Abstract

The multiple-spiral medallions are beautiful decorations of the Mandelbrot set. Computer graphics provide an invaluable tool to study the structure of these decorations with central symmetry, formed by an infinity of baby Mandelbrot sets that have high periods. Up to now, the external arguments of the external rays landing at the cusps of the cardioids of these baby Mandelbrot sets could not be calculated. Recently, a new algorithm has been proposed in order to calculate the external arguments in the Mandelbrot set. In this paper we use an extension of this algorithm for the calculation of the binary expansions of the external arguments of the baby Mandelbrot sets in the multiple-spiral medallions.

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1. Introduction

As is well known, the Mandelbrot set [1] can be defined by
\[ \mathcal{M} = \left\{ c \in \mathbb{C} : \lim_{k \to \infty} f_{c}^{k}(0) = \infty \right\} \],
where \( f_{c}^{k}(0) \) is the \( k \)-iteration of the parameter-dependent quadratic function \( f_{c}(z) = z^2 + c \) (\( z \) and \( c \) complex) for the initial value \( z_0 = 0 \) (the critical point).

Douady and Hubbard studied the Mandelbrot set by means of the external arguments theory [2-4] that was popularized as follows [5]. Imagine a capacitor made of an aluminum bar, shaped in such a way that its cross-section is \( \mathcal{M} \), placed along the axis of a hollow metallic cylinder (Fig. 1). Set the bar at potential 0 and the cylinder at a high potential. This creates an electric field in the region between the cylinder and the bar. Following the electric
field, one gets field-lines, called the external rays of \( \mathcal{M} \). Each external ray starts at a point \( x \) on the boundary of \( \mathcal{M} \), and reaches a point \( y \) of the cylinder. The position of \( y \) is identified by an angle, called the external argument of \( x \). The unity for external arguments is the whole turn. If there are several accesses to \( x \) from outside of \( \mathcal{M} \), the point \( x \) has several external arguments.

As is known, a hyperbolic component of period \( n \) in \( \mathcal{M} \) is a connected component of the open set consisting of all parameter values \( c \) such that \( f_c(z) \) has a (necessarily unique) attracting orbit of period \( n \) [6]. The distinguished parabolic point on the boundary of a hyperbolic component is called the root point of the hyperbolic component that can be both the cusp of a cardioid and the tangent point of a disc. A root point has two external arguments which are rational numbers with odd denominators. A hyperbolic component is narrow [7] if it contains no component of equal or lesser period in its wake (the region of the parameter plane enclosed by its two external rays).

On the other hand, as is also known, a Misiurewicz point is a parameter value for which the critical point or, equivalently, the critical value, is strictly preperiodic [7, 8]. A Misiurewicz point has one or several external arguments which are rational numbers with even denominators.

Examples of external arguments and external rays in \( \mathcal{M} \) are shown in Fig. 2. Figure 2(a) is a copy of the handmade one that appears in the seminal work by Douady and Hubbard [2] where we can read “La contemplation des dessins réalisés sur micro ou mini-ordinateur ... a été déterminante”. Note that the discovery of the Mandelbrot set occurred in 1980 [1] and Fig. 2(a), showing external rays and external arguments, was published in 1982 [2] only two years later. Figure 2(b) is a computer reproduction of Fig. 2(a). The equipotential lines (blue) are drawn by means of escape lines with escape radius \( r_e = 256 \) [9] whereas the external rays (red) have been drawn by means of a computer program by Jung [10], which uses the Böttcher coordinate. Until now it is still not possible to automatically draw, using a computer program, external rays near their landing points inside of the magnification of a small detail of \( \mathcal{M} \). For this reason, in the rest of the figures we will draw manually the external rays one by one, using the computer, by means of Bézier curves [11] taking into account that, as is well known, external rays and equipotential lines have to be perpendicular.

Let us consider the doubling function \( \theta_{n} = 2\theta \mod 1 \), where \( \theta \) is the external argument of an external ray landing at a point \( c \) of the Mandelbrot set. As is also known, if \( \theta \) has period \( n \) under doubling, then \( c \) is the root of a period-\( n \) hyperbolic component. If \( \theta \) is
preperiodic under doubling then \( c \) is a Misiurewicz point. For instance, when \( \theta = 1/7 \) the sequence \( 1/7, 2/7, 4/7, 1/7 \ldots \) has period 3 and \( \theta = 1/7 \) corresponds to a period-3 hyperbolic component. When \( \theta = 1/6 \) the sequence \( 1/6, 2/6, 4/6, 2/6 \ldots \) has preperiod-1 and period-2, and \( \theta = 1/6 \) corresponds to a Misiurewicz point.

An important assertion is that all the external arguments of external rays landing at the same Misiurewicz point have the same kneading sequence [7]. As is known, the period of a Misiurewicz point, obtained by direct iteration of its complex coordinates, and the period of the common kneading sequence of the external arguments of the external rays landing at the Misiurewicz point are the same. Suppose that a preperiodic external argument \( \theta \) has preperiod \( l \) and period \( n \). Then the kneading sequence \( K(\theta) \) has the same preperiod \( l \), and its period \( k \) divides \( n \). If \( n/k > 1 \), then the number of external rays landing at the same point as the external ray with external argument \( \theta \) is \( n/k \); if \( n/k = 1 \) the number of external rays is 1 or 2 [12]. For instance, let us consider the external argument \( 9/56 = 0.001\overline{010} \). We have \( K(9/56) = 1101\overline{11} = 110\overline{1} \), \( n/k = 3 \), and there are three rays (with external arguments \( 9/56 \), \( 11/56 \) and \( 15/56 \)) landing at the Misiurewicz point with this kneading sequence (Fig. 2).

According to Douady et al. “Baby Mandelbrot sets are born in cauliflowers” [13]. Informally, this statement has been completed with “... and the baby Mandelbrot sets (bMs’s) have a parent and a gene that are the same for each cauliflower” [14]. As is known, a multiple-spiral medallion is a generalization of a cauliflower and is a beautiful structure with central symmetry located near to its parent [14]. The multiple-spiral medallions are impossible to discover using only mathematical tools. However, they are evident using computer graphics. In Fig. 3 we can see examples of the location of some multiple-spiral medallions around the parent \( \left( \overline{p_1}, \overline{p_2} \right) = \left( \frac{1}{14}, \frac{1}{14} \right) = \left( 0.011, \overline{0100} \right) \). This parent is located at a branch of the shrub [15] that emerges from the Myrberg-Feigenbaum point of the period doubling cascade of the gene \( \left( \overline{g_1}, \overline{g_2} \right) = \left( \frac{1}{7}, \frac{2}{7} \right) = \left( 0.001, \overline{010} \right) \).

In general, a bMs has high period. Hence, we believe that to date, the binary expansions of the external arguments of a bMs were difficult to obtain by brute force, i.e. by ordering, because of the excessive number of bits of its binary expansions. Recently two general algorithms, the outer and inner algorithms [16], have been proposed in order to find the binary expansions of the external arguments of a hyperbolic component of \( \mathcal{M} \). In this
work we will extend these algorithms to calculate the binary expansions of the bMs’s in a multiple-spiral medallion.

2. Extensions of the outer and inner algorithms

Let \((\overline{b_1}, \overline{b_2}), (\overline{p_1}, \overline{p_2})\) and \((\overline{g_1}, \overline{g_2})\) be the binary expansions of the central bMs, the parent and the gene, \(\overline{g_1} < \overline{p_1} < \overline{p_2} < \overline{g_2}\), of a multiple-spiral medallion [14]. Note that in [14] the structure of the symbolic binary expansions of a bMs was found, but not the method to calculate them. In this paper we pretend to solve this lack. A useful extension of the outer and inner algorithms [16] can be made, with the limitation we will see later, to calculate the binary expansions of the external arguments of the bMs’s in a multiple-spiral medallion.

We extend the concept of narrow hyperbolic component [7] to locally narrow hyperbolic component. Likewise, we extend the notation of symbolic binary expansions to local binary expansions where the symbolic digits became local digits 0 and 1. For instance, let us consider the symbolic binary expansions \((\overline{b_1 p_1 p_2 p_2}, \overline{b_2 p_2 p_1})\), corresponding to a bMs in a double-spiral medallion. If we put \(b_1 = 0\), \(b_2 = 1\), \(p_1 = 0\) and \(p_2 = 1\) we obtain the local binary expansions \((0011, 0100)\). The local binary expansions locate the bMs’s inside the medallion. By definition, a bMs is locally narrow if contains no bMs of equal or lesser period in its wake inside the medallion, i.e., if the local binary expansions of the bMs only differ in one unity. For instance the bMs of binary expansions \((\overline{b_1 p_1 p_2 p_2}, \overline{b_2 p_2 p_1})\) is locally narrow whereas the bMs of binary expansions \((\overline{b_1 p_1 p_2}, \overline{b_2 p_2 p_1})\), corresponding to a bMs in a non-spiral medallion, is locally non-narrow.

3. Applications

3.1. Non-spiral medallions

Let \((\overline{p_1}, \overline{p_2})\) be the parent and let \((\overline{g_1}, \overline{g_2})\) be the gene of a non-spiral medallion. The symbolic binary expansions of the central bMs of a non-spiral medallion are [14]

\[
(\overline{h_1}, \overline{h_2}) = (\overline{p_1 p_1 g_2}, \overline{p_2 p_1 g_1}), \text{ with } i = 1, 2, 3, \ldots
\]
Examples of the location of the central bMs of some non-spiral medallions are shown in Table 1.

In Fig. 4(a) we can see the external rays of the central bMs’s of the non-spiral medallions \(i = 1, 2\) of Table 1. Note that the limit of Eq. (1) when \(i \to \infty\), \(\left(\overline{p_1 p_2}, \overline{p_2 p_1}\right)\), gives the external arguments of the tip of the parent. On the other hand, the limit of the harmonics of the parent \(\left(\overline{p_1 p_2}, \overline{p_2 p_1}\right)\) [17] also gives the external arguments of the tip of the parent. We can consider that there is a “mirror” located at the parent tip so that a harmonic of the parent is reflected in the mirror as the central bMs of a non-espiral medallion.

In Fig. 4(b) we can see the non-spiral medallion \(\left(\overline{b_1}, \overline{b_2}\right) = \left(\overline{p_1 p_2 g_2}, \overline{p_2 p_1 g_1}\right)\) of Table 1. The period of the central bMs is 23. This medallion has two period-27 bMs’s with binary expansions \(\overline{b p}\) (where \(b\) is \(b_1\) or \(b_2\) and \(p\) is \(p_1\) or \(p_2\)), four period-31 bMs’s with binary expansions \(\overline{b p p}\) ... and so on [14]. The binary expansions of these bMs’s can be calculated starting from the sketch of Fig. 4(c). Let us see some determinations. Using the inner algorithm one starts from a point of the period-27 ray \(\overline{b p}\) (a) and goes in the direction towards the inside of the hyperbolic component wake (on the left in the figure) until the first ray with period less than 27 is found (the period-23 ray \(\overline{b}\)). One starts again from a point of the ray \(\overline{b p}\) (a) and goes in the opposite direction to the previous one, until the first ray with period \(27 - 23\) is found (the period-4 ray \(\overline{p}\)). In this way we obtain \(\overline{b p}\) (a) = \(\overline{h_1 p_1}\). In the same manner, we have \(\overline{b p}\) (b) = \(\overline{b_2 p_2}\). Using the outer algorithm we obtain \(\overline{b p}\) (c) = \(\overline{h_1 p_2}\) and \(\overline{b p}\) (d) = \(\overline{h_2 p_1}\). Finally, using the inner algorithm we have \(\overline{b p}\) (a) = \(\overline{h_1 p_2}\), \(\overline{b p}\) (b) = \(\overline{h_2 p_2}\), \(\overline{b p}\) (c) = \(\overline{h_1 p_1}\) and \(\overline{b p}\) (d) = \(\overline{h_2 p_1}\).

As an application of this procedure, in Fig. 5 we can see the non-spiral medallion \(\left(\overline{b_1}, \overline{b_2}\right) = \left(\overline{p_1 p_2 g_2}, \overline{p_2 p_1 g_1}\right)\) of Table 1 with the external rays and its corresponding binary expansions until level \(\overline{b p p}\).

### 3.2. Single-spiral medallions

As is known, the binary expansions of the central bMs of a single-spiral medallion are given by [17]
\[
\left( \overline{b_1}, \overline{b_2} \right) = \left( \overline{p_1^{i}}, \overline{p_2^{i}} \right) \quad \text{with} \quad i = 1, 2, 3, \ldots, \quad (2)
\]

that seems the composition of an antiharmonic of the parent \( \left( \overline{p_1}, \overline{p_2} \right) \) with the gene \( \left( \overline{g_1}, \overline{g_2} \right) \). The single-spiral medallions are located near the cusp of the parent. Let us note that the limit of the Eq. (2) when \( i \to \infty \) is the pair of binary expansions of the parent \( \left( \overline{p_1}, \overline{p_2} \right) \). Examples of the location of the central bMs \( \left( \overline{b_1}, \overline{b_2} \right) \) of some single-spiral medallions are shown in Table 2.

In Fig. 6(a) we can see the external rays of the central bMs of the single-spiral medallion \( \left( \overline{b_1}, \overline{b_2} \right) = \left( \overline{p_1^{5}}, \overline{p_2^{5}} \overline{g_1}, \overline{p_2^{5}} \overline{g_2} \right) \) of Table 2, and in Fig. 6(b) a detail of this medallion is shown. The period of the central bMs is 23. The medallion has two period-27 bMs's (with binary expansions \( \overline{b, p}^* \)), four period-31 bMs's (with binary expansions \( \overline{b, p, p}^* \))... and so on.

Let us compute the binary expansions \( \overline{b, p, (a)} \) and \( \overline{b, p, (b)} \) by means of Fig. 6(c). Using the extension of the outer algorithm one starts from a point of the ray \( \overline{b, p, (a)} \), by going away from the hyperbolic component wake, until the first ray with period less than 27 is found (the period-23 ray \( \overline{b_1} \)). One starts again from a point of the ray \( \overline{b, p, (a)} \), and goes in the opposite direction to the previous one, until the first ray with period 27-23 is found (the period-4 ray \( \overline{p_1} \)). We obtain \( \overline{b, p, (a)} = \overline{b, p_1} \). The binary expansion \( \overline{b, p, (b)} \) can not be obtained by means of the extension of the outer algorithm because we obtain \( \overline{b, p, (b)} = \overline{p_1 b_1} \). Obviously, this binary expansion can not be valid because it does not begin with \( \overline{b_1} \) nor \( \overline{b_2} \). In this case we must use directly the outer algorithm. A detailed analysis shows that between the period-27 ray \( \overline{b, p, (b)} \) and the period-4 ray of the parent \( \overline{p_1} \) there is the period-26 ray \( \overline{b_1 010} \) as we can see in Fig. 6(d). This ray does not belong to the medallion because its period, 26, is not 23, 27, 31, 35 .... Now, if we apply the outer algorithm [16] we obtain \( \overline{b, p, (b)} = \overline{b_1 010} + \overline{0} = \overline{b_1 0100} = \overline{b_2 p_2} \). Note that the bMs \( \left( \overline{b_1 p_1}, \overline{b_2 p_2} \right) \) is locally narrow. We can verify that all the bMs's of a single-spiral medallion are locally narrow and, for this reason, it is sufficient to determine only one binary expansion of a given bMs. For instance, using the extension of the outer algorithm we find \( \overline{b, p, (d)} = \overline{b_2 p_2} \), and taking into
account that the pair \( \left( \overline{b.p.}, \overline{b.p.} \right) \) must be locally narrow we find \( \overline{b.p.} = \overline{b_2p_1} \). In the same manner we obtain \( \left( \overline{b.p.p.}, \overline{b.p.p.} \right) = \left( \overline{b_2p_1p_1}, \overline{b_2p_1p_2} \right) \).

In Fig. 7 we can see the single-spiral medallion \( \left( \overline{b_1}, \overline{b_2} \right) = \left( \overline{p_1g_1}, \overline{p_2g_2} \right) \) of Table 2 with the external rays and its corresponding binary expansions until level \( \overline{b.p.p.p.p} \).

3.3. Double-spiral medallions

Let \( \left( \overline{p_1}, \overline{p_2} \right) \) be the period-\( n \) parent of a double-spiral medallion. The external rays that land at the tangent point of the period-\( 2n \) disc attached to the cardioid of the parent, Fig. 8(a), have the binary expansions \( \left( \overline{p_1p_2}, \overline{p_2p_1} \right) \). Taking into account the conjecture 1 of [14] we can form, for example, the following structures for the binary expansions of the central bMs in a double-spiral medallion

\[
\left( \overline{p_1p_2p_1p_2g_2}, \overline{p_1p_2g_1} \right) \quad \text{and} \quad \left( \overline{p_2p_1p_2p_1g_2}, \overline{p_2p_1g_1} \right) \quad \text{with } i = 1, 2, 3, \ldots. \tag{3}
\]

Let us note that the limits of the expressions (3) when \( i \to \infty \) are the binary expansions \( \overline{p_1p_2} \) and \( \overline{p_2p_1} \). Examples of the location of the central bMs \( \left( \overline{b_1}, \overline{b_2} \right) \) of some double-spiral medallions are shown in Table 3.

In Fig. 8(a) we can see a general view of the external rays of the central bMs \( \left( \overline{b_1}, \overline{b_2} \right) = \left( \overline{p_1p_2p_1p_2g_2}, \overline{p_1p_2p_1g_1} \right) \) and in Fig. 8(b) a detailed view of this medallion is shown. The sketch of Fig. 8(c) is useful to calculate the binary expansions of the bMs’s. For instance, using the extension of the inner algorithm we have \( \overline{b.p.} \left( a \right) = \overline{b_1p_1} \) and \( \overline{b.p.} \left( b \right) = \overline{b_2p_2} \), and using the extension of the outer algorithm we obtain \( \overline{b.p.} \left( c \right) = \overline{b_2p_2} \) and \( \overline{b.p.} \left( d \right) = \overline{b_2p_1} \).
As an application of this procedure, in Fig. 9 we can see the double-spiral medallion \( \left( \overline{b_1}, \overline{b_2} \right) = \left( \overline{p_1p_2}, \overline{p_1p_2g_2}, \overline{p_1p_2g_1} \right) \) of Table 3 with the external rays and its corresponding binary expansions until level \( \overline{b,p,p,p,p} \).

### 3.4. Triple-spiral medallions

Let \( \left( \overline{p_1}, \overline{p_2} \right) \) be the period-\( n \) parent of a triple-spiral medallion as we can see in Fig. 10(a). The external rays landing at the tangent point of the two period-3\( n \) discs attached to the cardioid of the parent have binary expansions \( \left( \overline{p_1p_2}, \overline{p_1p_2p_1} \right) \) and \( \left( \overline{p_2p_1}, \overline{p_2p_1p_1} \right) \). Taking into account the conjecture 1 of [14] we can form, for example, the following structures of the binary expansions of the central bMs in a triple-spiral medallion

\[
\begin{align*}
&\left( \overline{p_1p_2}, \overline{p_1p_2g_2}, \overline{p_1p_2g_1} \right), \\
&\left( \overline{p_1p_2}, \overline{p_1p_2^i}, \overline{p_1p_2^ig_1} \right), \quad \text{with } i = 1, 2, 3, \ldots.
\end{align*}
\]

(4)

Examples of the location of the central bMs \( \left( \overline{b_1}, \overline{b_2} \right) \) of some triple-spiral medallions are shown in Table 4.

For instance, let us consider the triple-spiral medallion \( \left( \overline{b_1}, \overline{b_2} \right) = \left( \overline{p_1p_2}, \overline{p_1p_2g_2}, \overline{p_1p_2g_1} \right) \) that we can locate in Fig. 10(a). The binary expansions of the bMs’s are \( \overline{b,p}, \overline{b,p,p}, \overline{b,p,b}, \overline{b,p,g} \) ... and so on. Firstly, Fig. 10(b,c), we find \( \overline{b,p}(a) \) and \( \overline{b,p}(b) \). In this case the outer algorithm can not be used because the binary expansion must begin by \( \overline{b} \). However, the inner algorithm obtains \( \overline{b,p}(a) = \overline{b,p_1} \) and \( \overline{b,p}(b) = \overline{b,p_2} \). Secondly, by applying the outer algorithm we obtain \( \overline{b,p}(c) = \overline{b,p_2} \) and \( \overline{b,p}(d) = \overline{b,p_2} \).

As an application of this procedure, in Fig. 11 we can see the triple-spiral medallion \( \left( \overline{p_1p_2}, \overline{p_1p_2g_2}, \overline{p_1p_2g_1} \right) \) of Table 4, with the external rays and its corresponding binary expansions until level \( \overline{b,p,p,p,p} \).
4. Binary expansions in the spirals of a multiple-spiral medallion

4.1. Spirals of a single-spiral medallion

Let us consider the single-spiral medallion \((\vec{b}_1, \vec{b}_2)\) when \((\vec{p}_1, \vec{p}_2) = (0.0011, 0.1000)\) and \((\vec{g}_1, \vec{g}_2) = (0.001, 0.100)\) (Fig. 7). It is easy to obtain the binary expansion of the center of one of its spirals. For instance, the binary expansion of the center of the spiral located at the point \(Q\) is the limit of the binary expansions \(\vec{b}_1 \vec{p}_2, \vec{b}_1 \vec{p}_2 \vec{p}_2, \vec{b}_1 \vec{p}_2 \vec{p}_2 \vec{p}_2, \ldots\). This limit is the preperiod-35 and period-4 binary expansion \(\vec{b}_1 \vec{p}_2\). The kneading sequence \(K(\vec{b}_1 \vec{p}_2)\) has preperiod-35 and period-4 and, according to [7], only one ray lands at the point \(Q\).

4.2. Spirals of a triple-spiral medallion

Let us consider the triple-spiral medallion of Fig. 11 \((\vec{b}_1, \vec{b}_2)\) when \(\vec{b}_1 = (3.011, 0.101)\) and \((\vec{p}_1, \vec{p}_2) = (0.0011, 0.1000)\) and \((\vec{g}_1, \vec{g}_2) = (0.001, 0.100)\), centered at \(-0.153265851\ldots + 1.036705813\ldots i\). The medallion has two evident triple-spirals (\(\alpha\) and \(\beta\)) centered at \(-0.153267611\ldots + 1.036706962\ldots i\) and \(-0.153264095\ldots + 1.036704746\ldots i\). By means of an appropriate magnification it is easy to see there are thirteen external rays landing at the center of spiral \(\alpha\) (having the same kneading sequence) and the same occurs with the spiral \(\beta\).

We can obtain the preperiod and the period of each one of the Misiurewicz points corresponding to the centers of the spirals \(\alpha\) and \(\beta\) simply by iteration of the complex coordinates of its centers. The center of the spiral \(\alpha\) is a Misiurewicz point with preperiod 50 and period 4, and the center of the spiral \(\beta\) is a Misiurewicz point with preperiod 51 and period 4. At the last paragraph of this section we will see that this apparent contradiction (different preperiods in symmetrical Misiurewicz points) can easily be explained.

A sketch of the external rays landing at the centers of spirals \(\alpha\) and \(\beta\) is shown in Fig. 12, where the external rays \(\vec{b}_1 \vec{p}_1, \vec{b}_1, \vec{b}_1 \vec{p}_2, \vec{b}_2 \vec{p}_1, \vec{b}_2\) and \(\vec{b}_2 \vec{p}_2\) are drawn with thick lines whereas the external rays landing at the spiral centers are drawn with thin lines.
Let us consider the spiral $\alpha$. The period of the binary expansions of each one of the external rays landing at the center of the spiral has to be $13$ (number of rays) $\times 4$ (period of the Misiurewicz point) and the symbolic binary expansions of these rays can be configured with thirteen period-4 symbolic digits $p_i$. Taking into account that $p_2 < \bar{p}_1$, it is easy to find the binary expansion $\bar{b}_1 s_{a1} < \bar{b}_1$ nearest to $\bar{b}_1$. We write

$$\bar{b}_1 = \frac{p_1 p_2 p_1 p_3 p_1 p_4 p_1 p_1 p_2 p_1}{p_2},$$

$$\bar{b}_1 s_{a1} = \frac{p_1 p_1 p_2 p_1 p_3 p_1 p_4 p_1 p_1 p_2 p_1 p_2 p_1}{p_2},$$

and $\bar{b}_1 s_{a1}$ has preperiod $51$ and period $52$.

In the same manner, taking into account that $p_1 + g_1$ we obtain the binary expansion

$$\bar{b}_2 s_{a2} = \frac{b_2 p_1 p_2 p_1 p_3 p_1 p_4 p_1 p_1 p_2 p_1}{p_2 p_1}.$$

Let us note that the periodic part $s_{a2}$ can be also obtained by means of a cyclic rotation of $s_{a1}$ with a jump of ten symbolic digits. The cyclic rotation of the periodic parts of the binary expansions corresponding to the external rays landing at a same Misiurewicz point was first found in [18]. The common kneading sequence of the binary expansions $\bar{b}_1 s_{a1}$ and $\bar{b}_2 s_{a2}$ is

$$110011001100110011001100110011001100110011111100,$$

with preperiod $51$ and period $4$ dividing $52$ that coincides with the period of the Misiurewicz point (center of the spiral $\alpha$).

Now, let us consider the spiral $\beta$. We obtain $\bar{b}_1 s_{b1} = \frac{b_1 p_1 p_2 p_1 p_3 p_1 p_4 p_1 p_1 p_2 p_1 p_2 p_1}{p_2}$ and $\bar{b}_2 s_{b2} = \frac{b_2 p_1 p_2 p_1 p_3 p_1 p_4 p_1 p_1 p_2 p_1 p_2 p_1 p_2 p_1}{p_2}$. Note that the periodic part $s_{b1}$ can also be obtained by means of a cyclic rotation of $s_{b2}$ with a jump of ten symbolic digits. We complete the rotations of the periodic parts in the spirals $\alpha$ and $\beta$ in Table 5.

If we take a look at the expressions of the binary expansions in the spiral $\alpha$ in Table 5, we can observe the following. When the preperiod is $b_1$ the period ends at $p_2$, and when the preperiod is $b_2$ the period ends at $p_2$. In the medallion we are considering, the last digit of the preperiod ($0$ in $b_1$ and $1$ in $b_2$) coincides with the last digit of the period ($0$ in $p_2$ and $1$ in
and, therefore, a simplification of one bit in the preperiod is possible. However, it is not possible a similar simplification in the expressions of spiral $\beta$ in Table 5. This explains the different preperiods, 50 and 51, in the Misiurewicz points of spirals $\alpha$ and $\beta$. In Fig. 13 we can see a detail of the external rays landing at the center of spiral $\alpha$ with its symbolic binary expansions.

5. Conclusions

We have carried out a preliminary study of the concepts used in this paper: external argument theory of Douady and Hubbard, Misiurewicz points, kneading sequence, etc.

We have used an extension of the outer and inner algorithms in order to calculate the symbolic binary expansions of the bMs’s of the multiple-spiral medallions. Applications of the former calculations have been made to non-spiral, single-spiral, double-spiral and triple-spiral medallions.

Finally, we give a method to calculate the symbolic binary expansions of the external arguments of the rays that land at the centers of the spirals of a multiple-spiral medallions.

Acknowledgements

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References


Figure captions

Fig. 1. Model of Douady and Hubbard theory of external arguments in the Mandelbrot set.

Fig. 2. Examples of external arguments and external rays in the Mandelbrot set. (a) Seminal handmade figure by Douady and Hubbard (1982) [2]. (b) Reproduction of (a) using the computer.

Fig. 3. Location of some multiple-spiral medallions near the parent \((p_1, p_2) = (0.001, 0.010)\).

Fig. 4. Non-spiral medallions. (a) Location of the medallions \((\bar{p}_1, \bar{p}_2) = \left(\frac{p_1 p_2}{g_1}, \frac{-p_1 p_2}{g_1}\right)\), \(i = 1, 2\). (b) General view of the non-spiral medallion \(i = 4\). (c) Sketch to determine binary expansions of bMs’s.

Fig. 5. Non-spiral medallion \((\bar{p}_1, \bar{p}_2) = \left(\frac{p_1 p_2}{g_1}, \frac{-p_1 p_2}{g_1}\right)\) located at \(-0.168988004 + 1.042370722i\) when \((p_1, p_2) = (0.001, 0.010)\) and \((g_1, g_2) = (0.001, 0.010)\).

Fig. 6. Single-spiral medallion. (a) Location of the medallion \((\bar{p}_1, \bar{p}_2) = \left(\frac{p_1 p_2}{g_1}, \frac{-p_1 p_2}{g_1}\right)\) when \((p_1, p_2) = (0.001, 0.010)\) and \((g_1, g_2) = (0.001, 0.010)\). (b) Detail of the medallion. (c) Sketch to determine binary expansions of bMs’s. (d) Neighborhood of the medallion.

Fig. 7. Single-spiral medallion \((\bar{p}_1, \bar{p}_2) = \left(\frac{p_1 p_2}{g_1}, \frac{-p_1 p_2}{g_1}\right)\) located at \(-0.153756141 + 1.030383223i\) when \((p_1, p_2) = (0.001, 0.010)\) and \((g_1, g_2) = (0.001, 0.010)\).

Fig. 8. Double-spiral medallion. (a) Location of the medallion \((\bar{p}_1, \bar{p}_2) = \left(\frac{p_1 p_2}{g_1}, \frac{-p_1 p_2}{g_1}\right)\) located at \(-0.159855629 + 1.038099816i\) when \((p_1, p_2) = (0.001, 0.010)\) and \((g_1, g_2) = (0.001, 0.010)\).

Fig. 9. Triple-spiral medallion. (a) Location of the central bMs \((\bar{p}_1, \bar{p}_2) = \left(\frac{p_1 p_2}{g_1}, \frac{-p_1 p_2}{g_1}\right)\) located at \(-0.153265851 + 1.036705813i\) when \((p_1, p_2) = (0.001, 0.010)\) and \((g_1, g_2) = (0.001, 0.010)\).

Fig. 10. Sketch of the external rays in the two more evident spirals \(\alpha\) and \(\beta\) of the triple-spiral medallion \((\bar{p}_1, \bar{p}_2) = \left(\frac{p_1 p_2}{g_1}, \frac{-p_1 p_2}{g_1}\right)\).

Fig. 11. Detail of external rays landing at the center of the spiral \(\alpha\) in the triple-spiral medallion \((\bar{p}_1, \bar{p}_2) = \left(\frac{p_1 p_2}{g_1}, \frac{-p_1 p_2}{g_1}\right)\) with \((\bar{p}_1, \bar{p}_2) = (0.001, 0.010)\) and \((g_1, g_2) = (0.001, 0.010)\).
### Table 1.
Location of some non-spiral medallions when $(\overrightarrow{R}_1, \overrightarrow{R}_2) = (0.0011, 0.0100)$ and $(\overrightarrow{G}_1, \overrightarrow{G}_2) = (0.001, 0.010)$.

<table>
<thead>
<tr>
<th>$(\overrightarrow{R}_1, \overrightarrow{R}_2)$</th>
<th>$c$</th>
</tr>
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<tbody>
<tr>
<td>$(\overrightarrow{P}_1, \overrightarrow{G}_1, \overrightarrow{G}_2, ..., \overrightarrow{P}_2, \overrightarrow{G}_2)$</td>
<td>$0.127499973 + 0.987460909i$</td>
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<tr>
<td>$(\overrightarrow{P}_1, \overrightarrow{G}_1, \overrightarrow{G}_2, ..., \overrightarrow{P}_2, \overrightarrow{G}_2)$</td>
<td>$0.143322716 + 1.018962627i$</td>
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<tr>
<td>$(\overrightarrow{P}_1, \overrightarrow{G}_1, \overrightarrow{G}_2, ..., \overrightarrow{P}_2, \overrightarrow{G}_2)$</td>
<td>$0.152403537 + 1.030383223i$</td>
</tr>
</tbody>
</table>

### Table 2.
Location of some single-spiral medallions when $(\overrightarrow{P}_1, \overrightarrow{P}_2) = (0.0011, 0.0100)$ and $(\overrightarrow{G}_1, \overrightarrow{G}_2) = (0.001, 0.010)$.

<table>
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<td>$0.152403537 + 1.030383223i$</td>
</tr>
</tbody>
</table>

### Table 3.
Location of some double-spiral medallions when $(\overrightarrow{P}_1, \overrightarrow{P}_2) = (0.0011, 0.0100)$ and $(\overrightarrow{G}_1, \overrightarrow{G}_2) = (0.001, 0.010)$.

<table>
<thead>
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</tr>
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<td>$0.159855689 + 1.038099816i$</td>
</tr>
</tbody>
</table>
Table 4.
Location of some triple-spiral medallions when $\left( \overline{p}_1, \overline{p}_2 \right) =$
$(0.001, 0.010)$ and $\left( \overline{r}_1, \overline{r}_2 \right) = (0.001, 0.010)$.

<table>
<thead>
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<th>$\left( \overline{r}_1, \overline{r}_2 \right)$</th>
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Table 5.
Binary expansions of the external rays landing at the centers of spirals $\alpha$ and $\beta$.

<table>
<thead>
<tr>
<th>Spiral $\alpha$</th>
<th>Spiral $\beta$</th>
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$\beta = p_1p_2p_3p_4p_5p_6p_7p_8p_9$
Fig. 1
Fig. 6
Fig. 7
Fig. 8
Fig. 9