[
P(z) = H(z)G(z)
\]

is a linear-phase FIR Nyquist filter of even order \( N \) and [5]

\[
V(z) = \sum_{l=0}^{L-1} P(zW_l^i)z^{-N/2}, \quad W_L = e^{-j2\pi/L}.
\]

In the time domain, this gives the Nyquist-I criterion as [9–11]

\[
p(n) = \begin{cases} 1 & n = N \\ \frac{1}{I} & n = mI \\ 0 & n = ml \\ \text{arbitrary} & n \neq ml \end{cases}
\]

The transmit pulse \( h(n) \) is usually matched to the receive pulse \( g(n) \) as

\[
g(n) = h^*(N - n)
\]

where the superscript * denotes the complex conjugation. This paper assumes linear-phase FIR filters with real symmetric
coefficients and (4) thus becomes
\[ g(n) = h(N - n). \]  

(5)

As a result of (1) and (5) and if \( P(z) \) is a Nyquist filter, each of the filters \( H(z) \) and \( G(z) \) is usually called a square root Nyquist filter. There exist many methods to design and realize fixed and partially reconfigurable SRC structures for different applications, e.g., [12–15,4], but none of these consider cascaded linear-phase FIR filters. In addition, for each new SRC factor \( L \), many of these methods require filter redesign. Therefore, one has to resort to other methods so that reconfigurable Nyquist filters are obtained without filter redesign or a high arithmetic complexity.

1.1. Contribution of the paper

Generally, SRC can be performed in single or multiple stages [5]. In multi-stage realizations, the overall SRC ratio is factorized into multiple ratios thereby reducing the arithmetic complexity [16]. The two-stage realization of linear-phase FIR Nyquist filters was considered in [17] where fixed linear-phase FIR Nyquist filters were used. This idea was later extended to partially reconfigurable linear-phase FIR Nyquist filters where one of the stages was fixed [18]

The present paper further extends those earlier results to a new and fully reconfigurable structure for two-stage realization of linear-phase FIR Nyquist filters. In both stages, the zeroth polyphase component, of these general filters, is a Type I linear-phase FIR filter whereas the remaining polyphase components are realized by the Farrow structure. Therefore, both stages can independently be reconfigured. By combining these stages, we can perform reconfigurable two-stage SRC with a low arithmetic complexity. This reconfigurability needs simple modifications in some predetermined multipliers of the Farrow structure but it does not require filter redesign.

Besides this full reconfigurability, the present paper uses a new design method as compared to [17,18]. We minimize the reweighted \( \ell_1 \)-norm [19,20] so as to maximize the number of zero-valued fixed filter coefficients. This reduces the arithmetic complexity and it also results in Farrowsubfilters of different effective orders. This has not been treated earlier, in this context. The proposed structure is compared to the reconfigurable single-stage structures [21–23]. Through design examples, it is shown that the proposed structures have a lower arithmetic complexity. However, they generally have a longer delay than the reconfigurable single-stage structures.

1.2. Paper outline

Section 2 gives some prerequisites whereas Section 3 introduces the proposed two-stage SRC structures. In Section 4, the arithmetic complexity is discussed with the filter design algorithm outlined in Section 5. Section 6 provides some design examples. The concluding remarks are given in Section 7.

2. Prerequisites

2.1. Single-stage SRC

In Fig. 1(a) [(b)], interpolation [decimation] by \( L \) requires an upsampler [a downsampler] and an anti-imaging [anti-aliasing] filter \( H(z) [G(z)] \). The filters \( G(z) \) and \( H(z) \) usually have lowpass characteristics with a roll-off of \( 0 \leq \rho \leq 1 \) and passband/stopband edges as
\[ \omega_k T = \frac{1 - \rho}{L}, \quad \omega_k T = \frac{1 + \rho}{L}. \]  

(6)

If the output of Fig. 1(a) is connected to the input of Fig. 1(b), the filter \( P(z) = H(z)G(z) \) is sandwiched between upsampling and downsampling by \( L \). To exactly recover \( x(n) \), the causal decimated (by \( L \)) version of \( P(z) \) must be a pure delay [5]. Then, the filter \( P(z) \) is a Nyquist filter and (2) holds.

2.1.1. A note on the delay of \( P(z) \)

In an actual implementation, the delay of \( P(z) \) must be a multiple of \( L \). Otherwise, a delay term, say \( z^{-\Delta} \), with
\[ \Delta = \frac{N}{2} \mod L \]  

(7)

must be added so that \( z^{-\Delta} P(z) \) has a delay being a multiple of \( L \). In (7), \((N/2) \mod L \) is the remainder of \((N/2)/L \). Then, (2) becomes
\[ V(z) = \sum_{k=0}^{L-1} z^{-\Delta} P(zW_k^L) = z^{-(N/2 + \Delta)}. \]  

(8)

This is a realization issue and it affects neither the filter design nor the results of this paper. Throughout this paper, we have therefore assumed \( \Delta = 0 \).

2.2. Two-stage SRC

Fig. 1(a) [(b)] shows the single-stage equivalent of the structure in Fig. 2(a) [(b)] for interpolation [decimation] by \( L = L_1L_2 \) where [5]
\[ H(z) = H_1(z^{L_1})H_2(z), \quad G(z) = G_1(z^{L_2})G_2(z). \]  

(9)

By cascading the structures in Fig. 2(a) and (b), and considering (1) and (9), we have
\[ P(z) = H_1(z^{L_1})H_2(z)G_1(z^{L_2})G_2(z). \]  

(10)

Fig. 1. Interpolation (a) and decimation (b) by \( L \).

Fig. 2. Interpolation (a) and decimation (b) by \( L = L_1L_2 \).
As in Section 2.1, the filter \( P(z) \) should be a Nyquist filter of even order \( N \) so that

\[
V(z) = \sum_{l=0}^{L-1} H(zW_l^j)G(zW_l^j) = z^{-N/2}.
\] (11)

In this paper, and to reduce the number of optimization parameters, we assume (5) and

\[
G_1(z) = H_1(z), \quad G_2(z) = H_2(z).
\] (12)

Then, with \( P(z) \) being a Nyquist filter, the filters \( H(z) = G(z) \) are square root Nyquist filters where

\[
P(z) = H_1(z^2)H_2(z),
\] (13)

and

\[
V(z) = \sum_{l=0}^{L-1} H_1^2(z^j W_l^1)H_2^2(z W_l^j) = z^{-N/2}.
\] (14)

Note that this paper deals with integer SRC factors \( L_1 \) and \( L_2 \). If a rational SRC is desired, we need to replace one of the SRC stages with a single-stage rational SRC block as outlined in [23]. In other words, for rational two-stage SRC with a factor of say, \( L_1L_2/M_2 \), the second stage must be replaced with the structure proposed in Fig. 6 of [23]. Although this replacement does not change the principles of analysis and design, the implementation structures as well as the derivations of \( P(z) \) and \( V(z) \) will be different. However, we do not further discuss these issues as they are out of the scope of the present paper.

2.3. Farrow structure

The Farrow structure, shown in Fig. 3, consists of fixed linear-phase FIR subfilters \( F_k(z), k = 0, 1, \ldots, Q \). The transfer function is [24]

\[
F(z, \mu) = \sum_{k=0}^{Q} F_k(z)\mu^k, \quad |\mu| \leq 0.5.
\] (15)

The subfilters \( F_k(z) \) are designed so that \( F(z, \mu) \approx z^{-\mu} \), in the passband of the fractional-delay (FD) filter \( F(z, \mu) \) [25]. For simplicity, the sequel uses \( F(z) \) instead of \( F(z, \mu) \). In Fig. 3, the number of distinct multiplications in \( F(z) \) is [25]

\[
U_F(Q, N_F) = \left\{ \begin{array}{ll}
-\left[\frac{Q}{2}\right] + \sum_{k=1}^{Q} \left[\frac{N_F}{2}\right] + 1 & \text{even } N_F \\
\sum_{k=0}^{Q} N_F + 1 & \text{odd } N_F \\
\end{array} \right.
\] (16)

where \( N_F \) is the (largest) order of the subfilters \( F_k(z), k = 0, 1, \ldots, Q \). In practice, we can design the subfilters to have different orders at each branch [25].

2.4. Sets of \( L_1 \) and \( L_2 \)

This paper deals with reconfigurable two-stage structures where the values of \( L_{1u}, u=1,2 \), belong to a set \( L_u \). With a given ripple and roll-off factor, and as long as \( L_u \) does not change, our proposed method does not need filter redesign. Due to reconfigurability, \( L_1 \) and \( L_2 \) can arbitrarily be chosen from the sets \( L_1 \) and \( L_2 \), respectively. This necessitates to meet the filter specifications for all values in the sets \( L_1 \) and \( L_2 \) so that we can solve one offline filter design while being able to choose any value \( L_1 \in L_1 \) and \( L_2 \in L_2 \), online.

3. Reconfigurable two-stage SRC

The polyphase decomposition of a filter \( H_1(z) \) is [5]

\[
H_1(z) = \sum_{m=0}^{L_1-1} z^{-m}H_{1,m}(z^{L_1}).
\] (17)

If \( H_1(z) \) is an ideal lowpass filter of order \( N_1 \), we have

\[
H_{1,m}(z) = \begin{cases} z^{-N_1/2} & \text{in the passband} \\ 0 & \text{in the stopband.} \end{cases}
\] (18)

From (17) and (18), we get

\[
H_{1,m}(z) = \begin{cases} z^{-(N_1/2) - m}/L_1 & \text{in the passband} \\ 0 & \text{in the stopband.} \end{cases}
\] (19)

An \( N_1 \)-th order lowpass filter can hence be designed if \( H_{1,0}(z) \) is an \( N_{1,0} \)-th order Type I linear-phase FIR filter and if the Farrow structure realizes \( H_{1,m}(z) \), \( m = 1, 2, \ldots, L_1 - 1 \), of odd order \( N_{1,m} \) as [21–23]

\[
N_{1,0} = \frac{N_1}{L_1} = N_{1,1} + 1.
\] (20)

Then, we can use (15) to obtain

\[
H_{1,m}(z) = \sum_{k=0}^{Q_1} F_{1,k}(z)\mu_1^m
\] (21)

with

\[
\mu_{1,m} = -\frac{m}{L_1} + \frac{1}{2} \Rightarrow \mu_{1,m} = -\mu_{1,1} - m.
\] (22)

Consequently, we have a Type I linear-phase FIR filter

\[
H_1(z) = H_{1,0}(z^{L_1}) + \sum_{m=1}^{L_1-1} z^{-m} \sum_{k=0}^{Q_1} F_{1,k}(z^{L_1^2})\mu_1^m.
\] (23)

The same principle, as in (17)– (23), can be applied so that

\[
H_2(z) = H_{2,0}(z^{L_2}) + \sum_{m=1}^{L_2-1} z^{-m} \sum_{k=0}^{Q_1} F_{2,k}(z^{L_2^2})\mu_1^m
\] (24)

where

\[
N_{2,0} = \frac{N_2}{L_2} = N_{2,2} + 1
\] (25)

and

\[
\mu_{2,m} = -\frac{m}{L_2} + \frac{1}{2} \Rightarrow \mu_{2,m} = -\mu_{2,1} - m.
\] (26)

\[1\] With proper modifications, even-order filters can also be designed [21].
Inserting (23) and (24) into (13), we have

\[ P(z) = \left( H_{1,0}(z^2) \sum_{m=1}^{l-1} (z^2)^{m-1} \sum_{k=0}^{Q_1} F_{1,k}(z^2) \mu_{1,m}^k \right)^2 \times \left( H_{2,0}(z^2) \sum_{m=1}^{l-1} z^{-m} \sum_{k=0}^{Q_2} F_{2,k}(z^2) \mu_{2,m}^k \right)^2 \]

\[ = \left( H_{1,0}(z^2) \sum_{m=1}^{l-1} \sum_{k=0}^{Q_1} F_{1,k}(z^2) \mu_{1,m}^k \right)^2 \]

\[ \times \left( H_{2,0}(z^2) \sum_{m=1}^{l-1} \sum_{k=0}^{Q_2} F_{2,k}(z^2) \mu_{2,m}^k \right)^2. \]  

(27)

Similarly, inserting (23) and (24) into (14), we have

\[ V(z) = \sum_{l=0}^{L-1} \left( H_{1,0}(z^2) \sum_{m=1}^{l-1} (z^2)^{m-1} \sum_{k=0}^{Q_1} F_{1,k}(z^2) \mu_{1,m}^k \right)^2 \times \sum_{l=0}^{L-1} \left( H_{2,0}(z^2) \sum_{m=1}^{l-1} \sum_{k=0}^{Q_2} F_{2,k}(z^2) \mu_{2,m}^k \right)^2. \]

As

\[ (z^2 W_l^{2L})^{-l} = (z^2 W_l^{2L})^{-m} = (z^2 W_l^{2L})^{-m} \]

and

\[ (z^2 W_l^{2L})^{-m} = (z^2 W_l^{2L})^{-m} \]

we can then write (28) as:

\[ V(z) = \sum_{l=0}^{L-1} \left( H_{1,0}(z^2) \sum_{m=1}^{l-1} (z^2)^{m-1} \sum_{k=0}^{Q_1} F_{1,k}(z^2) \mu_{1,m}^k \right)^2 \times \left( H_{2,0}(z^2) \sum_{m=1}^{l-1} (z^2)^{m-1} \sum_{k=0}^{Q_2} F_{2,k}(z^2) \mu_{2,m}^k \right)^2. \]

(29)

and

\[ (z^2 W_l^{2L})^{-m} = (z^2 W_l^{2L})^{-m} \]

3.2. Filter order

With orders of \( H_1(z) \) and \( H_2(z) \) being \( N_1 \) and \( N_2 \), respectively, (9) gives the order of \( G(z) = H(z) \) as

\[ N_G = N_H = L_2 N_1 + N_2. \]  

Then,

\[ N = 2(L_2 N_1 + N_2) = 2L(N_{F_1} + 1) + 2L_2(N_{F_2} + 1). \]  

(33)

4. Arithmetic complexity

In this section, we discuss the multiplicative complexity of reconfigurable interpolation. As decimators are transposed structures of interpolators, their arithmetic complexity is equal to that of the interpolators. We will use the term \( S_k \) to represent the number of elements (required multipliers) in the set \( L \). In (23) and (24), the number of distinct multiplications, due to the coefficients of \( F_{u,k}(z) \) and \( H_{u,0}(z) \), with \( u = 1,2 \), is

\[ C_u = \frac{N_{u,0}}{2} + 1 + U_f(Q_u, N_{F_u}). \]  

(34)

As in Fig. 4 and besides these \( C_u \) multiplications, we also need

\[ P_u = (Q_u + 1)(L_u - 1) \]

(35)

predetermined multipliers \( \mu_{u,m}^k \) according to (22). Consider an example with \( Q_1 = 3 \) and \( L_1 \in \{2,3\} \). Then, we have

\[ P_1 = (Q_1 + 1) \left( \sum_{l=1}^{L_1} (L_1 - 1) \right) = 12 \]  

(36)

predetermined multipliers \( \mu_{1,m}^k \). These \( P_1 = 12 \) predetermined multipliers \( \mu_{1,m}^k \) have four values of \{1, 0, 0, 0\} if \( L_1 = 2 \) and eight values of \{1, 0, 0, 0, 0, 0, 0, 0\} if \( L_1 = 3 \). Ignoring multiplications by one or zero, we assume that the remaining predetermined multipliers \( \mu_{1,m}^k \) belong to a set \( P_1 \). In the above example, we have \( P_1 = \{1, 0, 0, 0, 0, 0, 0, 0\} \) giving \( S_{P_1} = 6 \).

According to the structure in Fig. 4, we can perform these multiplications in a matrix form with \( Q_u + 2 \) inputs, i.e., outputs of \( F_{u,k}(z) \) and \( H_{u,0}(z) \), as well as \( L_u \) outputs, i.e., inputs to the commutator. We can also share these multiplications [26–28] which may further reduce the arithmetic complexity but such details are not considered in this paper.

The number of multiplications, per input sample, for reconfigurable two-stage interpolation by \( L = L_1 L_2 \) hence becomes \( C_l = S_{P_1} + L_1 (C_2 + S_{P_2}) \). With sets \( L_1 \) and \( L_2 \), the total number of multiplications, per input sample, is thus

\[ C_{st} = \sum_{l_1 \in L_1, l_2 \in L_2} (C_1 + S_{P_1}) + L_1 (C_2 + S_{P_2}) \]  

(37)

The reconfigurable SRC by \( L_u = 1,2 \), can efficiently be realized using fixed filters \( F_{u,k}(z) \) and \( H_{u,0}(z) \), variable predetermined multipliers \( \mu_{u,m}^k \), and commutators [21–23]. We then use the structure of Fig. 4 for reconfigurable interpolation by \( L_1 L_2 \). By transposing the structure of Fig. 4, we can obtain reconfigurable decimation, by \( L = L_1 L_2 \), as shown in Fig. 5.

Fig. 4. Reconfigurable two-stage interpolation by \( L = L_1 L_2 \) using fixed filters, variable predetermined multipliers, and commutators. For details, see [21–23].
where we use the values of \(L_1\) and \(L_2\) with \(L = L_1L_2\) and \(L_1 \leq L_2\) [16]. The average multiplication ratio, for all values of \(L = L_1L_2\), is given as
\[
M_{tr} = \frac{1}{S_L} \sum_{\ell_1 \in L_1} \sum_{\ell_2 \in L_2} \left( C_{\ell_1} + S_{\ell_2} \right) + L_1 \left( C_{\ell_2} + S_{\ell_1} \right) .
\] (38)

4.1. Reconfigurable single-stage SRC

We can also use the structure of Fig. 4 for reconfigurable single-stage SRC with \(L \in \mathbb{L}\) [21–23]. To do so, we should utilize one of the stages. Discarding the terms for the second stage and inserting \(L = L_1\), \(Q = Q_1\), \(N_F = N_{F_1}\), and \(N_0 = N_{I_0}\) in (37), the total number of multiplications, per input sample, for reconfigurable single-stage SRC becomes
\[
C_{\ell_1} = \sum_{\ell_1 \in \mathbb{L}} \left( C + S_{\ell_1} \right) = \sum_{\ell_1 \in \mathbb{L}} \left( \frac{N_0}{2} + 1 + U_{\ell_1}(L,F,N_F) + S_{\ell_1} \right) .
\] (39)

The corresponding average multiplication ratio is
\[
M_{rn} = \frac{1}{S_L} \sum_{\ell_1 \in \mathbb{L}} \left( \frac{1}{L} \sum_{\ell_2 \in \mathbb{L}} \left( \frac{N_0}{2} + 1 + U_{\ell_2}(L,F,N_F) + S_{\ell_2} \right) \right) .
\] (40)

5. Filter design

With ideal filters, we have
\[
P(z) = \begin{cases} 
    z^{-N/2} & \text{in the passband} \\
    0 & \text{in the stopband}
\end{cases}
\] (41)
giving \(V(z) = z^{-N/2}\). With practical filters, we can approximate (41). In this paper, we use a reweighted \(\ell_1\)-norm minimization, having minimax constraints and with an objective function defined in
\[
\mathcal{O}_{\ell_1} = \sum_{k=0}^{L_F} \sum_{n=0}^{N_{F_1} + 1/2} W_{f_{1,k}}(n)|f_{1,k}(n)|
+ \sum_{k=0}^{L_F} \sum_{n=0}^{N_{F_2} + 1/2} W_{f_{2,k}}(n)|f_{2,k}(n)|
+ \sum_{n=0}^{N_{H_1}} W_{h_{1,0}}(n)|h_{1,0}(n)|
+ \sum_{n=0}^{N_{H_2}} W_{h_{2,0}}(n)|h_{2,0}(n)|.
\] (42)

In (42), the terms \(W_{f_{1,k}}(n), W_{f_{2,k}}(n), W_{h_{1,0}}(n),\) and \(W_{h_{2,0}}(n)\) are positive weighting functions. Ideally, we have \(\mathcal{O}_{\ell_1} = 0\) and all filter coefficients are zero. In a practical filter design, one aims to minimize \(\mathcal{O}_{\ell_1}\), which, in turn, maximizes the number of zero-valued fixed filter coefficients. Consequently, we formulate the filter design problem as follows:

**Approximation problem**: Find the unknowns \(f_{1,k}(n)\) for \(k = 0, 1, \ldots, Q_1\), \(n = 0, 1, \ldots, (N_{F_1} + 1)/2\); \(f_{2,k}(n)\) for \(k = 0, 1, \ldots, Q_2\), \(n = 0, 1, \ldots, (N_{F_2} + 1)/2\); \(h_{1,0}(n)\) for \(n = 0, 1, \ldots, N_{I_0}/2 + 1\); and \(h_{2,0}(n)\) for \(n = 0, 1, \ldots, N_{I_0}/2 + 1\) to minimize \(\mathcal{O}_{\ell_1}\) subject to \((\forall \ell_1 \in \mathbb{L}_1, \; \forall \ell_2 \in \mathbb{L}_2)\)

\[
|P(e^{j\omega T}) - e^{-jN\omega/2}| \leq \delta_c, \; \omega T \in [0, \pi]
\] for reconfigurable (a) single-stage, and (b) two-stage Nyquist filters with \(Q = 2, N_F = 9, Q_1 = 2, N_{F_1} = 7, Q_2 = 1, N_{F_2} = 3\), and \(\delta_c = 0.01\).

\[
L = L_1L_2, \quad L_1 \leq L_2.
\] (43)

In (43), \(\alpha \omega T\) is given by (6) with \(\delta_c\) and \(\delta_s\) being, respectively, the desired bounds on the approximation errors for the Nyquist criterion and the stopband attenuation.

Algorithm 1 describes the proposed iterative design method for reconfigurable two-stage SRC. Here, \(M_{q}\) denotes the total number of nonzero filter coefficients in \(f_{1,k}(n), f_{2,k}(n), h_{1,0}(n),\) and \(h_{2,0}(n)\) which is obtained at iteration \(q\). If a coefficient is zero, we set its corresponding weighting to a typical large value as, e.g., \(\frac{1}{10^{10}}\). This is required to avoid division by zero. As discussed in Section 4.1, we can use the structure of Fig. 4 to obtain reconfigurable single-stage SRC. This requires some changes in the filter design as summarized in Algorithm 2.
Algorithm 1. Pseudocode for designing reconfigurable two-stage Nyquist filters.

Determine $Q_1, N_{F_1}, Q_2, N_{F_2}, F_{1,k}(e^{j\omega})$, and $F_{2,k}(e^{j\omega})$ as in Section 5.1;
Set $q = 1$ and $Flg = 1$;
Set $H_{1,0}(z) = z^{-\frac{2\pi}{\omega}}$;
Set $H_{2,0}(z) = 0$;
Set $W_{f_1}(n) = \frac{1}{2\pi|m|}$;
Set $W_{f_2}(n) = 0$;
Set $W_{b_1}(n) = \frac{1}{2\pi|m|}$;
Set $W_{b_2}(n) = 0$;
while $Flg$ do
  if $M_q < M_{q_{req}}$ then
    Set $q = q + 1$;
    Update the weighting functions as follows:
    Set $W_{f_1}(n) = \frac{1}{2\pi|m|}$;
    Set $W_{f_2}(n) = 0$;
    Set $W_{b_1}(n) = \frac{1}{2\pi|m|}$;
    Set $W_{b_2}(n) = 0$;
  elseif $M_q = 0$ then
    Set $Flg = 0$;
  end
end
Set $L = L_1, Q = Q_1, N_{f} = N_{F_1}$, and $N_{l,0} = N_0$;
Save the results;

Algorithm 2. Pseudocode for designing reconfigurable single-stage Nyquist filters.

Set $L_2 = \{1\}, Q_2 = 0, N_{F_2} = 0$, and $N_{F_1} = 1$;
Determine $Q_1, N_{F_1}$, and $F_{1,k}(e^{j\omega})$ as in Section 5.1;
Set $q = 1$ and $Flg = 1$;
Set $H_{1,0}(z) = z^{-\frac{2\pi}{\omega}}$;
Set $H_{2,0}(z) = 0$;
Set $W_{f_1}(n) = \frac{1}{2\pi|m|}$;
Set $W_{f_2}(n) = 0$;
Set $W_{b_1}(n) = \frac{1}{2\pi|m|}$;
Set $W_{b_2}(n) = 0$;
while $Flg$ do
  if $M_q < M_{q_{req}}$ then
    Set $q = q + 1$;
    Update the weighting functions as following:
    Set $W_{f_1}(n) = \frac{1}{2\pi|m|}$;
    Set $W_{f_2}(n) = 0$;
    Set $W_{b_1}(n) = \frac{1}{2\pi|m|}$;
    Set $W_{b_2}(n) = 0$;
  elseif $M_q = 0$ then
    Set $Flg = 0$;
  end
end
Set $L = L_2, Q = Q_1, N_{f} = N_{F_1}$, and $N_{l,0} = N_0$;
Save the results;

5.1. Initial solutions

Ideally, we have [21,25]
\[ F_{k}(e^{j\omega}) = \frac{(-j\omega T)^k}{k!}. \]
(44)
This can be used to find the initial solutions using, e.g., the linprog routine of MATLAB, and it is a convex design critical. In other words, if these values are estimated to be too large, as compared to their optimal values, this only increases the design time, slightly.

The initial solutions for $H_{1,0}(z)$ and $H_{2,0}(z)$ can, respectively, be selected as $z^{-N_{l,0}/2}$ and $z^{-N_{l,0}/2}$. After obtaining the initial solutions, we can solve the nonconvex and nonlinear design problem, mentioned above, using, e.g., the fminimax routine of MATLAB.

![Fig. 8. Characteristics of $P(e^{j\omega})$ for reconfigurable (a) single-stage, and (b) two-stage Nyquist filters with $Q=3, N_0=11, Q_1=2, N_{F_1} = 9, Q_2 = 2, N_{F_2} = 5$, and $\delta_1 = \delta_2 = 0.001$.](image-url)
The main purpose of this paper is to compare the arithmetic complexity of different SRC structures. Thus, it is appropriate to compare the complexities of the filters meeting the same specification. For this purpose, the choice of specification types and design criteria is less crucial as it has no essential effect on the complexity comparisons. Here, we have minimax constraints and we then minimize the reweighted $\ell_1$-norm so as to reduce the arithmetic complexity.

### 6. Design examples and comparison

The characteristics of $|V(e^{j\omega T}) - e^{-jN\pi/2}|$ for reconfigurable (a) single-stage, and (b) two-stage Nyquist filters with $Q=3, N_F=11, Q_1=2, N_{F_1}=9, Q_2=2, N_{F_2}=5,$ and $\delta_c=\delta_s=0.001$.

---

**Fig. 9.** Characteristics of $|V(e^{j\omega T}) - e^{-jN\pi/2}|$ for reconfigurable (a) single-stage, and (b) two-stage Nyquist filters with $Q=3, N_F=11, Q_1=2, N_{F_1}=9, Q_2=2, N_{F_2}=5,$ and $\delta_c=\delta_s=0.001$.

---

**Fig. 10.** Characteristics of $H_1(z)$ (dashed–dotted), $H_2(z)$ (dashed), $H(z)$ (dotted), and $P(z)$ (solid line) for one case of the (a) single-stage, and (b) two-stage Nyquist filters of Fig. 6. (a) $L=4$ and (b) $L_1=2, L_2=2$.

---

**Fig. 11.** Characteristics of reconfigurable two-stage Nyquist filters with $Q_1=2, N_{F_1}=11, Q_2=3, N_{F_2}=9,$ and $\delta_c=\delta_s=0.01$ where $F_{\text{samp}}=80$ MHz.

---

**Fig. 12.** The constellation of a 64-QAM data after a cascade of interpolation and decimation by $L=L_1L_2$ where $L_1=5$ and $L_2=7$ as in Fig. 11.
The delay of $P(z)$ in the reconfigurable single- and two-stage cases. As can be seen, the two-stage case has a lower arithmetic complexity as compared to the single-stage case. However, the delay of the single-stage case is generally shorter than that of the two-stage case.

### 6.2. Case study

Here, we consider a scenario where the aim is to cover bandwidths of $B_w = \{1.4, 3, 5, 10, 15, 20\}$ MHz. In this example, we assume the effective bandwidth of a Nyquist filter to be in the range $[0, \pi(1+\rho)/L]$. With a sampling frequency of $F_{\text{samp}}$, this amounts to having

$$\frac{\pi}{\pi(1+\rho)} = \frac{F_{\text{samp}}}{B_w}$$

which then gives

$$L \approx \left[ \frac{(1+\rho)F_{\text{samp}}}{2B_w} \right].$$

Now, assume $F_{\text{samp}} = 80$ MHz. Then, we get $L = (2, 3, 5, 10, 16, 35)$ which can be obtained if $L_1 = (2, 4, 5)$ and $L_2 = (2, 3, 4, 5, 7)$. The choice of $F_{\text{samp}} = 80$ results in some oversampling but it gives smaller values (and hence arithmetic complexities) in the sets $L_1$ and $L_2$. With these choices of $L_1$ and $L_2$, we should only consider the cases where $L = L_1L_2$ belongs to the set $L = \{2, 3, 4, 9, 15, 32\}$. This simplifies the filter design as it reduces the number of constraints in (43). Fig. 11 shows the characteristics of the filters designed for this scenario. Fig. 12 shows the constellation of a 64-quadrature amplitude modulation (QAM) data where we have performed a cascade of interpolation and decimation by $L = L_1L_2$ with $L_1 = 5$ and $L_2 = 7$ as in Fig. 11. The resulting error vector magnitude (EVM), for this 64-QAM data, is about $-43$ dB.

### 7. Conclusion

Reconfigurable two-stage Nyquist filters, using the Farrow structure, were outlined. The Nyquist filter is composed of cascaded linear-phase FIR filters. As long as the filter specification, i.e., $\rho$, $\delta_r$, or $\delta_c$ does not change, reconfigurable Nyquist filters can be obtained by one offline filter design and through online adjustments in (i) the number of polyphase components and (ii) the variable predetermined multipliers of the Farrow structure. When compared to reconfigurable single-stage Nyquist filters, the proposed reconfigurable two-stage filters have a lower (longer) arithmetic complexity (delay).

### References


