Relations among the fractional chromatic, choice, Hall, and Hall-condition numbers of simple graphs

A. Daneshgar\textsuperscript{a, *}, A.J.W. Hilton\textsuperscript{b}, P.D. Johnson Jr.\textsuperscript{c, 2}

\textsuperscript{a}Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11365-9415, Tehran, Iran
\textsuperscript{b}Department of Mathematics, The University of Reading, Whiteknights, Reading RG6 2AX, UK
\textsuperscript{c}Department of Discrete and Statistical Sciences, 120 Math Annex, Auburn University, AL 36849, USA

Abstract

Hall’s condition for the existence of a proper vertex list-multicoloring of a simple graph $G$ has recently been used to define the fractional Hall and Hall-condition numbers of $G$, $h_f(G)$ and $s_f(G)$. Little is known about $h_f(G)$, but it is known that $s_f(G) = \max\{|V(H)|/\alpha(H); H \subseteq G\}$, where ‘$\subseteq$’ means ‘is a subgraph of’ and $\alpha(H)$ denotes the vertex independence number of $H$. Let $\chi_f(G)$ and $c_f(G)$ denote the fractional chromatic and choice (list-chromatic) numbers of $G$. (Actually, Slivnik has shown that these are equal, but we will continue to distinguish notationally between them.) We give various relations among $\chi_f(G)$, $c_f(G)$, $h_f(G)$, and $s_f(G)$, mostly notably that $\chi_f(G) = c_f(G) = s_f(G)$, when $G$ is a line graph. We give examples to show that this equality does not necessarily hold when $G$ is not a line graph. Relations among and behavior of the ‘$k$-fold’ parameters that appear in the definitions of the fractional parameters are also investigated. The $k$-fold Hall numbers of the claw are determined and from this certain conclusions follow—for instance, that the sequence $(h_k(G))$ of $k$-fold Hall numbers of a graph $G$ is not necessarily subadditive. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Throughout, $G$ is a finite simple graph, $C$ is an infinite set of ‘colors’ or symbols, $\mathcal{F}(C)$ is the collection of finite subsets of $C$, and $L : V(G) \rightarrow \mathcal{F}(C)$ is a list assignment to the vertices of $G$; in addition, $\kappa : V(G) \rightarrow \mathbb{N} = \{0, 1, 2, \ldots\}$ is a function. A proper $(L, \kappa)$-coloring of $G$ is a function $\phi : V(G) \rightarrow \mathcal{F}(C)$ satisfying

\begin{itemize}
  \item Corresponding author.
  \item \textit{E-mail addresses:} danshgar@math.sharif.ac.ir (A. Daneshgar), a.j.w.hilton@reading.ac.uk (A.J.W. Hilton), johnspd@mail.auburn.edu (P.D. Johnson Jr.).
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(i) \( \varphi(v) \subseteq L(v) \) for all \( v \in V(G) \);
(ii) \( |\varphi(v)| = \kappa(v) \) for all \( v \in V(G) \); and
(iii) if \( u, v \in V(G) \) are adjacent then \( \varphi(u) \cap \varphi(v) = \emptyset \).

This last requirement may be restated:

(iii)' for each \( \sigma \in C \), the set \( \text{supp}_G(\sigma, \varphi) = \{ v \in V(G); \sigma \in \varphi(v) \} \) is an independent set of vertices in \( G \).

For a subgraph \( H \) of \( G \), and \( \sigma \in C \), let \( \omega(\sigma, L, H) \) denote the independence number of \( (\text{supp}_H(\sigma, L))_H \), the subgraph of \( H \) induced by the vertices of \( H \) with \( \sigma \) on their \( L \)-lists. To put it the other way, \( \omega(\sigma, L, H) \) is the largest size of an \( H \)-independent subset of \( \text{supp}_H(\sigma, L) = \text{supp}_G(\sigma, L) \cap V(H) \). Since for any proper \((L, \kappa)\)-coloring \( \varphi \) of \( H \) and any \( \sigma \in C \), \( \text{supp}_H(\sigma, \varphi) \) is an \( H \)-independent subset of \( \text{supp}_H(\sigma, L) \), we have that

\[
\sum_{v \in V(H)} \kappa(v) = \sum_{\sigma \in C} |\text{supp}_H(\sigma, \varphi)| \leq \sum_{\sigma \in C} \omega(\sigma, L, H).
\]

We will say that \( G, L, \) and \( \kappa \) satisfy Hall’s condition if and only if, for each subgraph \( H \) of \( G \),

\[
\sum_{v \in V(H)} \kappa(v) \leq \sum_{\sigma \in C} \omega(\sigma, L, H). \tag{\ast}
\]

Note that for Hall’s condition to be satisfied, it suffices that \((\ast)\) hold for each induced subgraph \( H \) of \( G \), since removing edges from \( H \) does not affect \( \sum_{v \in V(H)} \kappa(v) \), and cannot diminish \( \sum_{\sigma \in C} \omega(\sigma, L, H) \).

Remarks preceding show that Hall’s condition is necessary for the existence of a proper \((L, \kappa)\)-coloring of \( G \). When \( G \) is a clique, it is also sufficient; indeed, this sufficiency is the content of the Halmos and Vaughan [7] improvement of Hall’s [6] theorem on systems of distinct representatives, and that is why we call this condition Hall’s condition. (Hall’s original theorem may be stated: Hall’s condition is sufficient for the existence of a proper coloring when \( G \) is a clique and \( \kappa \equiv 1 \).) For more on the sufficiency or non-sufficiency of Hall’s condition for a proper coloring, consult [1,2,8], or [10].

For \( k \in \mathbb{N} \), let \( k \) also stand for the constant function \( \kappa \) with constant value \( k \). For \( k \) positive, the \textit{kth Hall number} of \( G \), denoted by \( h^{(k)}(G) \), is the smallest positive integer among those \( m \geq k \) such that, whenever \( G, L, \) and \( k \) satisfy Hall’s condition, \textit{and} \( |L(v)| \geq m \) for all \( v \in V(G) \), then there is a proper \((L,k)\)-coloring of \( G \). (Observe that \( h^{(k)}(G) \geq k \), by definition. There is a reason for this.) The parameter \( h^{(1)} = h \), called the \textit{Hall number}, was introduced in [8] and further studied in [1,9,10]. The \textit{fractional Hall number} of \( G \), denoted by \( h_f(G) \), is \( \inf k \geq 1 k^{-1} h^{(k)}(G) \). Note that \( 1 \leq h_f(G) \leq h(G) \), so that if \( h(G) = 1 \), then \( h_f(G) = 1 \). It follows from the main result in [8] that \( h_f(G) = 1 \) for every \( G \) in which every block (maximal 2-connected subgraph) is a clique. We do not know if the converse holds.
Similarly, the \textit{kth Hall-condition number} of \( G \), denoted by \( s^{(k)}(G) \), is the smallest integer among those \( m \) such that \( G \), \( L \), and \( k \) will satisfy Hall’s condition whenever \( |L(v)| \geq m \) for all \( v \in V(G) \). (Note that there is no requirement that \( m \geq k \). No \( m < k \) could possibly satisfy the stated requirement; consider subgraphs \( H \) of \( G \) consisting of single vertices.) The parameter \( s^{(1)} = s \), called the \textit{Hall-condition number}, was introduced in [8] and further studied in [12]. The \textit{fractional Hall-condition number} of \( G \) is \( s_f(G) = \inf_{k \geq 1} k^{-1}s^{(k)}(G) \).

The various definitions of the fractional chromatic number, \( \chi_f \), and the fractional choice (list-chromatic) number, \( c_f \), are by now well known—see [14,16]. It suits our purposes here to follow the pattern established above. The \textit{kth chromatic number} (or, the \textit{k-fold chromatic number}) of \( G \), denoted by \( \chi^{(k)}(G) \), is the smallest positive integer among those \( m \) such that there is a proper \((\{1,\ldots,m\},k)\)-coloring of \( G \). (Here \( \{1,\ldots,m\} \) denotes the function \( L : V(G) \to \mathcal{F}(N) \) with constant value \( \{1,\ldots,m\} \).) The \textit{kth (or k-fold) choice number} of \( G \), denoted as \( c^{(k)}(G) \), is the smallest integer among those \( m \) such that there is a proper \((L,k)\)-coloring of \( G \), whenever \( |L(v)| \geq m \) for all \( v \in V(G) \). The fractional chromatic and choice numbers are defined by
\[
\chi_f(G) = \inf_{k \geq 1} k^{-1}\chi^{(k)}(G), \quad c_f(G) = \inf_{k \geq 1} k^{-1}c^{(k)}(G).
\]

Actually, it is well known that ‘\( \inf \)’ can be replaced by ‘\( \lim_{k \to \infty} \)’ in each case (see [14]). \( \chi^{(1)} = \chi \) and \( c^{(1)} = c \) are the ordinary chromatic and choice numbers, respectively.

The Hall and Hall-condition numbers are useful in searching for solutions \( G \) of the equation \( \chi(G) = c(G) \), because of certain relations among the parameters \( h, s, \chi, \) and \( c \) (see [10]). There is no analogous incentive for the study of \( h_f \) and \( s_f \), because Slivnik proved [16] that \( \chi_f(G) = c_f(G) \) for all \( G \). (A proof also appears in [14], where the result is attributed to Alon, Tuza, and Voigt.) However, the question of when \( \chi^{(k)}(G) = c^{(k)}(G) \) is quite interesting, and suggests many related conjectures and problems. (For instance, does \( \chi(G) = c(G) \) imply that \( \chi^{(k)}(G) = c^{(k)}(G) \) for all \( k \)?)

It turns out that \( h^{(k)} \), \( s^{(k)} \), \( \chi^{(k)} \), and \( c^{(k)} \) enjoy exactly the same fundamental relations as do \( h, s, \chi, \) and \( c \). These are given in Theorem 1, which, although easy to prove, we hope will prove useful. (In fact, we put it to the modest use in the proof of Theorem 2.)

Actually, determining values of the \( k \)-fold parameters for specific graphs (for instance, see the calculation in [16] of \( c^{(k)}(C_{2m+1}) \), \( m = 1,2,\ldots \)) is generally non-trivial (except for \( s^{(k)} \); see below), and surely the calculation of \( h^{(k)}(G) \) will be especially difficult for most \( G \), since even determining \( h(G) \) is quite difficult. From the definition of \( h^{(k)} \) ones sees that \( h^{(k)}(G) = k \) for those \( G \) with the property that the satisfaction of Hall’s condition by \( G \), \( L \), and \( k \), for all \( L \) and \( k \), suffices for the existence of a proper \((L,k)\)-coloring of \( G \). In [2,11] it is noted that this class of graphs (the HHV graphs—see [2]) include the complete graphs, by Halmos and Vaughan [7], and in [2] it is shown that this class also includes all paths, and graphs formed by joining two cliques at a cut-vertex. We do not know if \( h^{(k)}(G) = k \) for all \( k = 1,2,\ldots \) implies that \( G \in \text{HHV} \).
The smallest graph which is not HHV is $K_{1,3}$ (see [2]). In Theorem 2 the values $h(k)(K_{1,3})$, $k=1,2,\ldots$, are given. From these certain conclusions can be drawn. Here are two observations not formally included as corollaries of Theorem 2, in the next section:

1. The sequence $(h(k)(G))$ is not necessarily subadditive; i.e., it is not necessarily the case that $h(k+t)(G) \leq h(k)(G) + h(t)(G)$ for all positive integers $k$ and $t$. To see this, take $k = t = 1$ and $G = K_{1,3}$, or $t = 1$ and any $k \not\equiv 2 \mod 3$, and consult Theorem 2.
2. $h_f(G) = \inf k^{-1}h(k)(G)$ is not necessarily $\lim_{k \to \infty} k^{-1}h(k)(G)$. From Theorem 2 we have $h_f(K_{1,3}) = 1$ while $\lim_{k \to \infty} k^{-1}h(k)(K_{1,3}) = 5/3$. This example raises the possibility that $\lim_{k \to \infty} k^{-1}h(k)(G)$ might be a more interesting parameter than $\inf k^{-1}h(k)(G)$, and more suitable for the title of ‘fractional Hall number’. For now, we will stick to the original definition.

In fact, we do not know if $\lim_{k \to \infty} k^{-1}h(k)(G)$ always exists. A number of problems about $h_f$ and $h(k)$ are posed in [11], and clearly there are many more that can be posed.

But the most interesting problem we know of that might involve the $h(k)$ and the $s(k)$ is not directly about them, but about the $c(k)$: is the sequence $(c(k)(G))_{k \geq 1}$ necessarily subadditive? It is asserted that this is ‘easy to see’ in [14, p. 69] (beware: the notation is quite different), but there is no demonstration, and we do not see one.

The presumed subadditivity of the $c(k)$ is adduced in [14] to conclude that $\lim_{k \to \infty} k^{-1}c(k)(G)$ exists and is equal to $\inf k^{-1}c(k)(G)$ for all $G$. These conclusions are obtainable without appeal to subadditivity. They follow straightforwardly from the analogous assertions about the $\chi(k)$, together with the facts that $\chi(k) \leq c(k)$ for all $k$, and that $\chi_f(G) = c_f(G)$ for all $G$ (with $c_f(G)$ as defined here, $c_f(G) = \inf k^{-1}k^{-1}c(k)(G)$), neither of the two proofs of which alluded to above, in [14,16], presume the subadditivity of the $c(k)$.

So the subadditivity of the $c(k)$ is not ‘needed’ for some purposes. Nevertheless, it seems to us to be a deep and interesting question in the area of list-multicolorings, whether or not $(c(k)(G))_{k \geq 1}$ is always (eventually) subadditive, and, if not, for which $G$ it is. In attacking this question we hope that the results given in Theorem 1 will turn out to be useful.

The application of Theorem 1 to question about the $\chi(k)$ and the $c(k)$ might be greatly facilitated by the fact that the $s(k)$ are quite tractable. The following gives a ‘formula’ for $s(k)(G)$, proven in [11].

**Theorem A** (Hitton et al. [10]). $s(k)(G) = \max[k|V(H)|/\alpha(H)]; H$ is a subgraph of $G$, where $\alpha(H)$ denotes the vertex independence number of $H$.

**Corollary A** (Hitton et al. [10]). $s_f(G) = \max[|V(H)|/\alpha(H)]; H$ is a subgraph of $G$.

The subadditivity of $(s(k))$ follows from Theorem A.

Corollary A says that $s_f$ is an engineer’s dream of a fractional chromatic graph parameter; $s_f(G)$ is what many would expect or hope $\chi_f(G)$ to be. In Theorem 3 we
give three sufficient conditions for the equality \( s_f(G) = \chi_f(G) \). The most important of these, the first, is really just a revision of a result of Seymour and Stahl, and the other two are easy observations. Still, easy though they may be, we feel that this is the moment and the place to point out these conditions, to establish a baseline for the study of the equality \( s_f(G) = \chi_f(G) \).

In Theorem 4 we present some graphs for which \( s_f < \chi_f \). These graphs arise from a special case of a more general construction introduced by the first author [3,4] to show the sharpness of Xu’s Conjecture.

Our thanks are due to one of the referees for pointing out that the Grötzsch graph (see [14, p. 49]) \( G \) has \( \chi(G) = 4 \), \( \chi_f(G) = \frac{29}{11} \), and \( s_f(G) = \frac{8}{7} \). This is the only graph \( G \) we know of satisfying \( s_f(G) < \chi_f(G) \) and \( \chi(G) \leq 5 \).

Regarding the graphs of Theorem 4, observe that although \( \chi_f \) is larger than \( s_f \) for these graphs, it is not much larger, which suggests a question analogous to one asked in [12] for \( \chi \) and \( s \): how large can \( \chi_f(G)/s_f(G) \) be? The greatest ratio among the graphs described here (including the example provided by the referee) is 6/5.

2. Results

It should be noted that Theorem 1 in [10] is essentially Theorem 1 below for the case \( k = 1 \), and the proof here is essentially the same as the proof there (in [10]).

**Theorem 1.** (a) \( s^{(k)}(G) \leq \chi^{(k)}(G) \) and \( h^{(k)}(G) \leq c^{(k)}(G) \).
(b) \( h^{(k)}(G) \leq s^{(k)}(G) \) if and only if \( s^{(k)}(G) = c^{(k)}(G) \), in which case \( c^{(k)}(G) = \chi^{(k)}(G) \).
(c) \( h^{(k)}(G) \geq s^{(k)}(G) \) if and only if \( h^{(k)}(G) = c^{(k)}(G) \).
(d) If \( c^{(k)}(G) > \chi^{(k)}(G) \) then \( h^{(k)}(G) = c^{(k)}(G) \).

**Corollary 1.** For a graph \( G \) and a positive integer \( k \), exactly one of the following holds:

\[ c^{(k)}(G) = h^{(k)}(G) > \chi^{(k)}(G) \geq s^{(k)}(G); \]
\[ c^{(k)}(G) = \chi^{(k)}(G) = h^{(k)}(G) \geq s^{(k)}(G); \]
\[ c^{(k)}(G) = \chi^{(k)}(G) = s^{(k)}(G) > h^{(k)}(G). \]

**Corollary 2.** \( h^{(k)}(G) \geq \chi^{(k)}(G) \) if and only if \( h^{(k)}(G) = c^{(k)}(G) \), and \( h^{(k)}(G) \leq \chi^{(k)}(G) \) if and only if \( c^{(k)}(G) = \chi^{(k)}(G) \).

**Corollary 3.** \( s_f(G), h_f(G) \leq \chi_f(G) \) and if \( h_f(G) = \chi_f(G) \), then either \( h_f(G) = \min_{k \geq 1} k^{-1}h^{(k)}(G) \) or \( h^{(k)}(G) = c^{(k)}(G) > \chi^{(k)}(G) \) for infinitely many \( k \).

The first conclusion of Corollary 3 holds if the current definition of \( h_f(G) \) is replaced by \( \limsup_{k \to \infty} k^{-1}h^{(k)}(G) \), or by \( \liminf_{k \to \infty} k^{-1}h^{(k)}(G) \). If \( \chi_f(G) = \liminf_{k \to \infty} k^{-1}h^{(k)}(G) \)
(G), then \( \lim_{k \to \infty} k^{-1} h^{(k)}(G) \) exists (and is equal to \( \chi_f(G) \)), but we do not know if this limit is actually achieved in this case.

**Theorem 2.** If \( G = K_{1,3} \) and \( k \) is a positive integer, then \( \chi^{(k)}(G) = s^{(k)}(G) = c^{(k)}(G) = 2k \), \( h^{(1)}(G) = 1 \), and \( h^{(k)}(G) = 2k - \lfloor k/3 \rfloor \) for \( k \geq 2 \).

**Corollary 4.** If \( h_f(G) = 1 \) then either every block of \( G \) is a clique or \( G \) is claw-free.

**Corollary 5.** If \( \lim \inf_{k \to \infty} k^{-1} h^{(k)}(G) = 1 \) then \( G \) is claw-free.

**Theorem 3.** Each of the following is a sufficient condition for the equality \( \chi_f(G) = s_f(G) = c_f(G) \):

(a) \( G \) is the line graph of a simple graph;
(b) \( \chi(G) = \omega(G) \), the clique number of \( G \);
(c) \( \chi(G) = |V(H)|/\alpha(H) \) for some subgraph \( H \) of \( G \).

To state Theorem 4, we need a definition.

For \( m \geq 3 \), let \( G_m \) be the graph partially depicted in Fig. 1. Let the vertices of the central clique, \( K_{m-1} \), be denoted by \( v_0, \ldots, v_{m-2} \). The adjacencies (not depicted) between the \( a_i \) and the \( v_j \), and the \( b_i \) and the \( v_j \), are as follows: \( a_i \) is adjacent to every \( v_j \) except \( v_i \), and \( b_i \) is adjacent to every \( v_j \) except \( v_i \) and \( v_{i+1} \) [where \( i + 1 \) is an interpreted mod\((m - 1)\)].

**Theorem 4.** \( \chi^{(k)}(G_m) = mk \) for every \( m \geq 3 \) and \( k \geq 1 \).

**Corollary 6.** \( \chi_f(G_m) = m \) for all \( m \geq 3 \), so \( \chi_f(G_m) > s_f(G_m) = m - 1 \) for \( m \geq 6 \).

For \( m = 3, 4, 5 \), we have \( s_f(G_m) = \chi_f(G_m) = m \). To see this, note that \( s_f(G_m) \leq \chi_f(G_m) = m \) by Corollaries 3 and 6; the inequality the other way follows from Corollary A by considering \( H_3 = G_3, H_4 = G_4 - a_0, \) and \( H_5 = G_5 - a_0 - a_2 \).
3. Proofs

**Proof of Theorem 1.** (a) That $h^{(k)}(G) \geq c^{(k)}(G)$ is straightforward from the definitions. That $s^{(k)}(G) \leq \chi^{(k)}(G)$ follows from Theorem A: let $m = \chi^{(k)}(G)$; the fact that there is a proper $(\{1, \ldots, m\}, k)$-coloring of $G$ implies that $G$, $\{1, \ldots, m\}$, and $k$ satisfy Hall’s condition, so for any subgraph $H$ of $G$,

$$k|V(H)| \leq \sum_{i=1}^{m} \alpha(i, \{1, \ldots, m\}, H) \leq m\alpha(H),$$

so $[k|V(H)|/\alpha(H)] \leq m = \chi^{(k)}(G)$.

(b) Clearly $\chi^{(k)}(G) \leq c^{(k)}(G)$, so if $s^{(k)}(G) = c^{(k)}(G)$, then $c^{(k)}(G) = \chi^{(k)}(G)$, by part (a).

Suppose that $h^{(k)}(G) \leq s^{(k)}(G)$. If $|L(v)| \geq s^{(k)}(G)$ for all $v \in V(G)$, then $G$, $L$, and $k$ satisfy Hall’s condition (by the definition of $s^{(k)}(G)$); therefore, because $|L(v)| \geq h^{(k)}(G)$ for all $v \in V(G)$, there must be a proper $(L, k)$-coloring of $G$. Thus $c^{(k)}(G) = \chi^{(k)}(G)$ for all $v \in V(G)$, and we have $h^{(k)}(G) \leq \chi^{(k)}(G)$.

(c) Suppse that $h^{(k)}(G) \geq s^{(k)}(G)$. Suppose that $|L(v)| \geq h^{(k)}(G)$ for all $v \in V(G)$. Then $G$, $L$, and $k$ satisfy Hall’s condition (by the definition of $s^{(k)}(G)$), so $|L(v)| \geq h^{(k)}(G)$ for all $v \in V(G)$ implies that there is a proper $(L, k)$-coloring of $G$. Thus $h^{(k)}(G) \geq c^{(k)}(G)$, so $h^{(k)}(G) = \chi^{(k)}(G)$, by (a).

On the other hand, if $h^{(k)}(G) = c^{(k)}(G)$, we have $h^{(k)}(G) = c^{(k)}(G) \geq c^{(k)}(G)$, using (a).

(d) If $c^{(k)}(G) > \chi^{(k)}(G)$ then $c^{(k)}(G) - 1 \geq \chi^{(k)}(G)$. By the definition of $c^{(k)}(G)$, there is a list assignment $L$ satisfying $|L(v)| \geq c^{(k)}(G) - 1 \geq \chi^{(k)}(G) \geq s^{(k)}(G)$ (using (a), again) for all $v \in V(G)$, such that there is no proper $(L, k)$-coloring of $G$. Since $|L(v)| \geq s^{(k)}(G)$ for all $v \in V(G)$, $G$, $L$, and $k$ satisfy Hall’s condition. Therefore, because $|L(v)| \geq c^{(k)}(G) - 1$, for all $v \in V(G)$, and there is no proper $(L, k)$-coloring of $G$, it follows that $h^{(k)}(G) > c^{(k)}(G) - 1$, so $h^{(k)}(G) \geq c^{(k)}(G)$. Thus $h^{(k)}(G) = c^{(k)}(G)$, by (a).

Corollary 1 is really a restatement of Theorem 1, in perhaps a more memorable way, Corollary 2 follows from it easily.

**Proof of Corollary 3.** That $s_f(G), h_f(G) \leq \chi_f(G)$ follows from Theorem 1(a) and the fact that $\chi_f(G) = c_f(G)$.

Now suppose that $h_f(G) = \chi_f(G)$. It is well known (see [14]) that $\chi_f(G) = k^{-1}\chi^{(k)}(G)$ for infinitely many $k$. For any such $k$, if $h_f(G) = \chi_f(G)$, we would have $h_f(G) = \inf_t t^{-1}h^{(t)}(G) \leq k^{-1}h^{(k)}(G) \leq k^{-1}\chi^{(k)}(G) = \chi_f(G) = h_f(G)$, so we would have that $h_f(G) = k^{-1}h^{(k)}(G) = \min_t t^{-1}h^{(t)}(G)$. Therefore, if ‘$\inf$’ in the definition of $h_f(G)$ cannot be replaced by ‘$\min$’, it must be that $h^{(k)}(G) \leq \chi^{(k)}(G)$ for infinitely many $k$; by Corollary 2, $h^{(k)}(G) = c^{(k)}(G)$ for every such $k$. □
Proof of Theorem 2. Clearly $G$ can be properly $k$-colored from any assignment of sets of size $2k$ to its vertices, so $c^k(G) \leq 2k$. On the other hand, $s^{(k)}(G) = 2k$ by Theorem A. Thus $c^{(k)}(G) = \chi^{(k)}(G) = s^{(k)}(G) = 2k$ by Theorem 1.

Since $G$ is a graph every block of which is a clique, as previously remarked we have $h^{(1)}(G) = 1$ by the main result in [8]. Next, we show that $h^{(k)}(G) \geq 2k - \lceil k/3 \rceil$ for each $k = 2, 3, \ldots$ by providing a list assignment to $V(G)$, satisfying Hall’s condition with $G$ and $k$, from which no proper $k$-coloring is possible, with every list of length (cardinality) at least $2k - \lceil k/3 \rceil - 1$. For $k = 2$ and 3 these list assignments are given in Fig. 2. (We leave the verification of Hall’s condition and non-colorability to the reader.)

Now suppose that $k = 3r + i > 3$, with $i \in \{0, 1, 2\}$. Then $2k - \lceil k/3 \rceil - 1 = 5r + 2i - 1$. Let the central vertex of $G$ be $u$, and the other vertices be $x$, $y$, and $z$. Set

- $L(u) = \{1, \ldots, 6r + 2i\}$,
- $L(x) = \{r + 2, \ldots, 6r + 2i\}$,
- $L(y) = \{1, \ldots, r + 1, 2r + 3, 2r + 4, \ldots, 6r + 2i\}$,

and

- $L(z) = \{1, \ldots, 2r + 2, 3r + 4, \ldots, 6r + 2i\}$.

Suppose that there is a proper $(L, k)$-coloring $\phi$ of $G$. Then

$$|\phi(x) \cap \phi(y) \cap \phi(z)| \leq |L(x) \cap L(y) \cap L(z)| = 3r + 2i - 3 = k - (3 - i).$$

It follows that

$$|\phi(x) \cup \phi(y) \cup \phi(z)| \geq k - (3 - i) + \psi(3 - i),$$

where $\psi(1) = 2$, $\psi(2) = 3$, and $\psi(3) = 5$. Then

$$|\phi(u)| \leq |L(u) \setminus (\phi(x) \cup \phi(y) \cup \phi(z))| \leq 2k - (k - (3 - i) + \psi(3 - i)) < k,$$

so, in fact, there is no such $\phi$; $G$ is not $(L, k)$-colorable.

To see that $G$, $L$, and $k$ satisfy Hall’s condition, observe that $G - v$ is properly $(L, k)$-colorable for each $v \in V(G)$ (not completely trivial, but straightforward).
Therefore, the only subgraph \( H \) for which (*) needs to be checked is \( G \) itself:

\[
\sum_{1 \leq i \leq 6r+2i} \alpha(i, L, G) = 15r + 6i - 3 \geq 4k = 12r + 4i
\]
because \( 3r + 2i > k > 3 \).

Now suppose that \( k = 3r + i \), as before, but we require only \( k > 1 \), not \( k > 3 \).
We will finish the proof by showing that if \( L \) is a list assignment to \( V(G) \) satisfying
\( |L(v)| \geq 5r + 2i \) for all \( v \in V(G) \) and \( G, L, k \) satisfy Hall’s condition, then there is a proper \((L, k)\)-coloring of \( G \). Let \( u, x, y, z \) be as above.

We may as well suppose that \( |L(v)| \leq 2k - 1 \) for \( v \in \{x, y, z\} \), because if, say, \( |L(x)| \geq 2k \), we can properly \((L, k)\)-color \( G - x \) by either of the main results in [2] (i.e., \( G - x = P_3 \in HHV \)) and then finish the coloring of \( G \) by coloring \( x \) with \( k \) elements of \( L(x) \) not in the color set on \( u \). By the way, this completes the proof for \( k = 2 \), so assume \( k \geq 3 \). Let \( \mu_v \) be defined by \( |L(v)| = 5r + 2i + \mu_v, v \in \{x, y, z\} \). By preceding remarks, we can assume \( 0 \leq \mu_v < r \).

Applying (*) to the case when \( H \) is a single edge with end-vertices \( u \) and \( v \), we see that \( |L(u) \cup L(v)| \geq 2k = 6r + 2i \). Then \( |L(u) \setminus L(v)| \geq 6r + 2i - (5r + 2i + \mu_v) = r - \mu_v \), for all \( v \in \{x, y, z\} \). Let \( A_i \) be a subset of \( L(u) \setminus L(v) \) with \( |A_i| = r - \mu_v \). Choose a subset \( \varphi(u) \subseteq L(u) \) with \( |\varphi(u)| = k = 3r + i \) and with \( A_i \cup A_j \cup A_k \subseteq \varphi(u) \). For each \( v \in \{x, y, z\} \), \( A_i \subseteq \varphi(u) \setminus L(v) \), so \( 3r + i = |\varphi(u)| = |\varphi(u) \cap L(v)| + |\varphi(u) \setminus L(v)| \geq |\varphi(u) \cap L(v)| + r - \mu_v \), whence \( |\varphi(u) \cap L(v)| \leq 2r + i + \mu_v \). Therefore,

\[
|L(v) \setminus \varphi(u)| = |L(v)| - |L(v) \cap \varphi(u)|
\]

\[\geq 5r + 2i + \mu_v - (2r + i + \mu_v)\]

\[= 3r + i = k.
\]

Thus it is possible to extend \( \varphi \) to a proper \((L, k)\)-coloring of \( G \). \( \square \)

**Lemma.** If \( H \) is an induced subgraph of \( G \), and \( k \) is a positive integer, then \( h^{(k)}(H) \leq h^{(k)}(G) \).

The proof of this is just like the proof of the special case \( k = 1 \) in [11], so we omit it here.

**Proof of Corollary 4.** If not every block of \( G \) is a clique, then \( h(G) \geq 2 \), as mentioned before, by the main result in [8]. If, in addition, \( G \) has an induced \( K_{1,3} \) subgraph, then \( h^{(k)}(G) \geq h^{(k)}(K_{1,3}) \geq \frac{k}{3} k \) for \( k \geq 2 \), by the Lemma and Theorem 2. Thus \( h_f(G) = \inf_k k^{-1} h^{(k)}(G) \geq \frac{k}{3} > 1 \). \( \square \)

**Proof of Corollary 5.** Corollary 5 follows from the Lemma and Theorem 2 just as Corollary 4 did. \( \square \)

**Proof of Theorem 3.** (a) Let \( G_0 \) be the simple graph of which \( G \) is the line graph. Using usual notation for denoting edge analogues of parameters defined with respect
to vertices, we have \( \chi_f(G) = \chi'_f(G_0) \), and \( s_f(G) = s'_f(G_0) = \max \{|E(H)|/\alpha'(H); H \text{ is a subgraph of } G_0 \text{ with } E(H) \neq \emptyset \}. \)

Now, there is a result of Seymour [15] and Stahl [17], derived from the matching polytope theorem of Edmonds [5] (see [13, Theorem 7.4.6, p. 288] for a succinct account) which gives a ‘formula’ for \( \chi'_f(G_0) \): \( \chi'_f(G_0) = \max \{\Delta(G_0), \max(2|E(H)|/(|V(H)| - 1); H \text{ is a subgraph of } G_0, |V(H)| \geq 3 \} \). We will show that this formula boils down to the formula for \( s'_f(G_0) \) given above. By Corollary 3, it suffices to show that \( \chi'_f(G_0) \leq s'_f(G_0) \).

If \( H = K_{1, \Delta(G_0)} \) then \( |E(H)|/\alpha'(H) = \Delta(G_0) = 1 = A(G_0) \). Since \( G_0 \) contains a \( K_{1, \Delta(G_0)} \), it follows that \( \Delta(G_0) \leq s'_f(G_0) \). Now, suppose that \( H \) is a subgraph of \( G_0 \) of odd order, with \(|V(H)| \geq 3\); then \( \alpha'(H) \leq (|V(H)| - 1)/2 \), so \( s'_f(G_0) \geq |E(H)|/\alpha'(H) \geq 2|E(H)|/(|V(H)| - 1) \). This completes the proof that \( s'_f(G_0) \geq \chi'_f(G_0) \).

(b) Suppose \( x = \chi(G) = \omega(G) \), so that \( G \) contains a \( K_e \). Then \( s_f(G) = |V(K_e)|/\alpha(K_e) = x = \chi(G) \geq \chi_f(G) \); the result now follows by Corollary 3.

Note that in case \( \chi(G) = \omega(G) \), we have \( \chi_f(G) = s_f(G) = \chi(G) = \omega(G) \).

(c) If \( \chi(G) = \alpha(H) \) for some \( H \subseteq G \) then \( \chi_f(G) \leq \chi(G) = \alpha(G) \leq s_f(G) \), so the result follows by Corollary 3.  

Note that in the circumstances of Theorem 3(c), \( \chi_f(G) = s_f(G) = \chi(G) \). Corollary 6 will show that \( \chi(G) = \chi_f(G) \) is not sufficient for \( s_f(G) = \chi_f(G) \).

**Proof of Theorem 4.** First we show that \( \chi(G_m) \geq n \). In the next paragraph we show that \( \chi(G_m) \leq m \). We have \( \chi(G_1) \geq \omega(G_1) = 3 \), so assume that \( m \geq 4 \). Clearly, \( \chi(G_m) \geq \omega(G_m) = m - 1 \). Suppose that \( G_m \) can be properly colored with \( 0, \ldots, m - 2 \). Without loss of generality, \( v_i \) (the \( i \)th vertex in \( K_{m-1} \)) is colored \( i \). Then \( a_i \) must be colored \( i \), \( i = 0, \ldots, m - 2 \). But then no \( b_i \) can be colored; \( b_0 \), for instance, is adjacent to vertices colored \( 0, 1, 2, \ldots, m - 2 \).

On the other hand, \( G_m \) can be colored with \( 0, \ldots, m - 1 \): color \( v_i \) with \( i \), \( i = 0, \ldots, m - 2 \), each \( a_i \) with \( m - 1 \), and \( b_i \) with \( i \), \( i = 0, \ldots, m - 2 \). So \( \chi(G_m) = m \). By subadditivity, \( \chi(k)(G_m) \leq k\chi(G) = km \), for any \( k > 1 \).

Suppose that there is a proper \( \{0, \ldots, km - 2\} \) coloring \( \phi \) of \( G_m \). Without loss of generality, \( \phi(v_i) = \{ik, \ldots, (i + 1)k - 1\} \), \( i = 0, \ldots, m - 2 \).

Let \( S_j = \{u \in V(G_m); j \in \phi(u)\} \). Then \( \sum_{i=0}^{km-2} |S_j| = k|V(G_m)| = 3k(m-1) \).

For \( 0 \leq j \leq (m - 1)k - 1 \), since \( j \in \phi(v_i) \) for some \( i, j \) could only possibly color \( a_i, b_{i-1} \), and \( b_i \), besides \( v_i \). But these three induce a \( K_3 \). Therefore, \( |S_j| \leq 2 \) for \( 0 \leq j \leq (m - 1)k - 1 \). For \( (m - 1)k \leq j \leq mk - 2 \), \( j \) appears as a color only on the \( a_i \) and \( b_j \) possibly, so \( |S_j| \) is no greater than the vertex independence number of the subgraph of \( G_m \) induced by these. It is straightforward to see that vertex independence number is \( m - 1 \). Thus

\[
3k(m-1) = \sum_{j=0}^{km-2} |S_j| \leq 2(m - 1)k + (m - 1)(k - 1) = (3k - 1)(m - 1),
\]

a contradiction.  

Proof of Corollary 6. In view of Theorem 4, what remains to be shown is that 
\( s_f(G_m) = m - 1 \) for \( m \geq 6 \). We use Corollary A. Since \( G_m \) contains a \( K_{m-1} \), \( s_f(G_m) \geq m - 1 \). Since \( |V(G_m)| = 3(m - 1) \) and \( G_m \) contains no \( K_m \), to show that \( |s_f(G_m)| = \max_{H \subseteq G_m} |V(H)|/\chi(H) \leq m - 1 \), it suffices to consider those \( H \subseteq G_m \) with \( \chi(H) = 2 \).

Could such a subgraph \( H \) have order \( \geq 2(m - 1) + 1 \)? If \( |V(H)| \geq 2m - 1 = (3m - 3) - (m - 2) \) then \( H \) contains all but, at most, \( m - 2 \) of \( G_m \)'s vertices. Since \( a_0, \ldots, a_{m-2} \) are independent, and \( \chi(H) = 2 \), \( H \) would have to miss all but, at most, two of the \( a_i \).

It cannot miss them all, because \( |V(H)| \geq 2m - 1 \). If \( H \) contains exactly one \( a_i \), then it contains all \( m - 1 \) \( b_j \), so, considering the \( m - 3 \) \( b_j \) that the \( a_i \) is not adjacent to, \( \chi(H) \geq 1 + \lceil (m - 3)/2 \rceil \geq 3 \), since \( m \geq 6 \). If \( H \) contains exactly two of the \( a_i \), then it misses at most one \( b_j \); counting that \( b_j \) and the four, at most, that the two \( a_i \) are adjacent to, there is at least one \( b_l \in V(H) \) which is adjacent to neither of the two \( a_i \)'s in \( H \). Thus \( \chi(H_m) \geq 3 \), again, a contradiction to \( \chi(H) = 2 \). □

References