Numerical treatment of stochastic models used in statistical systems and financial markets

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A B S T R A C T

In this paper, by means of the variational iteration method, numerical solutions are computed for some stochastic models, without any linearization or weak assumptions. Two stochastic models, the Fokker–Planck equation for non-equilibrium statistical systems and the Black–Scholes model for pricing stock options, are solved numerically. In this approach, the solution is found in the form of a convergent series with easily computed components. The behavior of the approximate solutions is shown graphically.

1. Introduction

Stochastic partial differential equations are essentially partial differential equations that have additional random terms. They can be exceedingly difficult to solve. However, they have strong connections with quantum field theory and statistical mechanics [1]. The study of stochastic partial differential equations is an exciting topic which brings together techniques from probability theory, functional analysis, and the theory of partial differential equations. Stochastic partial differential equations appear in several different applications [2]: study of random evolution of systems with a spatial extension (random interface growth, random evolution of surfaces, fluids subject to random forcing), study of stochastic models where the state variable is infinite dimensional (for example, a curve or surface). The solution to stochastic partial differential equations may be viewed in several manners. One can view a solution as a random field (set of random variables indexed by a multidimensional parameter).

The present work is aimed at producing analytic and approximate solutions, which are obtained in rapidly convergent series, with elegantly computable components, by the variational iteration method (shortly, VIM). Two types of special interests will be discussed here to illustrate the use of the VIM.

The Fokker–Planck equation (FPE) is currently one of the most important equations in the study of the dynamic process of non-equilibrium statistical systems. There are many models in natural science, including biology, chemistry and physics, whose analysis reduces to the problem of finding the solution of FPE. The FPE was first used by Fokker and Planck (see, [3]) to describe the Brownian motion of particles. If a small particle of mass \(m\) is immersed in a fluid, the equation of motion for the distribution function \(F(x, t)\) is given by

\[
\frac{\partial F}{\partial t} = \gamma \frac{\partial F}{\partial v} + \frac{KT}{m} \frac{\partial^2 F}{\partial v^2},
\]

where \(v\) is the velocity for the Brownian motion of a small particle, \(t\) is the time, \(\gamma\) is the fraction constant, \(K\) is the Boltzmann’s constant, and \(T\) is the temperature of the fluid [3]. Eq. (1.1) used in the force filed for studying stabilities of a collisions plasma and Brownian particle moving through a medium at some fixed temperature.

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A more general form of FPE, the nonlinear FPE, which has important applications in various areas, such as, plasma physics, population dynamics, biophysics, engineering, laser physics and marketing (see, for example [4]). In one variable case, the nonlinear Fokker–Planck equation is written in the following form [3]

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x, t, u) u + \frac{\partial^2}{\partial x^2} B(x, t, u) u\right]$$

(1.2)

with the initial condition given by

$$u(x, 0) = f(x), \quad x \in \mathbb{R}$$

(1.3)

where $u(x, t)$ is the unknown distribution function. In Eq. (1.2), $B(x, t, u) > 0$ is called the diffusion coefficient, and $A(x, t, u) > 0$ is called the drift coefficient. A special case of Eq. (1.2) is the FPE from plasma physics which has the form [5]

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left(\frac{1}{2} x^{1-2\epsilon} p\right) - \frac{\partial}{\partial x} \left(\frac{1}{4} x^{-2\epsilon} p\right)$$

(1.4)

$p(x, t)$ is the probability density function. Making use of the replacement

$$p(x, t) = \sqrt{2} x^{(2e-1)/2} \omega(\tau, y), \quad \tau = t, \quad y = \frac{2\sqrt{2}}{2e + 1} x^{(2e+1)/2}.$$ 

(1.5)

Eq. (1.4) reduces to the FPE for the linear Brownian motion as

$$\frac{\partial \omega(\tau, y)}{\partial \tau} = -\frac{\partial}{\partial y} \left(\frac{\omega}{2y}\right) + \frac{1}{2} \frac{\partial^2 \omega}{\partial y^2}$$

(1.6)

with initial condition

$$\omega(0, y) = \text{erf}\left[\frac{1}{2\sqrt{2y}}\right] e^{y^2/2} + e^{y^2/2}.$$ 

(1.7)

The stochastic analysis has interesting applications in mathematical modeling of financial market option pricing [5]. In option pricing theory, the Black–Scholes equation for the determination of the fair value of a call option or derivative security on the market, is one of the most effective models [6,7]. For European options, the Black–Scholes equation can be solved in terms of a diffusion equation boundary value problem, or directly using the Mellin transform [1].

The most well-known stochastic model for the equilibrium condition between the expected return on the option, the riskless interest rate is the Black–Scholes equation [8]

$$\frac{\partial}{\partial t} C(s, t) = \frac{v^2}{2} \frac{\partial^2 C(s, t)}{\partial s^2} + rs \frac{\partial C(s, t)}{\partial s} - rC(s, t)$$

(1.8)

where $s$ is the assert price, which undergoes geometric Brownian motion, $C(s, t)$ is the call price, $v$ is the velocity and $r$ is the reckless interest rate.

Previous works on the numerical solutions for some stochastic models, include, [9] where the Adomian decomposition method was proposed for solving the Fokker–Planck equation and some similar equations. The authors in [10] present numerical solutions for the space and time fractional Fokker–Planck equation using both, the VIM and Adomian decomposition method. In [11], the operator method has been used to solve fractional Fokker–Planck equation. A semi-discrete Galerkin formulation was employed in [12] for the approximate solution of the Black–Scholes equation. The Mellin transform was used in [6,7] to solve a modified form of Black–Scholes equation, which models the valuation of stock options with discrete dividend payments.

The variational iteration method (VIM) established in 1999 by He (see, [13,14]) is thoroughly used by many researchers to handle linear and nonlinear models. Recently, the method was employed for solving KdV equations [15,16], Burgers–Fisher’s equations [17], hyperbolic differential equations [18], and the generalized nonlinear Boussinesq equation [19]. Yusufoglu [20] presented an alternative framework of VIM to construct some compact and noncompact structures of Klein–Gordon equations. Several researchers have used VIM to solve effectively, easily and accurately a large class of nonlinear problems in fluid flows and thermoelasticity [21–23]. Authors of [24] solved the Fokker–Planck equations using the variation iteration technique. The method has been also used by many authors in [25–27] and the references therein to handle a wide variety of scientific and engineering applications.

The motivation of this paper is to extend the analysis of VIM proposed by He [13,14] to produce analytic and approximate solutions for the Fokker–Planck Eq. (1.2), and the Black–Scholes Eq. (1.8). It is worth noting that the VIM handles effectively the linear and nonlinear problems. For nonlinear problems, the VIM is extremely efficient in supplying analytical approximation which converges very rapidly. It solves these types of problems without requiring linearization, or perturbation. The VIM method is capable of greatly reducing the size of calculations while still maintaining high accuracy of the numerical solution.

The rest of the article is structured as follows: Section 2 reviews basic procedure of the variational iteration method. To show the efficiency of the VIM, we present applications on stochastic models and numerical results in Section 3. A conclusion is given in Section 4.
2. Basic idea of VIM

Variational iteration method (VIM) was first proposed by the Chinese mathematician He [13,14]. This method has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions. Some advantages of this technique are

1. The initial condition can be chosen freely with some unknown parameters.
2. The unknown parameters in the initial condition can be easily identified.
3. The calculation is simple and straightforward.

This approach is successfully and effectively applied to various equations, see for example [10,13,14,25–27], and the reference therein.

The idea of this method is constructing a correction functional by a general Lagrange multiplier. The multiplier in the functional should be chosen such that its correction solution is superior to its initial approximation, called trial function, and is the best within the flexibility of trial function, accordingly we can identify the multiplier by the variational theory [13,14]. A complete review of the VIM is available in [28].

The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing the boundary/initial conditions. To illustrate the procedure of this approach, we consider the following general partial differential equation

\[ L[u(x, t) + N(u(x, t))] = f(x, t) \]  \hspace{1cm} (2.1)

where \( L \) is a linear operator, \( N \) is a nonlinear operator, and \( f(x, t) \) is an inhomogeneous term. According to the variational iteration method [13,14], the terms of a sequence \( \{u_n\} \) are constructed such that this sequence converges to the exact solution, \( u_n \)'s are calculated by a correction functional as follows:

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\tau) \left\{ L[u_n(x, \tau)] + N(\tilde{u}_n(x, \tau)) - f(x, \tau) \right\} \, d\tau \]  \hspace{1cm} (2.2)

where \( \lambda \) is general Lagrangian multipliers, which can be identified optimally via the variational theory [13], the subscript \( n \) denotes the \( n \)th order approximation. The second term, involving the integral, on the right-hand side of Eq. (2.2) is called the correction. Under suitable restricted variational assumption (i.e., \( \tilde{u}_n \) is considered as a restricted variation), we can assume that the above correctional functional are stationary (i.e., \( \delta \bar{u}_n = 0 \)). The successive approximations \( u_{n+1}(x, t) \), \( n \geq 0 \) of the solution \( u(x, t) \) will be readily obtained upon using Lagrange multipliers, and by using the selective function \( \bar{u} \). The initial condition \( u(x, 0) \) is usually used for selecting the zeroth approximation \( u_0 \). With \( \lambda \) determined, then several approximations \( u_n(x, t) \), \( n \geq 0 \), follow immediately, the exact solution may be obtained by using

\[ u(x, t) = \lim_{n \to \infty} u_n(x, t). \]

For linear problems, its exact solution can be obtained by only one iteration step, this is due to the fact that the Lagrange multipliers can be exactly identified, see [13]. He’s technique provides a sequence of functions which converges to the exact solution of the problem [14].

In fact, the solution of the partial differential equation (2.1) is considered as the fixed point of the functional (2.2) under suitable choice of the initial approximation. For the convergence proof of (2.2), we state the following known result that is useful to support the convergence of our iteration.

**Theorem 2.1** ([2]). For a Banach space \( X \), suppose the nonlinear mapping \( A : X \to X \) satisfy

\[ \|A[u] - A[\bar{u}]\| \leq \gamma \|u - \bar{u}\|, \quad u, \bar{u} \in X \]

for some constant \( \gamma < 1 \). Then \( A \) has a unique fixed point. Furthermore, the sequence \( u_{n+1} = A[u_n] \) with arbitrary choice of \( u_0 \in X \), converges to the fixed point of \( A \), and

\[ \|u_k - u_j\| \leq \|u_1 - u_0\| \sum_{\ell=j}^{k-2} \gamma^{\ell}. \]

According to this theorem, for the nonlinear mapping

\[ A[u] = u(x, t) + \int_0^t [L[u(x, \tau)] + N(u(x, \tau))] - f(x, \tau) \, d\tau. \]

A sufficient condition for the convergence of the VIM is strictly contraction of \( A \). Furthermore, the sequence (2.2) converges to the fixed point of \( A \), which is also the solution of the differential equation in Eq. (2.1). In what follows, we will apply the VIM for two stochastic models, to illustrate the strength of the method and to establish approximations of high accuracy for these models.
3. Applications

In this section, we will apply the VIM to solve some stochastic models, to illustrate the strength of the method and to establish approximations of high accuracy for two stochastic partial differential equations.

3.1. The Fokker–Planck equation

According to the VIM, we can construct the correction functional for Eq. (1.2) as follows

\[ u_{n+1}(x, t) = u_n(x, 0) + \lambda \int_0^t \left\{ \frac{\partial u_n}{\partial t} - \left[ -\frac{\partial}{\partial x} A(x, \tau, (\tilde{u}_n)) + \frac{\partial^2}{\partial x^2} B(x, \tau, (\tilde{u}_n)) \right] (\tilde{u}_n) \right\} d\tau \]  

(3.1)

where \( \lambda \) is the general Lagrange multipliers, \( \tilde{u}_n \) to denote restricted variables, i.e., \( \delta \tilde{u}_n = 0 \). Making the above correction functionals stationary, we can obtain the following stationary conditions:

\[ \lambda'(\tau) = 0, \quad 1 + \lambda(\tau)|_{\tau = t} = 0. \]

The above two conditions are called, Lagrange–Euler equation, natural boundary condition, respectively. The Lagrange multipliers, therefore, can be identified as \( \lambda = -1 \). Substituting this into the correction functional equation (3.1) results into the following iteration formula:

\[ u_{n+1}(x, t) = u_n(x, 0) - \int_0^t \left\{ \frac{\partial u_n}{\partial t} - \left[ -\frac{\partial}{\partial x} A(x, \tau, (\tilde{u}_n)) + \frac{\partial^2}{\partial x^2} B(x, \tau, (\tilde{u}_n)) \right] (\tilde{u}_n) \right\} d\tau. \]  

(3.2)

We start with an initial approximation \( u_0(x, t) = u(x, 0) \). However, in practice all the terms of the iteration formula cannot be determined, and the solution will be approximately by a series.

To show the efficiency of the new method described previously, in what follows six examples will be considered. Some of these tests are chosen such that there exist analytical solutions for them to give an obvious overview of the VIM. The computer application program Mathematica was used to execute the algorithms that were used to solve the given examples.

Example 3.1 ([3]). Linear Fokker–Planck equation.

Consider Eq. (1.3) with \( f(x) = x \), \( x \in \mathbb{R} \), and in Eq. (1.2), let \( A(x, t, u) = -1 \) and \( B(x, t, u) = 1 \). Using the discussion presented previously, we obtain the recurrence relation

\[ u_{n+1}(x, t) = u_n(x, 0) - \int_0^t \left\{ \frac{\partial u_n}{\partial t} - \left[ -\frac{\partial}{\partial x} (4u - x) + \frac{\partial^2}{\partial x^2} u \right] (\tilde{u}_n) \right\} d\tau. \]

According to the VIM, we can obtain a few first terms being calculated:

\[ u_0(x, t) = x, \quad u_1(x, t) = x + t. \]

Therefore, \( u(x, t) = x + t \), which is the exact solution of the problem, that agrees with our discussion, regarding the solution of linear problems using VIM, in Section 2.

Example 3.2 ([3]). Nonlinear Fokker–Planck equation.

In this example, we consider the case that \( A(x, t, u), B(x, t, u) \) depend on \( x \) and \( u \). Consider (1.2) with \( f(x) = x^2 \), \( x \in \mathbb{R} \) and \( A(x, t, u) = \frac{4u}{\tau} - \frac{x}{\tau}, \) and \( B(x, t, u) = u \). Eq. (1.2) with the above conditions, has the exact solution \( u(x, t) = x^2 e^t \). Using the variational iteration formula (3.2), we obtain the following recurrence relation

\[ u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left\{ \frac{\partial u_n}{\partial t} - \left[ -\frac{\partial}{\partial x} \left( \frac{4u}{\tau} - \frac{x}{\tau} \right) + \frac{\partial^2}{\partial x^2} u \right] (\tilde{u}_n) \right\} d\tau. \]

(3.3)

We start with an initial approximation \( u_0(x, t) = u(x, 0) \), given by \( u(x, 0) = f(x) = x^2 \), we can obtain directly the other components as

\[ u_1(x, t) = x^2 (1 + t) \]
\[ u_2(x, t) = x^2 \left( 1 + t + \frac{t^2}{2} \right) \]
\[ u_3(x, t) = x^2 \left( 1 + t + \frac{t^2}{2} + \frac{t^3}{6} \right) \]

and so on, in the same manner, the rest of components of the iteration formula (3.6) were obtained by the Mathematica Package. In order to verify numerically that the proposed methodology is efficient, Table 1 shows the difference between the analytical solution and the approximate solution evaluated at some points. It is obvious that only three iterations are used to obtain the approximation \( u_3(x, t) \). Other methods, require many iterations to get this result.

There are situations that are characterized by an anomalous diffusion in anisotropic media, such as a crystal with randomly distributed topological defects [29,30]. More generally, the time-dependent probability distribution may depend
Results for Example 3.2, for \( t = 0.1 \) and different values of \( x \)

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>VIM error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.62415E−06</td>
</tr>
<tr>
<td>0.2</td>
<td>6.49661E−06</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0000146174</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0000259864</td>
</tr>
<tr>
<td>0.5</td>
<td>0.000046038</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0000584695</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0000795835</td>
</tr>
<tr>
<td>0.8</td>
<td>0.000103946</td>
</tr>
<tr>
<td>0.9</td>
<td>0.000131556</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000162415</td>
</tr>
</tbody>
</table>

Table 1

on a set of \( N \) macrovariables \( x_i \). The general form of such Fokker–Planck Equation is

\[
\frac{\partial}{\partial t} p(x, t) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left\{ D_{ij} \frac{\partial}{\partial x_j} [p(x, t)]^i \right\} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} [F_i(x, t)p(x, t)]
\]

(3.4)

where \( x = (x_1, x_2, \ldots, x_n) \), \( F_i(x, t) \) is an external force, and \( D_{ij} \) are the diffusion coefficients, which can be time and spatial dependent in general. In particular, Eq. (3.4) has been employed in many physical situations, such as percolation of gases through porous media \((\nu \geq 2)\), thin saturated regions in porous media \((\nu = 2)\), see [29,30]. Similar to solutions obtained above, the recurrent scheme of VIM for Eq. (3.4) can be expressed as

\[
u_{n+1}(x, t) = \nu_n(x, t) - \int_0^t \left[ \frac{\partial}{\partial \tau} p(x, \tau) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left\{ D_{ij} \frac{\partial}{\partial x_j} [p(x, \tau)]^i \right\} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} [F_i(x, \tau)p(x, \tau)] \right] d\tau.
\]

(3.5)

Example 3.3 ([51]). Fokker–Planck equation from plasma physics.

Consider the Fokker–Planck equation from plasma physics in terms of quite complicated integrals, the governing equation is as in (1.4). In order to find the solution of Eq. (1.4) by VIM, we simply solve Eq. (1.6). Knowing the recurrence relation to be

\[
w_{n+1}(y, \tau) = w_n(y, \tau) + \int_0^t \lambda \left( \frac{\partial \omega_n(y, \tau)}{\partial \tau} + \frac{\partial}{\partial y} \left( \tilde{\omega}_n(y, \tau) \right) - \frac{1}{2} \frac{\partial^2 \tilde{\omega}_n(y, \tau)}{\partial y^2} \right) d\tau.
\]

(3.6)

Calculating variation with respect to \( \omega_n \), noticing that, the restricted variation \( \delta \tilde{\omega}_n = 0 \). Under this condition, with the stationary conditions of the above correction functional, the Lagrange multiplier, therefore, can be easily identified to be \( \lambda = -1 \). Leading to the following iteration formula:

\[
w_{n+1}(y, \tau) = w_n(y, \tau) - \int_0^t \left( \frac{\partial \omega_n(y, \tau)}{\partial \tau} + \frac{\partial}{\partial y} \left( \omega_n(y, \tau) \right) - \frac{1}{2} \frac{\partial^2 \omega_n(y, \tau)}{\partial y^2} \right) d\tau.
\]

(3.7)

If we begin with the initial condition given by Eq. (1.7), i.e.,

\[
\omega_0(y, \tau) = \text{erf} \left[ \frac{1}{2} \sqrt{2} y \right] e^{y^2/2} + e^{y^2/2}.
\]

(3.8)

The first two iterations \( \omega_1(y, t), \omega_2(y, t) \) can be calculated as

\[
\omega_1(y, t) = \frac{2 \sqrt{\tau} y (y^2 - 1) + e^{y^2} \sqrt{\pi} (y^4 + 2y^2 + t) + 2e^{y^2} \sqrt{\pi} (y^4 + 2y^2 + t) \text{erf} \left( \frac{y}{\sqrt{2}} \right)}{4 \sqrt{\pi} y^2}
\]

and,

\[
\omega_2(y, t) = \frac{1}{16} \left[ \left( e^{y^2} (y^8 + 4y^6 + 2y^4 - 4y^2 + 9) \left( 2\text{erfc} \left( \frac{y}{\sqrt{2}} \right) - 3 \right) - 2 \sqrt{\pi} y (y^6 + 3y^4 + y^2 - 9) \right) t^2 \right]
\]

\[
+ 4e^{y^2} y^2 t + 8 \sqrt{\pi} y t + 8e^{y^2} y^2 \text{erf} \left( \frac{y}{\sqrt{2}} \right) t + \frac{8e^{y^2} \text{erf} \left( \frac{y}{\sqrt{2}} \right) t}{y^2} - 8 \sqrt{\pi} y + 4e^{y^2} t \right]
\]

\[
+ 16e^{y^2} \text{erf} \left( \frac{y}{\sqrt{2}} \right) \right] .
\]
Substituting these into Eq. (1.5) gives the solution of the probability density $p(x, t)$ of the Fokker–Planck equation from plasma physics. The behavior of the probability density $p(x, t)$ versus $x$ for different values of time is shown in Fig. 1. It is obvious that only two iterations are used to obtain the approximation $\omega_2(y, t)$.

Example 3.4. Itô stochastic differential equation.

The Fokker–Planck equation can be used for computing the probability densities of other stochastic differential equations. Consider Itô stochastic differential equation

$$\text{d}X_t = \mu(X_t, t)\text{d}t + \sigma(X_t, t)\text{d}W_t$$

(3.9)

where $X_t \in \mathbb{R}^N$ is the state, and $W_t \in \mathbb{R}^M$ is a standard $M$-dimensional Wiener process. If the initial distribution is $X_0 \sim p(x, 0)$, then the probability density $p(x, t)$ of the state $X_t$ is given by the FP equation (3.4) with $F_i(x, t) = \mu_i(x, t)$, and diffusion terms $D_{ij} = \frac{1}{2} \sum_k \sigma_{ik}(x, t)\sigma_{jk}(x, t)$. A standard scalar Wiener process is generated by the stochastic differential equation

$$\text{d}x_t = \text{d}W_t.$$ 

Now the drift term $\mu$ is zero, and the diffusion coefficient is $\frac{1}{2}$, and thus the corresponding FP equation is the simplest form of diffusion equation

$$\frac{\partial p(x, t)}{\partial t} = \frac{\partial^2 p(x, t)}{\partial x^2}.$$ 

3.2. The Black–Scholes equation

Stochastic analysis have interesting applications in mathematical modelling and financial market option pricing. The most well-known stochastic model for the equilibrium condition between the expected return on the option, the expected return on the stock and the riskless interest rate is the Black–Scholes equation (1.8). Following [5], we reformulate (1.8) by introducing a new dependent variable

$x = \ln s$, $\text{p}(x, t) = e^x \text{C}(s, t)$, where $p(x, t)$ is the probability density function. As a result, Eq. (1.8) transfers to a diffusion-convection-reaction equation of Brownian motion.

Example 3.5 ([8]). The Black–Scholes model.

The Black–Scholes model for time evolution of the call price option $C(s, t)$, as a function of the underlying asset price $s$ and time $t$, is given by Eq. (1.8). To solve (1.8) by VIM, the correction functional reads as

$$C_{n+1}(s, t) = C_n(s, t) + \int_0^t \lambda \left[ \frac{\partial}{\partial t} C_n(s, \tau) - \frac{\nu^2 s^2}{2} \frac{\partial^2}{\partial s^2} C_n(s, \tau) - rs \frac{\partial}{\partial s} C_n(s, \tau) + rC_n(s, \tau) \right] \text{d}\tau.$$ 

(3.10)

After imposing the stationary conditions, Lagrange multiplier can be identified as $\lambda = -1$. Therefore, the solution of (1.8) can be obtained via the iteration formula

$$C_{n+1}(s, t) = C_n(s, t) - \int_0^t \left[ \frac{\partial}{\partial t} C_n(s, \tau) - \frac{\nu^2 s^2}{2} \frac{\partial^2}{\partial s^2} C_n(s, \tau) - rs \frac{\partial}{\partial s} C_n(s, \tau) + rC_n(s, \tau) \right] \text{d}\tau.$$ 

(3.11)
where the initial condition is taken to be
\[ C_0(s, 0) = s + \frac{1}{s^{7/5}}. \]

Proceeding as before, we can select \( C_0(s, t) = s + \frac{1}{s^{7/5}} \). Using this selection in Eq. (3.11) we obtain the following successive approximations:

\[
\begin{align*}
C_0(s, t) &= s + \frac{1}{s^{7/5}}, \\
C_1(s, t) &= \left\{ s - \frac{6t (10r - 7v^2)}{25s^{7/5}} + \frac{1}{s^{7/5}} \right\}, \\
C_2(s, t) &= \left\{ \frac{18t^2 (10r - 7v^2)^2}{625s^{7/5}} - \frac{6t (10r - 7v^2)}{25s^{7/5}} + s + \frac{1}{s^{7/5}} \right\}, \\
C_3(s, t) &= \left\{ -\frac{36t^3 (10r - 7v^2)^3}{15625s^{7/5}} + \frac{18t^2 (10r - 7v^2)^2}{625s^{7/5}} - \frac{6t (10r - 7v^2)}{25s^{7/5}} + s + \frac{1}{s^{7/5}} \right\}, \\
&\vdots
\end{align*}
\]

and so on, in the same manner the rest of components of the iteration formula (3.11) were obtained using the Mathematica Package. The behavior of the solution \( p(x, t) \) using only \( C_3(s, t) \) obtained above, (for \( \nu = 0.2, r = 0.01 \)) is shown in Fig. 2.

**Example 3.6.** The Crash model.

An extension of the Black–Scholes is proposed in [31]. The stochastic differential equation representing the underlying stock price dynamics is coupled to a post-crash index that exhibits exponentially decaying oscillatory motion, and given by the following stochastic partial differential equation for the option price:

\[
\frac{\partial C}{\partial t} = rs \frac{\partial C}{\partial s} - rC + \frac{1}{2} (\nu s + \gamma g(t))^2 \frac{\partial^2 C}{\partial s^2}. \tag{3.12}
\]

The boundary condition for the European call are given by [32]

\[ C(s, T) = \max(s - X, 0) \]

where \( T \) is the time to expiry, and \( X \) is the strike price.

The PDE for the call price in Eq. (3.12) with time dependent parameters has no known closed form solution, which also can be reduces to the Black–Scholes equation (1.8) for \( \gamma = 0 \) as expected for an uncoupled stock price, \( r \) represents the free interest rate, \( \nu \) is the volatility of the stock price, and \( g(t) \) is the forcing function on the dynamics of the stock price, which has the general form

\[ g(t) = A + Be^{\alpha t} \sin(\omega t). \]
The exponentially decaying sinusoidal function \( g(t) = -10 + 5e^{-2t} \sin 10t \), that represents the market Crash in Eq. (3.12).

To solve Eq. (3.12), according to the VIM, we consider the correction functional in \( t \)-direction in the following form

\[
C_{n+1}(s, t) = C_n(s, t) + \int_0^t \lambda \left( \frac{\partial C(s, \tau)}{\partial \tau} + rs \frac{\partial \tilde{C}_n(s, \tau)}{\partial s} - rC_n(s, \tau) + \frac{1}{2} \left[ \nu s + tg(t) \right]^2 \frac{\partial^2 C_n(s, \tau)}{\partial s^2} \right) d\tau
\]  

(3.13)

where again \( \lambda \) is the general Lagrange multiplier, in which its optimal value is found using variational theory [13, 14]. \( C_0(s, t) \) is an initial approximation, and \( \tilde{C}_n \) is the restricted variation, i.e., \( \delta \tilde{C}_n = 0 \). In this case the optimal value of \( \lambda \) is \(-1\). Thus, the solution of Eq. (3.13) is given as a convergent sequence, which has the following iteration formula

\[
C_{n+1}(s, t) = C_n(s, t) - \int_0^t \left( \frac{\partial C(s, \tau)}{\partial \tau} + rs \frac{\partial C_n(s, \tau)}{\partial s} - rC_n(s, \tau) + \frac{1}{2} \left[ \nu s + tg(t) \right]^2 \frac{\partial^2 C_n(s, \tau)}{\partial s^2} \right) d\tau.
\]  

(3.14)

Using this iteration, we find the approximation of the exact solution for sufficiently large values of \( n \). The first two iterations are computed. The results using VIM presented in Figs. 4 and 5, show that the crash model (3.12) systematically produces lower call price than the Black–Scholes model (1.8). Also Figs. 4 and 5 show that the stronger the coupling of the stock to the market index as represented by \( \gamma \), the larger the effect of the crashing index on the call price. In Fig. 5 (where \( \gamma = 1 \)), the difference between the crash model price and the Black–Scholes price is bigger than the difference in Fig. 4, where \( \gamma = 0.25 \).

4. Conclusions

In this work, we presented an analytic framework for handling linear and nonlinear stochastic models. He’s variational iteration method successfully worked to give an infinite power series solutions, which can, in turn, be expressed in a closed form solution for these models. He’s variational iteration method gives several successive approximations through using the iteration of the correction functional. In this method, there is no specific need to handle nonlinear terms. He’s variational method provides an efficient method for handling nonlinear stochastic models arise frequently in statistical systems, and financial markets.
Fig. 5. The upper curve is the original Black–Scholes price equation (1.8), and the lower curve is the option price given by Eq. (3.12) for $v = 0.2$, $\gamma = 1$, $r = 0.01$, $T = 1$, $s = 60$.

References