Abstract

We develop a High Frequency (HF) trading strategy where the HF trader uses her superior speed to process information and to post limit sell and buy orders. We introduce a multi-factor self-exciting process which allows for feedback effects in market buy and sell orders and the shape of the limit order book (LOB). The model accounts for arrival of market orders that influence activity, trigger one-sided and two-sided clustering of trades, and induce temporary changes in the shape of the LOB. The resulting strategy outperforms the Poisson strategy where the trader does not distinguish between influential and non-influential events.

Keywords: Algorithmic Trading, High Frequency Trading, Self-Exciting Processes, Hawkes processes, Limit Order Book, Order Cancellations

1. Introduction

Most of the traditional stock exchanges have converted from open outcry communications between human traders to electronic markets where the activity between participants is handled by computers. In addition to those who have made the conversion, such as the New York Stock Exchange and the London Stock Exchange, new electronic trading platforms have entered the market, for example NASDAQ in the US and Chi-X in Europe. Along with the exchanges, market participants have been increasingly relying on the use of computers to handle their trading needs. Initially, computers were employed to execute trades, but nowadays computers manage inventories and make trading decisions; this modern way of trading in the electronic markets is known as Algorithmic Trading (AT).

Despite the substantial changes that markets have undergone in the recent past, some strategies used by investors remain the same. When asked about how to make money in the stock market, an old adage responds: “Buy low and sell high”. Although in principle this sounds like
a good strategy, its success relies on spotting opportunities to buy and sell at the right time. Surprisingly, more than ever, due to the incredible growth in computing power, a great deal of the activity in the US and Europe’s stock exchanges is based on trying to profit from short-term price predictions by buying low and selling high. The effectiveness of these computerized short-term strategies, a subset of AT known as High Frequency (HF) trading, depends on the ability to process information and send messages to the electronic markets in microseconds, see Cartea and Penalva (2011). In this paper we develop an HF strategy that profits from its superior speed advantage to decide when and how to enter and exit the market over extremely short time intervals.

The increase in computer power has made it easier for market participants to deploy ever more complicated trading strategies to profit from changes in market conditions. Key to the success of HF strategies is the speed at which agents can process information and news events to take trading decisions. A unique characteristic to HF trading is that the strategies are designed to hold close to no inventories over very short periods of time (seconds, minutes, or at most one day) to avoid both exposure to markets after close and to post collateral overnight. Thus, profits are made by turning over positions very quickly to make a very small margin per roundtrip transaction (buy followed by a sell or vice-versa), but repeating it as many times as possible during each trading day.

In the past, markets were quote driven which means that market makers quoted buy and sell prices and investors would trade with them. Nowadays, there are limit order markets where all participants can post limit buy or sell orders; i.e. behave as market makers in the old quote driven market. The limit orders show an intention to buy or sell and must indicate the amount of shares and price at which the agent is willing to trade. The limit buy (sell) order with the lowest (highest) price tag is known as the best bid (best offer). During the trading day, all orders are accumulated in the limit order book (LOB) until they find a counterparty for execution or are canceled by the agent who posted them. The counterparty is a market order which is an order to buy or sell an amount of shares, regardless of the price, which is immediately executed against limit orders resting in the LOB at the best execution prices.

As expected, changes in the way trading is conducted in modern electronic markets, coupled with the advent of AT in general and HF trading in particular, is reflected both in the changes of price distributions (see Cvitanic and Kirilenko (2010)) and in the dramatic changes observed in LOB activity. In any one day it is possible to observe tens of thousands of messages being submitted to the LOB of a single stock. Take for instance NASDAQ’s first trading day in February 2008. On that day, between 7am and 8pm IBM’s book received 162,935 buy and sell orders out of which only 7,954 became trades (full or partial execution) and 156,824 were canceled. Furthermore, although IBM’s book received 330,991 messages during that trading day, the maximum number of outstanding buy or sell orders throughout the day was only 1,479 (at 3.43pm approximately).\footnote{The number of outstanding buy an sell orders is defined as the running cumulative sum of buy plus sell orders minus the running cumulative sum of the cancelations and executions (in full).} This is a typical day in NASDAQ for IBM or other tickers
and in general, data show that the trend is for trades to get faster, for the volume per trade to reduce, for the number of submissions of limit orders to increase, and for the number of order cancelations to increase.

The goal of this paper is to develop a particular dynamic HF trading strategy based on optimal postings and cancelations of limit orders to benefit from short-term predictions of the activity in the LOB. The strategy we develop consists in submitting limit orders to maximize expected utility of terminal wealth over a fixed horizon $T$. Our strategy falls within the category of HF trading because the trading horizon $T$ is at most one trading day, all the limit orders are canceled an instant later if not filled, and inventories are optimally managed and drawn to zero by $T$. Early work on optimal postings by a securities dealer is that of Ho and Stoll (1981) and more recently Avellaneda and Stoikov (2008) study the optimal HF submission strategies of bid and ask limit orders.

Intuitively, the HF dynamic strategy we find maximizes the expected utility of profits resulting from roundtrip trades by specifying how deep on the sell and buy side the limit orders are placed in the book. Clearly, the closer the limit orders are to the best bid and best offer, the higher the probability of being executed, but the profits from a roundtrip are also lower. At the same time, the optimal dynamic strategy induces mean reversion to zero in the inventories which avoids having to post collateral overnight. For example, if the probability of the next market order being a buy or sell is the same, and inventories are positive, the limit sell orders are posted closer to the best ask and the buy orders are posted further away from the best bid so that the probability of the offer being lifted is higher than the bid being hit. And as the dynamic trading strategy approaches the terminal date $T$, orders are posted nearer the midquote to guarantee that all positions are unwound so that inventories are drawn to zero at $T$.

Key to the success of the particular HF strategy we develop here, and for most AT strategies in general, is to model the arrival of market orders. Trade initiation may be motivated by many reasons which have been extensively studied in the literature, see for example Sarkar and Schwartz (2009). Some of these include: asymmetric information, differences in opinion or differential information, and increased proportion of impatient (relative to patient) traders. Similarly, trade clustering can be the result of various market events, see Cartea and Jaimungal (2010). For instance, increases in market activity could be due to shocks to the fundamental value of the asset, or the release of public or private information that generates an increase in trading (two-sided or one-sided) until all information is impounded in stock prices. However, judging by the sharp rise of AT in the recent years and the explosion in volume of submissions and order cancelations it is also plausible to expect that certain AT strategies that generate trade clustering are not necessarily motivated by the reasons mentioned above. An extreme example occurred during the flash crash of May 6 2010 where it is clear that trading between HF traders generated more trading giving rise to the ‘hot potato’ effect.\(^2\)

\(^2\)During that day between 13:45:13 and 13:45:27 CT HF traders traded over 27,000 contracts which accounted for approximately 49% of the total trading volume, while their net position changed by only about 200 contracts.
The profitability of these low latency AT strategies depends on how they interact with the dynamics of the LOB, and, more importantly, how these AT strategies coexist with each other. The recent increase in the volume of orders shows that fast traders are dominating the market and it is very difficult to link news arrival or other classical ways of explaining motives for trade to the activity one observes in electronic markets. Superfast algorithms make trading decisions in split milliseconds. This speed, and how other superfast traders react, makes it difficult to link trade initiation to private or public information arrival, a particular type of trader, liquidity shock, or any other market event.

Therefore, as part of the model we develop here, we propose a reduced-form model for the intensity of the arrival of market sell and buy orders. The novelty we introduce is to assume that market orders arrive in two types. The first type of orders are influential orders which excite the market and induces other traders to increase the amount of market orders they submit. For instance, the arrival of an influential market sell order increases the probability of observing another market sell order over the next time step and also increases (to a lesser extent) the probability of a market buy order to arrive over the next time step. On the other hand, when non-influential orders arrive the intensity of the arrival of market orders does not change. This reflects the existence of trades that the rest of the market perceives as not conveying any information that would alter their willingness to submit market orders. In our model we also incorporate the arrival of public news as a state variable that increases the intensity of market orders. In this way, our model for the arrival of market orders is able to capture trade clustering which can be one-sided or two-sided and allow for the activity of trading to show the positive feedback that algorithmic trades seem to have brought to the market environment.

Naturally, the LOB is also affected by the arrival of market orders. In our model the arrival of influential orders has a transitory effect on the shape of both sides of the LOB. More specifically, since some market makers anticipate changes in the intensity of both the sell and buy market orders, the shape of the buy and sell side of the book will also undergo a temporary change due to market makers repositioning their limit orders in anticipation of the increased expected market activity.

2. Arrival of Market Orders and Price Dynamics

Very little is known about the details of the strategies that are employed by AT desks or the more specialized proprietary HF trading desks. Algorithms are designed for different purposes and to seek profits in different ways, Bouchard et al. (2011). For example, there are algorithms that are designed to find the best execution prices for investors who wish to minimize the price impact of large buy or sell orders, Almgren (2003), Kharroubi and Pham (2010) and Bayraktar and Ludkovski (2011). There are HF strategies that specialize in arbitraging across different

(see Kirilenko et al. (2010)).
trading venues whilst others seek to profit from short-term deviations in stock prices. And finally, there are trading algorithms that seek to profit from providing liquidity by posting bids and offers simultaneously, Guilbaud and Pham (2011).

A pillar of capital markets is the provision of liquidity to investors when they need it. As compensation for providing immediacy, market makers or liquidity providers earn the realized spread, which is the market maker’s expected gain from a roundtrip trade. These expected gains depend, among other things, on the architecture of the LOB, and on the ability that market makers have to hold inventories which gives them the opportunity to build strategic long or short positions.

In the LOB, limit orders are prioritized first according to price and then according to time. For example, if two sell (buy) orders are sent to the exchange at the same time, the one with the lowest (highest) price is placed ahead in the queue. Similarly, orders that improve the prices for buy or sell will jump ahead of others regardless of how long they have been resting in the book. Thus, based on the price/time priority rule the LOB stacks on one side all buy orders (also referred to as bids) and on the other side all sell orders (also referred to as offers). The difference between the best offer and best bid is known as the spread and their mean is referred to as the midquote price. Another dimension of the book is the quantities on the sell and buy side for each price tick which give ‘shape’ to the LOB.

The HF trading strategy we develop here is designed to profit from the realized spread where we allow the HF trader (HFT) to build inventories. To this end, before we formalize the HFT’s optimization problem, we require a number of building blocks to capture the most important features of the market dynamics. Since the HFT maximizes expected utility from wealth over

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3 Some exchanges also pay rebates to liquidity providers, see Cartea and Jaimungal (2010).

4 Although we focus on a HF trading market making algorithm, the framework we develop here can be
a finite horizon $T$ and she is continuously repositioning buy and sell limit orders, the success of the strategy depends on optimally picking the ‘best places’ in the bid and offer queue which requires us to model: (i) The dynamics of the fundamental value of the traded stock. (ii) The arrival of market buy and sell orders. And (iii) how market orders cross the resting orders in the LOB. In this section we focus on (i) and (ii), then in Section 3 we discuss (iii) and after that we present the formal optimal control problem that the HFT solves.

We assume that the midprice (or fundamental price) of the traded asset follows

$$S_t = S_0 + \sigma W_t,$$

where $W_t$ is a $\mathbb{P}$-standard Brownian Motion, and $S_0 > 0$ and $\sigma > 0$ are constants. This assumption is consistent with market data over the short periods of time on which our trading strategy is specified. Some traders may have a particular view that the asset price will be rising or falling over the next short time interval. This view can be incorporated into the analysis by including a linear drift into the midprice process or by assuming that the midprice is mean reverting. Moreover, it is not difficult to incorporate jumps (independent of all else) into the midprice dynamics.

2.1. Self-exciting incoming market order dynamics

Markets tend to follow an intraday pattern. Usually after the market opens and before the market closes there is more activity than during the rest of the day. Figure 2 shows the historical intensity of trade arrival, buy and sell, for IBM over a three minute period (starting at 3.30pm, February 1 2008). The historical intensities are calculated by counting the number of buy and sell market orders over the last 1 second. The fitted intensities are computed using our model (see Equation (2)) under the specific assumption that all trades are influential – see Appendix A for more details. From the figures we observe that market orders may arrive in clusters and that there are times when the markets are mostly one-sided (for instance the first 60 seconds of trading for IBM is more active on the buy side than on the sell side) and that these bursts of activity die out rather quickly and revert to around 5 events per second for IBM and 1 event per second for PCP.

Why are there bursts of activity on the buy and sell sides? It is difficult to link all these short-lived increases in the levels of activity to the arrival of news. One could also argue that trading algorithms, including HF, are also responsible for the sudden changes in the pace of the market activity, including bursts of activity in the LOB, and most of the times these algorithms act on information which is difficult to link to public news. Thus, here we take the

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adapted for other types of AT strategies.

5Unless otherwise stated, all random variables and stochastic processes are on the completed filtered probability space $(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ and where $\mathbb{P}$ is the real-world probability measure. What generates the filtration will be defined precisely in Section 4. Simply put, it will be generated by the Brownian motion $W_t$, counting processes corresponding to buy/sell market orders, news events and the indicator of whether a trade is influential or not.
view that some market orders generate more trading activity in addition to the usual effect of news increasing the intensity of market orders.

In our model market orders arrive in two types. The first are influential orders which excite the state of the market and induce other traders to increase their trading activity. We denote the total number of arrivals of influential sell/buy market orders up to (and including) time $t$ by the processes $\{ M_t^-, M_t^+ \}$. The second type of orders are non-influential orders. These are viewed as arising from players who do not excite the state of the market. We denote the total number of arrivals of non-influential sell/buy market orders up to (and including) time $t$ by the processes $\{ ˜M_t^-, ˜M_t^+ \}$. Note that the type indicator of an order is not an observable. Rather all one can observe is whether the market became more active after that trade. Therefore we assume that, conditional on the arrival of a market order, the probability that the trade is influential is a constant $\rho \in [0, 1]$.

Clearly, public bad (good) news increases the sell (buy) activity, but what is not clear is whether market participants always interpret news in the same way. If there is disagreement in how to interpret news or news is ambiguous, then both sides of the market will show an increase in the intensity of buy and sell market orders.$^6$

Thus, we model the intensity of sell, $\lambda_t^-$, and buy, $\lambda_t^+$, market orders by assuming that they solve the coupled system of stochastic differential equations:

$$
\begin{align*}
\frac{d\lambda_t^-}{dt} &= \beta(\theta - \lambda_t^-) + \eta d\bar{M}_t^- + \nu d\bar{M}_t^+ + \bar{\eta} d\bar{Z}_t^- + \bar{\nu} d\bar{Z}_t^+ , \\
\frac{d\lambda_t^+}{dt} &= \beta(\theta - \lambda_t^+) + \eta d\bar{M}_t^+ + \nu d\bar{M}_t^- + \bar{\eta} d\bar{Z}_t^+ + \bar{\nu} d\bar{Z}_t^-,
\end{align*}
$$

$^6$There are automated news feeds designed for AT that already classify the news as good, bad, and neutral.
where $Z_i^\pm$ are Poisson processes (independent of all other processes), with constant activity rate $\mu^\pm$, which represent the total amount of good and bad news that have arrived until time $t$, and recall that $M_t^+$ and $M_t^-$ are the total number of influential buy and sell orders up until time $t$. Moreover, $\beta, \theta, \eta, \nu, \tilde{\eta}, \tilde{\nu}$ are non-negative constants satisfying the constraint $\beta > \rho(\eta + \nu)$.

Market orders are self-exciting because their arrival rates $\lambda^\pm$ jump upon the arrival of influential orders (note that the arrival of non-influential orders do not affect $\lambda^\pm$). If the influential market order was a buy (so that a sell limit order was lifted), the jump activity on the buy side increases by $\eta$ while the jump activity on the sell side increases by $\nu$. On the other hand, if the influential market order was a sell (so that a buy limit order was hit), the jump activity on the sell side increases by $\eta$ while the jump activity on the buy side increases by $\nu$. Typically one would expect $\nu < \eta$ so that jumps on the opposite side of the book are smaller than jumps on the same side of the book (this bears out in the calibration as well as in the moving window activities reported in Figures 2).

News also affects market activity, but does not cause self-excitations. In our model we include cross effects to capture the fact that market participants do not always interpret news in the same way, for example good news increases the intensity of buy market orders by $\tilde{\nu}$ but also affects the intensity of market sell orders by $\tilde{\eta}$.

Trading intensity is mean reverting. Jumps in activity decay back to its long run level of $\theta$ at an exponential rate $\beta$. Figure 3 illustrates a sample path during which no news arrives, but some of the market orders that arrive are influential and induce jumps in the activity level. The lower bound condition on $\beta$ is required for the activity rate to be ergodic. To see this, define the mean future activity rate $m_t^\pm(u) = \mathbb{E}[\lambda^\pm_u|\mathcal{F}_t]$ for $u \geq t$. For the processes $\lambda_t^\pm$ to be ergodic, $m_t^\pm(u)$ must remain bounded as a function of $u$, for each $t$, and the following Lemma
provides a justification for the constraint.

**Lemma 1. Lower Bound on Mean-Reversion Rate.** The mean future rate \( m_t^\pm(u) \) remains bounded for all \( u \geq t \) if and only if \( \beta > \rho(\eta + \nu) \). Furthermore,

\[
\lim_{u \to \infty} m_t^\pm(u) = A^{-1}\zeta,
\]

where

\[
A = \begin{pmatrix}
\beta - \eta \rho & -\nu \rho \\
-\nu \rho & \beta - \eta \rho
\end{pmatrix}
\]

and

\[
\zeta = \begin{pmatrix}
\beta \theta + \tilde{\eta}_\mu^- + \tilde{\nu}_\mu^+ \\
\beta \theta + \tilde{\nu}_\mu^- + \tilde{\eta}_\mu^+
\end{pmatrix}.
\]

**Proof.** See Appendix B. \( \square \)

The intuition for the constraint is that when a market order arrives the activity will jump either by \( \eta \) or by \( \nu \) and this occurs with probability \( \rho \). Note that since both sell and buy influential orders induce self-excitations, the decay rate \( \beta \) must be strong enough to compensate for both jumps to pull the process towards its long-run level of \( \theta \).

### 3. Limit Quote Arrival Dynamics and Fill Rates

The LOB can take on a variety of shapes and changes dynamically throughout the day, see Rosu (2009) and Cont et al. (2010). Market orders eat into the LOB until all the volume specified in the order is filled. Limit orders in the tails of the LOB are less likely to be filled than those within a couple of ticks away from the midprice \( S_t \). Another important feature of the LOB dynamics is how quickly the book recovers from a large market order; i.e. the quoted spread returns to previous values. This is known as the resilience of the LOB.

Therefore, the decision where to post limit buy and sell orders depends on a number of characteristics of the LOB and on the market orders. Some of the LOB features are: shape of the LOB, resiliency of the LOB, and how the LOB changes in between the arrival of market orders. These features, combined with the size and rate of the incoming market orders, determine the fill rates of the limit orders. The fill rate is the rate of execution of a limit order. Intuitively, a high (low) fill rate indicates that a limit order is more (less) likely to be filled by a market order.

Here we model the fill rate facing the HFT in a general framework where we allow the depth and shape of the book to fluctuate. The fill rate depends on where the HFT posts the limit buy and sell orders, that is at \( S_t - \delta_t^- \) and \( S_t + \delta_t^+ \) respectively, where \( \delta^\pm \) denotes how far away from the midprice the orders are posted.
Assumption 1. The fill rates are of the form $\Lambda_t^\pm \equiv \lambda_t^\pm h_\pm(\delta; \kappa_t)$, where the non-increasing function $h_\pm(\delta; \kappa_t) : \mathbb{R} \mapsto [0, 1]$ is $C^1$ in $\delta$ (uniformly in $t$ for $\kappa_t \in \mathbb{R}^n$, fixed $\omega \in \Omega$) and $C^3$ in an open neighborhood of the risk-neutral optimal control, and $\lim_{\delta \to \infty} \delta h_\pm(\delta; \kappa_t) = 0$ for every $\kappa_t \in \mathbb{R}^n$. Moreover, the functions $h_\pm(\delta; \kappa_t)$ satisfy: $h_\pm(\delta; \kappa_t) = 1$ for $\delta \leq 0, \kappa_t \in \mathbb{R}^n$.

Assumption 1 allows for very general dynamics on the LOB through the dependence of the fill probabilities (FPs) $h_\pm(\delta; \kappa_t)$ on the process $\kappa_t$. The FPs can be viewed as a parametric collection with the exponential class $h_\pm(\delta; \kappa_t) = e^{-\kappa_t \delta^\pm}$ and power law class $h_\pm(\delta; \kappa_t) = (1 + (\kappa^\pm \delta^\pm)^\alpha)^{-1}$ being two prime examples. The process $\kappa_t$ introduces dynamics into the collection of FPs reflecting the dynamics in the LOB itself. The differentiability requirements in assumption 1 are necessary for the asymptotic expansions we carry out later on to be correct to order $o(\gamma)$. The limiting behavior for large $\delta^\pm$ implies that the book thins out sufficiently fast so that it is not optimal to place orders infinitely far away from the midprice. Finally, the requirement that $h_\pm(\delta; \kappa_t) = 1$ for $\delta \leq 0$ and $\forall \kappa_t \in \mathbb{R}^n$ is a financial one. A trader wanting to maximize her chances of being filled the next time a market order arrives, must post the limit orders at the midprice, i.e. $\delta^\pm = 0$, or she can also cross the midprice, i.e. $\delta^\pm < 0$. In these cases we suppose that the fill rate is $\Lambda_t^\pm = \lambda_t^\pm$, i.e. it equals the rate of incoming market orders. This assumption makes crossing the midprice a suboptimal decision because the trader cannot improve the arrival rate of market orders, thus she will always post limit orders that are $\delta^\pm \geq 0$ away from the midprice. Additionally, this condition is more desirable than explicitly restricting the controls $\delta^\pm$ to be non-negative, since it is not necessary to check the boundary condition at $\delta^\pm = 0$; it will automatically be satisfied. Moreover, we have the added bonus that the optimal control satisfies the first-order condition.

Assumption 2. The dynamics for $\kappa_t$ satisfy

$$
\begin{cases}
    d\kappa_t^- = \beta_\kappa (\theta^- - \kappa_t^-) \, dt + \eta_\kappa \, d\mathcal{M}_t^- + \nu_\kappa \, d\mathcal{M}_t^+,
    \\
    d\kappa_t^+ = \beta_\kappa (\theta^+ - \kappa_t^+) \, dt + \eta_\kappa \, d\mathcal{M}_t^+ + \nu_\kappa \, d\mathcal{M}_t^-,
\end{cases}
$$

where $\eta_\kappa$, and $\nu_\kappa$ are non-negative constants and $\theta_\kappa$ and $\beta_\kappa$ are strictly positive constants.

Assumption 2 is a specific modeling assumption\textsuperscript{7} on $\kappa_t$ which allows for incoming influential market orders to have an impact on the FPs. An increase (decrease) in the fill rate can be due to two main reasons: (i) a decrease (increase) in limit order book depth or (ii) an increase (decrease) in the distribution of market order volumes (in a stochastic dominance sense). This is a one-way effect because influential market orders cause jumps in the $\kappa_t$ process, but jumps in the FP do not induce jumps in market order arrivals. While it is possible to allow such feedback, empirical investigations (such as those in Large (2007)) demonstrate that the

\textsuperscript{7}It is in principle possible to include more general dynamics on $\kappa$; however, we opt to work with this assumption for two reasons (i) the results are easily interpreted and (ii) it reflects the behavior actually observed in the LOB dynamics.
incoming market orders influence the state of the LOB and not the other way around. The mean-reversion term draws $\kappa_t^\pm$ to the long-run mean of $\theta_\kappa$ so that the impact of influential orders on the LOB is only temporary. Typically, we expect that the rate of mean-reversion $\beta_\kappa$ for the LOB to be slower than the rate of mean-reversion $\beta$ of the market order activity. In other words, the impact of influential orders persists in the LOB on a longer time scale compared to their effect on market order activity.

Observe that if market order volumes are iid, then the $\kappa_t^\pm$ processes can be interpreted as parameters directly dictating the shape of the limit order book. In particular, if the market order volumes are iid exponentially distributed and the shape of the LOB is flat, then the probability that a limit order at price level $S_t + \delta_t^\pm$ is executed (given that a market order arrives) is equal to $e^{-\kappa_t^\pm \delta_t^\pm}$. Consequently, $\kappa_t^\pm$ can be interpreted as the exponential decay factor for the fill rate of orders placed away from the midprice. In order to satisfy the $C^1$ condition at $\delta^\pm = 0$ and the condition that $h_\pm(\delta, \kappa_t) = 1$ for $\delta^\pm \leq 0$, it is necessary to smooth the exponential function at $\delta = 0$. This is always possible, since there clearly exists smooth functions which has sup distance to the target function less than any positive $\epsilon$.

Moreover, it is easy to see that immediately after an influential market buy/sell order arrives (which eats into the sell/buy side of the book), then the probability (given that a market order arrives) that a limit order at price level $S_t + \delta_t^\pm$ is executed is, for the same $\delta^\pm$, smaller than the probability of being filled before the influential order arrived. The intuition is the following. Immediately after an influential market order arrives, market participants react in anticipation of the increase of market activity they will face and decide to send limit orders to the book. Since many market participants react in similar way, the probability of limit orders being filled, conditioned on a market order arriving, decreases.\(^8\)

Figure 4 illustrates the shape of the fill rates at time $t$ describing the rate of arrival of market orders which fill limit orders placed at price levels $S_t \pm \delta_t^\pm$. Notice that these rates peak at zero spread at which point they are equal to the arrival rate of market orders. In the figure these rates are asymmetric and decay at differing speeds because we have assumed different parameters for the buy and sell side, $\kappa_t^+ = 2$, $\kappa_t^- = 1$, $\lambda_t^+ = 0.75$ and $\lambda_t^- = 1$. In general these rates will fluctuate throughout the day.

\(^8\)It is also possible to have markets where, conditioned on the arrival of a market order, the probability of a limit order being filled increases immediately after the arrival of an influential order. We can incorporate this feature in our model. Note also that in our general framework, immediately after the influential buy/sell market order arrives, the intensities $\lambda^\pm$ increase and the overall effect of an influential order on the fill rates $\Lambda^\pm = \lambda_t^+ h_\pm(\delta, \kappa_t)$ is ambiguous when $\lambda_t^+$ and $h_\pm(\delta, \kappa_t)$ move in opposite directions after the arrival of an influential order, for example when $h_\pm(\delta, \kappa_t) = e^{-\kappa_t^\pm \delta_t^\pm}$. \(^\dagger\)
4. The High Frequency Trader’s Optimization Problem

So far, we have specified counting processes for market orders and dynamics of the LOB through the FP; however, we also require a counting process for the agent’s filled limit orders. To this end, let $N_t^+$ and $N_t^-$ denote the number of the agent’s limit sell and buy orders, respectively, that were filled up to and including time $t$ and the process $q_t = N_t^- - N_t^+$ is the agent’s total inventory. Note that the arrival rate of these counting processes can be expressed as $\Lambda_t^\pm \equiv \lambda_t^\pm h_\pm(\delta; \kappa_t)$, as in Assumption 1. Finally, the agent’s cash process (i.e., excluding the value of the $q_t$ shares she currently holds) satisfies the SDE

$$dX_t = (S_t + \delta_t^+) dN_t^+ - (S_t - \delta_t^+) dN_t^-.$$  (4)

4.1. Formulation of the HF investment problem

The HFT wishes to place sell/buy limit orders at the prices $S_t \pm \delta_t^\pm$ at time $t$ such that the utility of her expected terminal wealth is maximized. The HFT is continuously repositioning her limit orders in the book by canceling the limit orders that were not filled and sending new ones. Specifically, her value function is

$$\Phi(t, x, S, \lambda, \kappa) = \sup_{(\delta^\pm, \delta^\pm)} \mathbb{E} [U(X_T + q_T S_T) \mid \mathcal{F}_t],$$  (5)

where the supremum is taken over all non-negative $\mathcal{F}_t$-progressively measurable functions and $U(x)$ is the utility function. Moreover, $\mathcal{F}_t$ is the natural (and completed) filtration generated by the collection of processes $\{S_t, M_t^\pm, N_t^\pm\}$ and the extended filtration $\mathcal{F}_t = \mathcal{F}_t \vee \sigma(\{M_u\}_{0 \leq u \leq t})$, note that $\lambda_t$ and $\kappa_t$ are progressively measurable with respect to this expanded filtration. We will often suppress the dependence on many of the variables in $\Phi(\cdot).$
We assume that the HFT’s utility function is \( U(x) = -\exp\{-\gamma x\} \) where \( \gamma > 0 \) is the degree of risk aversion. For small levels of \( \gamma \) we write \( U(x) \sim x - \frac{\gamma}{2} x^2 + o(\gamma) \) so that maximizing expected utility is equivalent to maximizing expected terminal wealth while penalizing variance of terminal wealth. This intuition will play out in our asymptotic expansion of the value function and the corresponding optimal strategy that we derive in Corollary 4.

The above control problem can be cast into a discrete-time controlled Markov chain as carried out in Bäuerle and Rieder (2009). Classical results from Bertsekas and Shreve (1978) imply that a dynamic programming principle holds and that the value function is the unique viscosity solution of the HJB equation

\[
(\partial_t + \mathcal{L}) \Phi + \frac{1}{2} \sigma^2 \Phi_{ss} + \lambda^- \sup_{\delta^-} \left\{ \rho h_-(\delta^-; \kappa) \left[ S_{q,\lambda}^- \Phi(t, x - s + \delta^-) - \Phi \right]
\right.
\]
\[
\left. + (1 - \rho) h_-(\delta^-; \kappa) \left[ S_{q}^- \Phi(t, x - s + \delta^-) - \Phi \right] + \rho \left( 1 - h_-(\delta^-; \kappa) \right) [S_{\lambda}^- \Phi - \Phi] \right\}
\]

\[
+ \lambda^+ \sup_{\delta^+} \left\{ \rho h_+(\delta^+; \kappa) \left[ S_{q,\lambda}^+ \Phi(t, x + s + \delta^+) - \Phi \right]
\right.
\]
\[
\left. + (1 - \rho) h_+(\delta^+; \kappa) \left[ S_{q}^+ \Phi(t, x + s + \delta^+) - \Phi \right] + \rho \left( 1 - h_+(\delta^+; \kappa) \right) [S_{\lambda}^+ \Phi - \Phi] \right\} = 0,
\]

with boundary condition \( \Phi(T, \cdot) = -\exp\{-\gamma(x + qs)\} \), and the integro-differential operator \( \mathcal{L} \) is the part of the generator of the processes \( \lambda_t, \kappa_t \), and \( Z_t^\pm \) which do not depend on the controls \( \delta^\pm \). Explicitly,

\[
\mathcal{L} = \beta(\theta - \lambda^-) \partial_{\lambda^-} + \beta(\theta - \lambda^+) \partial_{\lambda^+}
\]
\[
+ \beta_\kappa(\theta_\kappa - \kappa^-) \partial_{\kappa^-} + \beta_\kappa(\theta_\kappa - \kappa^+) \partial_{\kappa^+} + \mu^- \left( \tilde{S}_{\lambda^-}^1 - 1 \right) + \mu^+ \left( \tilde{S}_{\lambda^+}^1 - 1 \right).
\]

Moreover, we have introduced the following shift operators:

\[
S_{q,\lambda}^\pm \Phi(t, x, s, q, \lambda, \kappa) = \Phi(t, x, s, q \mp 1, \lambda, \kappa), \quad (8a)
\]
\[
S_{q}^\pm \Phi(t, x, s, q, \lambda, \kappa) = \Phi(t, x, s, q \mp 1, \lambda + (\nu, \eta)' \kappa + (\nu_\kappa, \eta_\kappa)', \kappa), \quad (8b)
\]
\[
S_{\lambda}^\pm \Phi(t, x, s, q, \lambda, \kappa) = \Phi(t, x, s, q \mp 1, \lambda + (\eta, \nu)' \kappa + (\eta_\kappa, \nu_\kappa)', \kappa), \quad (8c)
\]
\[
S_{q,\lambda}^\pm \Phi(t, x, s, q, \lambda, \kappa) = S_{q}^\pm S_{\lambda}^\pm \Phi(t, x, s, q, \lambda, \kappa), \quad (8d)
\]
\[
\tilde{S}_{\lambda}^- \Phi(t, x, s, q, \lambda, \kappa) = \Phi(t, x, s, q \mp 1, \lambda + (\tilde{\nu}, \tilde{\eta})', \kappa), \quad (8e)
\]
\[
\tilde{S}_{\lambda}^- \Phi(t, x, s, q, \lambda, \kappa) = \Phi(t, x, s, q \mp 1, \lambda + (\tilde{\eta}, \tilde{\nu})', \kappa). \quad (8f)
\]

The terms of the operator \( \mathcal{L} \) have the usual interpretations: the first and second terms cause the activity rates to decay back to the long run level \( \theta \). The third and fourth terms pull \( \kappa^\pm \) to their long run level. The fifth and sixth terms cause market order activities to jump upon public news arrival. Furthermore, the various terms in the HJB equation represent the
jumps in the activity rate and/or a limit order being filled together with the diffusion of the asset price through the term $\frac{1}{2}\sigma^2\Phi_{ss}$. More specifically, the sup over $\delta^-$ contain the terms due to the arrival of a market sell order (which are filled by limit buy orders). The first of the three terms represents the arrival of an influential market order which fills the limit order, the second term represents the arrival of a non-influential market order which fills the limit order, while the third term represents the arrival of an influential market order which does not reach the limit order’s price level. The sup over $\delta^+$ contain the analogous terms for the market buy orders (which are filled by limit sell orders).

4.2. The Feedback Control of the optimal trading strategy

In general, an exact optimal control is not analytically tractable – one exception is the case of an exponential fill rate where the optimal control admits an exact analytical solution which we present in Appendix D.1. For the general case, we provide an approximate optimal control via an asymptotic expansion which is correct to $o(\gamma)$. In principle, the expansion can be carried to higher orders if so desired.

Proposition 2. Optimal Trading Strategy, Feedback Control Form. The value function $\Phi$ admits the decomposition $\Phi = -\exp\{-\gamma(x + qs + g(t, q, \lambda, \kappa))\}$ with $g(T, \cdot) = 0$. Furthermore, assume that $g$ can be written as an asymptotic expansion in $\gamma$ as follows

$$g(t, q, \lambda, \kappa) = g_0(t, \lambda, \kappa) + \gamma g_1(t, q, \lambda, \kappa) + o(\gamma)$$

with boundary conditions $g_0(T, \cdot) = g_1(T, q, \cdot) = 0$. Then, the feedback controls of the optimal trading strategy for the HJB equation (6) admit the expansion

$$\delta_t^{\pm} = \delta_0^{\pm} + \gamma \delta_1^{\pm} + o(\gamma),$$

where

$$\delta_1^{\pm} = A(\delta_0^{\pm} : \kappa) - B(\delta_0^{\pm} : \kappa) \left( \rho(S_{q, \lambda}^{\pm} - S_{\lambda}^{\pm}) g_1 + (1 - \rho) \Delta_q^{\pm} g_1 \right),$$

the coefficients

$$A(\delta_0^{\pm} : \kappa) = \frac{1}{2} \delta_0^{\pm} h_+^{\pm}(\delta_0^{\pm} : \kappa) C(\delta_0^{\pm} : \kappa), \quad B(\delta_0^{\pm} : \kappa) = \frac{h_+^{\pm}(\delta_0^{\pm} ; \kappa)}{C(\delta_0^{\pm} : \kappa)},$$

$$C(\delta_0^{\pm} : \kappa) = 2h_+^{\prime}(\delta_0^{\pm} ; \kappa) + \delta_0^{\pm} h_+^{\prime\prime}(\delta_0^{\pm} ; \kappa),$$
\[ \Delta_{q}^{\pm} = \mathcal{S}_{q}^{\pm} - 1, \] and the shift operators \( \mathcal{S}_{q,\lambda}^{\pm} \) and \( \mathcal{S}_{\lambda}^{\pm} \) are as in (8). Moreover, \( \delta_{0}^{\pm} \) is a strictly positive solution to
\[ \delta_{0}^{\pm} h_{\pm}'(\delta_{0}^{\pm}; \kappa) + h_{\pm}(\delta_{0}^{\pm}; \kappa) = 0. \] (13)

A solution to (13) always exists. Furthermore, the exact optimal controls are non-negative.

\textbf{Proof.} See Appendix C. \qed

The function \( g \) introduced in the above ansatz for \( \Phi \) is related to the so-called reservation price of the asset as in Ho and Stoll (1981). In particular, the reservation bid price \( r_{t}^{-} \) satisfies \( \Phi(t, x - r_{t}^{-}, s, q + 1, \lambda, \kappa) = \Phi(t, x, s, q, \lambda, \kappa) \) implying \( r_{t}^{-} = S_{t} - (1 - \mathcal{S}_{q}^{-}) g \). Similarly, the reservation ask price \( r_{t}^{+} \) satisfies \( \Phi(t, x + r_{t}^{+}, s, q - 1, \lambda, \kappa) = \Phi(t, x, s, q, \lambda, \kappa) \) implying \( r_{t}^{+} = S_{t} + (\mathcal{S}_{q}^{+} - 1) g \). In other words, the reservation bid (ask) price is the price at which the HFT is indifferent between buying (selling) and the inventory increasing (decreasing) by one unit or not trading at all.

In the next subsection we use the optimal controls derived here to solve the nonlinear HJB equation and obtain an analytical expression for \( g_{1} \) which is the last step we require to determine \( \delta_{t}^{\pm*} \). Before proceeding we discuss a number of features of the optimal control \( \delta_{t}^{\pm*} \) given by (10). First, note that \( \delta_{t}^{\pm*} \) is independent of the midprice \( S_{t} \) because in our model the midprice dynamics are independent of the arrival of market orders and the dynamics of the LOB.

Moreover, the first two terms on the right-hand side of equation (10) show how the optimal postings are decomposed into a risk-neutral component and a risk-averse component. The risk-neutral component, given by \( \delta_{0}^{\pm} \), does not depend on the arrival rate of market orders. To see the intuition behind this result, we note that a risk-neutral HFT seeks to maximize the probability of being filled at every instant in time. Therefore, the HFT chooses \( \delta_{t}^{\pm} \) to maximize the expected spread conditional on a market order hitting the appropriate side of the book, i.e. maximizes \( \delta_{t}^{\pm} h_{\pm}(\delta_{t}^{\pm}; \kappa_{t}) \). The first order condition of this optimization problem is given by (13) where we see that \( \lambda^{\pm} \) plays no role in how the limit orders are calculated.\(^9\)

The second term on the right-hand side of (10) reflects how the HFT penalizes variability in the outcomes of her strategy and depends on the arrival rate of market orders and the inventories \( q_{t} \). For example, the first term on the right-hand side of the expression for \( \delta_{1}^{\pm} \) in (11), given by the function \( A(\delta_{0}^{\pm}; \kappa) \) in (12), shows how, compared to the risk-neutral case, the HFT decreases the depth at which orders are posted in the LOB to mitigate the volatility of the terminal wealth \( X_{T} + q_{T} S_{T} \).\(^{10}\)

\(^{9}\)If there are multiple solutions to (13) the HFT chooses the \( \delta_{t}^{\pm} \) that yields the maximum of \( \delta_{t}^{\pm} h_{\pm}(\delta_{t}^{\pm}; \kappa_{t}) \).

\(^{10}\)Note that the numerator of \( A(\delta_{0}^{\pm}; \kappa) \) is the maximum of the FP, and that the denominator contains \( C(\delta_{0}^{\pm}; \kappa) < 0 \) which is the second order condition of the optimization problem solved by the risk-neutral HFT.
Finally, the first order term $g_0$ in the expansion of $g$ plays no role in the optimal strategy. The reason is that the spreads are set by how the value function changes when orders are filled and, since $g_0$ is independent of the agent’s inventory, its dependence drops out.

4.3. The asymptotic solution of the optimal trading strategy

Armed with the optimal feedback controls, our remaining task is to solve the resulting non-linear HJB equation to this order in $\gamma$. The following Theorem contains a stochastic characterization of the asymptotic expansion of the value function. This characterization can be computed explicitly in certain cases and then plugged into the feedback control to provide the optimal strategies.

**Theorem 3. Solving The HJB Equation.** The solution to the non-linear equation for $g_0$ is given by

$$g_0(t, \lambda, \kappa) = E \left[ \int_t^T \left\{ \lambda_u^+ \delta_{0,u}^{-} h_-(\delta_{0,u}^{-}; \kappa_u) + \lambda_u^+ \delta_{0,u}^{+} h_+ (\delta_{0,u}^{+}; \kappa_u) \right\} du \right| F_t ] .$$  

Moreover, the solution to the non-linear equation for $g_1$ can be written in the form

$$g_1 = a(t, \lambda, \kappa) + q b(t, \lambda, \kappa) + q^2 c(t, \lambda, \kappa) ,$$  

where

$$a(t, \lambda, \kappa) = E \left[ \int_t^T f(u, \lambda_u, \kappa_u, ) du \right| F_t ],$$  

$$b(t, \lambda, \kappa) = \sigma^2 E \left[ \int_t^T \{ h_+ (\delta_{0,u}^{+}; \kappa_u) \lambda_u^+ - h_- (\delta_{0,u}^{-}; \kappa_u) \lambda_u^- \} (T - u) du \right| F_t ] , \quad \text{and}$$  

$$c(t, \lambda, \kappa) = -\frac{1}{2} \sigma^2 (T - t) .$$  

Here, the function $f(t, \lambda, \kappa)$ is given by (arguments are suppressed for compactness)

$$f = \rho (\lambda^- h_- \Delta^- b - \lambda^+ h_+ \Delta^+ b) + c(\lambda^- h_- + \lambda^+ h_+) + b(\lambda^- h_- - \lambda^+ h_+) + f_0$$  

where $f_0$ is a function of $g_0$ provided in equation (D.3), and the difference operators are

$$\Delta_q^+ = S_q^+ - 1, \quad \Delta_{\lambda}^+ = S_{\lambda}^+ - 1, \quad \text{and} \quad \Delta_{q,\lambda}^+ = S_{q,\lambda}^+ - 1 .$$  

**Proof.** See Appendix D.
The asymptotic expansion of the optimal controls now follows as a straightforward corollary to Theorem 3. Note that the function \( b \) in (16b) is the only function which appears in the optimal controls below. As well, the function \( b \) can be computed explicitly under certain assumptions of the FPs \( h \). In particular, if \( h \) is exponential or power, it is computable in closed form and Proposition 6 in Appendix D.1 provides explicit formulae for a general class which includes these two cases.

**Corollary 4. Asymptotic Optimal Limit Orders.** The asymptotic expansion of the optimal controls to first order in \( \gamma \) is

\[
\delta^{\pm} = \delta^{\pm}_0 + \gamma \left\{ A(\delta^{\pm}_0; \kappa) + B(\delta^{\pm}_0; \kappa) \left( \frac{1}{2} \mp q \right) \sigma^2(T - t) \right. \\
\left. \quad \pm B(\delta^{\pm}_0; \kappa) \left[ \rho b \left( t, \lambda + \chi^{\pm}, \kappa + \xi^{\pm} \right) + (1 - \rho) b(t, \lambda, \kappa) \right] \right\},
\]

where \( \delta^{\pm}_0 \) satisfies (13), and we have \(|\delta^{\pm}_{opt} - \delta^{\pm}| = o(\gamma)\). In the above, \( A \) and \( B \) are provided in (12), \( \chi^- = (\eta, \nu)' \) and \( \chi^+ = (\nu, \eta)' \) and \( \xi^- = (\eta_\kappa, \nu_\kappa)' \) and \( \xi^+ = (\nu_\kappa, \eta_\kappa)' \). Time dependencies have been suppressed. Furthermore, the optimal controls \( \max\{\delta^{\pm}, 0\} \) are also of order \( o(\gamma) \).\(^{11}\)

**Proof.** Applying the explicit form for \( g_1 \) in Theorem 3 to Equations (10) and (11) of Proposition 2 and using the fact that \( a, b \) and \( c \) are independent of \( q \), after some tedious computations, \( \delta^{* \pm} \) reduces to (18). Finally, let \( \delta^{\pm}_{opt} \) denote the exact optimal controls. By the Proposition 2 we have that \( \delta^{\pm}_{opt} \) is non-negative and \(|\delta^{*} - \delta^{\pm}_{opt}| = o(\gamma)\) therefore,

\[
|\max\{\delta^{\pm*}, 0\} - \delta^{\pm}_{opt}| \leq |\delta^{\pm*} - \delta^{\pm}_{opt}| = o(\gamma),
\]

and we are done. \( \square \)

**5. Exponential and Power Fill Probabilities: Special Cases of Interest**

For completeness, we provide here the asymptotic expansion of the special case of exponential FPs,\(^{12}\) \( h_\pm(\delta^{\pm}, \kappa^{\pm}) = e^{-\kappa^{\pm}_t \delta^{\pm}} \), as well as power law fill rates, \( h_\pm(\delta^{\pm}, \kappa^{\pm}) = (1 + (\kappa^{\pm}_t \delta^{\pm})^{\alpha})^{-1} \)

\(^{11}\)Note that the exact solution of the optimal control is non-negative as discussed in Assumption 1, but this is not necessarily the case in the asymptotic solution, thus we write the optimal control as \( \max\{\delta^{\pm*}, 0\} \).

\(^{12}\)Note that exponential FPs do not have a maximum of 1 and therefore violates Assumption 1. However, as commented earlier, one can always modify the exponential fill rate so that it remains at 1 for negative \( \delta \) and is differentiable at 0 by a suitably differentiable function so that Assumption 1 is satisfied. Such a modification leads to the control \( \max(\delta_0 + \gamma \delta_1, 0) \) which Corollary 4 shows is of \( o(\gamma) \).
for $\alpha > 1$. To this end, the risk-neutral component of the optimal controls reduce to

$$
\delta^\pm_0 = \begin{cases} 
(k_t^\pm)^{-1}, & \text{exponential,} \\
(\alpha - 1)^{1/\alpha}(k_t^\pm)^{-1}, & \text{power law,}
\end{cases}
$$

and the coefficients of the first order correction in (11) reduce to

$$
A = -\frac{1}{2}(k_t^\pm)^{-2}, \quad B = 1, \quad \text{exponential,}
$$

$$
A = -\frac{1}{2}(\alpha - 1)^{1-2/\alpha}(k_t^\pm)^{-2}, \quad B = (\alpha - 1)^{-1}, \quad \text{power law.}
$$

Notice how similar these controls are. They also allow us to provide some simple financial interpretations of the expansion (10). The leading term is the risk-neutral optimal control and is inversely proportional to the available volume in the market. If there are many available quotes close to the midprice, reflected in a large value of $k_t^\pm$, then it is less likely that an order deep in the book will be executed; consequently the HFT places limit orders closer to the midprice. The second term is an adjustment for risk aversion and is made up of three components, see (11). The first is independent of the activity of market orders, but does depend again on the depth of the book. The second and third terms, on the other hand, do depend on the current market activity and represent the expected change in $g(\cdot)$ due to one of the limit orders being filled. The word ‘expected’ is used since the corresponding market order can be either influential or non-influential and the coefficients of each term is weighted by their respective probabilities.

In Figure 5, we illustrate how the limit order spreads $\delta^\pm_t$ vary through time for three different influential trade probabilities $\rho$, while holding the other parameters fixed. The dotted lines show the strategy that results when $\rho = 0$ so there is no self excitation in the arrival of market orders and LOB dynamics, we label this strategy with $\rho = 0$ the Poisson strategy. Panels (b) and (c) depict what happens when $\rho > 0$, which we label the adapted strategy, and shows that compared to the Poisson case spreads are wider because the HFT anticipates that after a market order arrives, the intensity of market orders increases (with probability $\rho$) and therefore

Figure 5: Behavior of $\delta^\pm_*$ as a function of time for three different levels of influential trade probability $\rho$. Dotted lines represent the case of no influential trades $\rho = 0$. The agents time horizon is $T = 30$ min and the model parameters are $\beta = 60$, $\theta = 1$, $\eta = 40$, $\nu = 10$, $\lambda^\pm = 1$, $\kappa^+ = 50$, $\kappa^- = 75$, $\mu^\pm = 0$, $\sigma = 0.005$ and $\gamma = 0.5$. Moreover, $\kappa^\pm$ are assumed constant here so that $\eta = \nu = 0$, $\theta^\pm = \kappa^\pm$ and $\nu = 0$. 

$\rho = 0$

$\rho = 0.5$

$\rho = 1$
the trader can offset the new acquired position and profit from a larger realized spread. Hence, \( \delta \) is increasing in \( \rho \) due to the expected increase in activity rates. The asymmetry in the results is due to the assumption that \( \kappa^+ < \kappa^- \) implying that the LOB has more concentration near the mid-price on the buy side compared to the sell side. Consequently, the optimal limit orders are ordered \( \delta^+ > \delta^- \). As maturity approaches, the adaptive strategy converges to the Poisson strategy because even if an arrival of an influential market order increases the arrival rate of market orders, the HFT has very little time left to complete the roundtrip trade.

In Figure 6, we illustrate how the spreads move immediately after a limit order has been filled by an influential order where we assume the HFT’s inventory before the arrival of the trade is zero. The figure shows three panels. The first panel shows the optimal spreads around the midprice before a market order arrives. The second panel shows how the spreads are adjusted immediately after a sell market order arrives (which is filled by the HFT’s buy limit order) and the third panel shows how the spreads are adjusted if the incoming market order is a buy which fills the HFT’s limit sell order. In the three panels, the \( x \)-axis shows the evolution of time and the strategy matures at \( T = 30 \) minutes.

To understand the intuition behind the second panel assume that a market sell order arrives and hits the HFT’s buy limit order. Immediately, the sell market order activity rate jumps by 40, the buy market order activity jumps by 10, and the HFT’s inventory level increases to 1. At the same time the HFT adjusts her strategy: the buy spread \( \delta^- \) increases from the zero inventory case while the sell spread \( \delta^+ \) decreases relative to the zero inventory case. The resulting strategy pushes the HFT to sell the asset and return to a neutral position. The third panel, Figure 6c, has a similar interpretation but when the HFT adjusts her strategy right after being lifted by a market buy order. These effects are also seen in the Poisson model, however, the changes in the adapted case are larger than in the Poisson case because the intensities of market orders jump.

Note that here we have not included the jump in the FP parameter \( \kappa \) in order to keep the interpretations simple. However, in real situations when an influential trade arrives, \( \kappa \) will increase inducing a downward pressure on the spread (since increasing \( \kappa \) decreases the prob-
ability of being filled at a fixed spread). This will compete with the effect of the influential order increasing the activity rate and the compounded result is difficult to interpret. In the simulations in the next section we do however include both effects.

6. Simulation study with exponential and power fill rates

In this section we apply a simulation study of the HF strategy where buy and sell market orders are generated over a period of 5 minutes. The HFT is rapidly submitting (and canceling) limit orders and are filled according to exponential and power FPs. The optimal postings are calculated using (19) and (20) together with Corollary 4 and the explicit form for \( b \) in Proposition 6. The market activity rates and \( \kappa_t \) processes are updated appropriately and the terminal cash-flows are stored to produce histograms of the profit and loss (PnL) generated from these strategies. To generate the PnL histograms we assume that the final inventory is liquidated at the midprice with different transactions costs per share: 0 basis points (bp), 1 bp and 10 bps.\(^{13}\) In each simulation the process is repeated 5,000 times to obtain the histograms in Figures 8 to 10. More details on the simulation procedure are in Appendix E.

In the simulations we assume that the HFT uses her superior computer power to process information and submit orders. In particular, the HFT is able to estimate the parameter \( \rho \) which, conditional on the arrival of a market orders, is the probability that the order is influential.\(^{14}\)

We assume that news does not arrive during the simulation and assume the following values for the parameters of the arrival rate of market orders \( \lambda^\pm_t \) and the dynamics of \( \kappa^\pm_t \): \( \rho = 0.7, \beta = 0, \theta = 1, \eta = 40, \nu = 10, \beta_\kappa = 10, \theta_\kappa = 50, \nu_\kappa = 5 \), and \( \eta_\kappa = 25 \). A snippet from one sample path of the simulation is shown in Figure 7.

We compare the results from the adapted strategy used by the HFT with a benchmark HFT that uses a Poisson strategy because she cannot distinguish between influential and non-influential orders. More specifically, she assumes that all orders are non-influential so that \( \rho = 0 \). Not being able to make the distinction between types of incoming market orders affects how the benchmark HFT perceives the intensity of market arrivals and also affects her perception of the parameters \( \kappa^+_t \) and \( \kappa^-_t \) she uses. Thus, we assume that the benchmark HFT uses the long-run mean \( \kappa^+_t = \bar{m}^+_t(\infty) \) given in Lemma 9 and the long-run mean \( \lambda^+_t = m^+_t(\infty) \) given in Lemma 1. Below we also refer to the benchmark case as the Poisson case.

In Figure 8(a), we see the effect that the risk aversion parameter, \( \gamma \), has on the PnL distribution. As expected, higher aversion to risk leads to less expected profit but with smaller

\(^{13}\)The transaction costs are computed on a percentage basis and since the starting mid-price in the simulations is $100, these correspond to approximately 0, 1, and 10 cents per share, respectively.

\(^{14}\)The details on how to estimate \( \lambda \), and how to update it online, will appear in a forthcoming article which focuses on statistical calibration issues.
Figure 7: A sample path generated by the strategy under an exponential fill rate. When influential trades arrive, the activity of both buy and sell orders increase but by differing amounts, the FP parameters also jump, and both decay to their long-run levels. Circles indicate the arrival of an influential market order, while squares indicate the arrival of non-influential trades.

variance. As demonstrated in Figure 8(b), the adapted strategy, where the HFT posts orders using information about influential trades, outperforms the Poisson strategy for both $\gamma = 0.1$ and $\gamma = 1$.

We repeat the simulation study using a stochastic power law fill rate function with $\alpha^\pm = 1.5$ and $\alpha^\pm = 3$ (motivated by Chakraborti et al. (2011), Section 3.3.1). The same qualitative results are found as in the case of exponential fill rates and shown in Figures 9 and 10. Increasing $\alpha^\pm$ reduces the profitability of the $\gamma = 1$ adapted strategy, but also reduces PnL variance. Moreover, the difference in PnL of the adapted strategy over the Poisson benchmark is more prominent when $\alpha^\pm = 3$ for both values of $\gamma$.

Although both the adaptive and Poisson strategy are designed to penalize reaching the terminal date $T$ with inventories different from zero, there are many scenarios where this is not the case. When this happens, the simulations above assumed that the inventories are unwound by the HFT at the midprice $S_T$ at the terminal date $T$. In practice the HFT will bear some costs when unwinding a large quantity which could be in the form of a temporary price impact (a
result of submitting a large market order) and by paying a fee to the exchange for taking liquidity in the form of an aggressive market order. Therefore, we also show the performance of the HIF strategies when inventories are liquidated at the midprice plus a transaction cost. In Figure 11 we assume that the cost of unwinding is 1 bp per share, and in Figure 12 the cost is 10 bps per share. We observe that even after including realistic liquidation costs the results are similar to those above; however, as expected the PnL histograms are slightly shifted to the left due to the liquidation cost.
7. Conclusions

We develop an HF trading strategy where an HFT uses her superior speed advantage to process information and to send orders to the LOB to profit from roundtrip trades over very short-time scales. One of our contributions is to differentiate market orders between influential and non-influential. The arrival of influential market orders increases market order activity and also affects the shape and dynamics of the LOB. On the other hand, when non-influential market orders arrive they eat into the LOB but have no effect on the demand or supply of shares in the market.

The trading strategy that the HFT employs is given by the solution of an optimal control
problem where the trader is constantly submitting and canceling limit orders to maximize expected utility of profits over a short time interval $T$. The strategy shows how to optimally post (and cancel) buy and sell orders and is continuously updated to incorporate information of the arrival of market orders, news (good, bad and ambiguous), and the size and sign of inventories. The optimal strategy captures one of the key characteristics that differentiate HFTs from other algorithmic traders: strong mean reversion of inventories to zero. For example, if inventories are positive then, everything else equal, the optimal strategy posts sell limit orders nearer to the midprice and limit buy orders further away from the midprice. In this way the HFT is more likely to be filled on the sell side which would reduce inventories. Moreover, as we approach the terminal date of the strategy, limit orders move nearer to the midprice so that inventories are drawn to zero at the end of the trading horizon.

Our framework allows us to derive asymptotic solutions of the optimal control problem under very general assumptions of the dynamics of the LOB. We perform a simulation study of our strategy using exponential and power fill rates. We show that our reduced form model with positive feedback (influential orders affect the intensity of the arrival of trades and the shape of the book) outperforms a strategy based on a model where the arrival of market orders is Poisson and there are no self-affecting effects on the demand and supply of assets. Our model depicts a more realistic picture of the dynamics of both the LOB and arrival of market orders because it allows for trade clustering as well as transitory changes in the competition for position in the LOB.
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8. References


Appendix A. Fitting the Model

When all market orders are influential (i.e., when $\rho = 1$), the path of the intensity process is fully specified once the times at which the buy and sell trades have been given. Because of this, the likelihood function can be written explicitly, and a straightforward maximum likelihood estimation (MLE) can be used. To be specific, suppose we have observations $\{t_1, t_2, \ldots, t_n\}$ of trade times (with $t_n \leq T$ the time of the last trade) and the indicators $\{B_1, B_2, \ldots, B_n\}$ which are 0 if the trade is a market sell and 1 if the trade is a market buy. Then it is easy to see that hazard rates at time $t$ are given by

$$
\lambda_t^+ = \theta + \sum_{i=1}^{n} [B_i \eta + (1 - B_i) \nu] e^{-\beta(t-t_i)}, \quad \text{and} \quad (A.1a)
$$

$$
\lambda_t^- = \theta + \sum_{i=1}^{n} [B_i \nu + (1 - B_i) \eta] e^{-\beta(t-t_i)}. \quad (A.1b)
$$

As well, the integrated hazard rates are

$$
\int_0^t \lambda_u^+ du = \theta t + \sum_{i=1}^{n} [B_i \eta + (1 - B_i) \nu] \frac{1 - e^{-\beta(t-t_i)}}{\beta}, \quad \text{and} \quad (A.2a)
$$

$$
\int_0^t \lambda_u^- du = \theta t + \sum_{i=1}^{n} [B_i \nu + (1 - B_i) \eta] \frac{1 - e^{-\beta(t-t_i)}}{\beta}. \quad (A.2b)
$$
Finally, the log-likelihood

$$\mathcal{L} = -2\theta T - \sum_{i=1}^{n} \left\{ B_i \log(\lambda_{i}^{+}) + (1 - B_i) \log(\lambda_{i}^{-}) + (\eta + \nu) \frac{1 - e^{-\beta(T-t_i)}}{\beta} \right\}. \quad (A.3)$$

Maximizing this log-likelihood we obtain the MLE estimates of the model parameters, and upon back substitution into Equation (A.1) provides us with the estimated path of activity. Finally, integrating this activity over the last second, i.e., $\int_{t-1}^{t} \lambda_{u}^{\pm} du$ provides us with the path labeled ‘Fitted’ in Figure 2 – which is the smoothed version of the intensity. This is directly comparable to the 1 second historical intensity, labeled as ‘Historical’ in Figure 2.

Estimated parameters obtained in this manner for the time window 3:30pm to 4:00pm on Feb 1, 2008 for IBM are as follows:

$$\hat{\beta} = 180.05, \quad \hat{\theta} = 2.16, \quad \hat{\eta} = 64.16, \quad \text{and} \quad \hat{\nu} = 55.73.$$ 

Notice that the spikes in the historical intensity are often above the fitted intensities. The reason for this difference is that (as explained above) the fitted intensities are based on the assumption that all trades are influential (i.e., $\rho = 1$). Consequently, the size of the jump in intensities must be smaller than the true jump size to preserve total mean activity of trades. When a full calibration is carried out – in which $\rho$ is not necessarily 1 and the influential/non-influential nature of the event must be filtered – the jump sizes are indeed larger. We will report on this more involved estimation procedure and filtering problem in a forthcoming paper.

Appendix B. Proof of Lemma 1

Integrating both sides of (2), taking conditional expectation, applying Fubini’s Theorem, and then taking derivative gives the following coupled system of coupled ODEs for $m_{t}^{\pm}(u)$

$$\begin{pmatrix}
\frac{dm_{t}^{-}(u)}{du} \\
\frac{dm_{t}^{+}(u)}{du}
\end{pmatrix} + \begin{pmatrix}
\beta - \eta \rho & -\nu \rho \\
-\nu \rho & \beta - \eta \rho
\end{pmatrix} \begin{pmatrix}
m_{t}^{-}(u) \\
m_{t}^{+}(u)
\end{pmatrix} - \begin{pmatrix}
\beta \theta + \tilde{\eta} \mu^{-} + \tilde{\nu} \mu^{+} \\
\beta \theta + \tilde{\nu} \mu^{-} + \tilde{\eta} \mu^{+}
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix} \quad (B.1)$$

with initial conditions $m_{t}^{\pm}(t) = \lambda_{t}^{\pm}$. This is a standard matrix equation and, if $A$ has no zero eigenvalues, admits the unique solution

$$\begin{pmatrix}
m_{t}^{-}(u) \\
m_{t}^{+}(u)
\end{pmatrix} = e^{-A(u-t)} \left[ \begin{pmatrix}
\lambda_{t}^{-} \\
\lambda_{t}^{+}
\end{pmatrix} - A^{-1} \zeta \right] + A^{-1} \zeta. \quad (B.2)$$

Since $A$ is symmetric, it is diagonalizable by an orthonormal matrix $U$. Furthermore, its eigenvalues are $\beta - (\eta \pm \nu)\rho$. Clearly, in the limit $u \to \infty$, $m_{t}(u)$ converges if and only if $\beta - (\eta \pm \nu)\rho > 0$ which implies $\beta > (\eta + \nu)\rho$ since $\eta, \nu, \rho \geq 0$. 

27
The remaining case is if \( \mathbf{A} \) has at least one zero eigenvalue. However, it is easy to see that in this case, the solution to (B.1) has at least one of \( m_i^\pm(u) \) growing linearly as a function of \( u \). Furthermore, if one eigenvalue is zero, then either \( \beta = (\eta - \nu)\rho \) or \( \beta = (\eta + \nu)\rho \), which lie outside the stated bounds. Finally, if both eigenvalues are zero, then we must have \( \beta = \nu = \eta = 0 \). Once again outside of the stated bounds. □

Appendix C. Proof of Proposition 2

Applying the ansatz on the form on \( \Phi \), dividing out \(-\Phi\), differentiating inside the supremum in (6) with respect to \( \delta \pm \), expanding \( g \) using the specified ansatz, writing \( \delta_{\pm*} = \delta_0^\pm + \gamma \delta_1^\pm + o(\gamma) \), and setting the resulting equation to 0 gives our first-order optimality condition. We provide the first-order condition for one side of the book, and suppress all dependencies on \( \kappa \) and \( \pm \) signs on the difference operators (which is the only difference between the two sides).

\[
\begin{align*}
 h(\delta_0) + \delta_0 h'(\delta_0) + \gamma \left\{ h'(\delta_0) \left[ \rho \delta_1 + \rho \Delta_{q,\lambda} g_1 - \frac{1}{2} \rho (\delta_0 + \Delta_{q,\lambda} g_0)^2 + \frac{1}{2} \rho (\Delta_{\lambda} g_0)^2 \right] ight. \\
 &
 \left. + (1 - \rho) (\delta_1 + \Delta_{q} g_1 - \frac{1}{2} \delta_0^2) - \rho \Delta_{\lambda} g_1 \right] \\
 + \delta_1 h''(\delta_0) \left[ \delta_0 + \rho \Delta_{q,\lambda} g_0 - \rho \Delta_{\lambda} g_0 \right] - h(\delta_0) \left[ \delta_0 + \rho \Delta_{q,\lambda} g_0 \right] + \delta_1 h'(\delta_0) \right\} + o(\gamma) = 0.
\end{align*}
\]

(C.1)

Observe that the Taylor expansion of \( h(\delta) \) about \( \delta_0 \) requires the \( C^3 \) regularity condition to keep the error of the correct order. The \( C^1 \) regularity condition ensures that the global maximizer satisfies (C.1). Setting the constant term to zero yields (13). Using the fact that \( \delta_0 \) satisfies (13) allows for elegant simplifications. Setting the coefficient of \( \gamma \) to zero and solving for \( \delta_1 \) gives (10). The finiteness of the optimal control correct to this order is ensured by the last condition in Assumption 1.

The existence of a solution to (13) is clear by noticing that (13) is a critical point of the function \( \delta h(\delta) \). The critical point exists since \( \delta h(\delta) \) is non-positive for \( \delta \leq 0 \), strictly positive on an open interval of the form \((0, d)\) due to \( C^1 \), and goes to 0 in the limit by Assumption 1.

To see that the exact values of the optimal controls are non-negative, observe that the value function is concave and increasing in \( x \). Therefore, \( \Phi(t, x + \delta, \cdot) < \Phi(t, x, \cdot) \) for any \( \delta < 0 \). Since the shift operators appearing in the argument of the supremum are linear operators, and \( h(\delta; \kappa) \) is bounded above by 1 and attains this maxima at \( \delta = 0 \), the \( \delta = 0 \) strategy dominates all strategies which have \( \delta < 0 \). □
Appendix D. Proof of Theorem 3

Note that using the expansion for \( g \) given in (9) and also writing \( \delta^\pm = \delta_0^\pm + \gamma \delta_1^\pm + o(\gamma) \), we have (with dependence on remaining arguments suppressed to keep the expressions compact)

\[
\frac{S^\pm_q \Phi(t, x \pm S + \delta^\pm, \cdot)}{\Phi(t, x, \cdot)} = 1 - \gamma \delta_0^\pm + \gamma^2 \left( \frac{1}{2} (\delta_0^\pm)^2 - (\delta_1^\pm + \Delta_q^\pm g_1) \right) + o(\gamma^2),
\]

(D.1a)

\[
\frac{S^\pm_{q, A} \Phi(t, x \pm S + \delta^\pm, \cdot)}{\Phi(t, x, \cdot)} = 1 - \gamma (\delta_0^\pm + \Delta_A^\pm g_0)
\]

+ \gamma^2 \left( \frac{1}{2} (\delta_0^\pm + \Delta_A^\pm g_0)^2 - (\delta_1^\pm + \Delta_{q, A}^\pm g_1) \right) + o(\gamma^2),
\]

(D.1b)

\[
\frac{S^\pm_A \Phi(t, x, \cdot)}{\Phi(t, x, \cdot)} = 1 - \gamma \Delta_A^\pm g_0 - \gamma^2 \left( \Delta_A^\pm g_1 - \frac{1}{2} (\Delta_A^\pm g_0)^2 \right) + o(\gamma^2),
\]

(D.1c)

where we the difference operators are given by (17).

Inserting the above expansion, together with the feedback controls (20), into the HJB equation (6), and carrying out tedious but ultimately straightforward expansions, to order \( \gamma \), equation (6) reduces to

\[
0 = (\partial_t + L) g_0 + \rho \lambda^+ \Delta_A^+ g_0 + \rho \lambda^- \Delta_A^- g_0 + \lambda^+ \delta_0^+ h_+(\delta_0^+; \kappa) + \lambda^- \delta_0^- h_-(\delta_0^-; \kappa)
\]

+ \gamma \left\{ (\partial_t + L) g_1 - \frac{1}{2} \sigma^2 q^2 + f_0 
\]

+ \rho h_+(\delta_0^-; \kappa) \lambda^- \Delta_q^+ g_1 + h_+(\delta_0^+; \kappa) (1 - \rho) \lambda^+ \Delta_q^+ g_1 + (1 - h_+(\delta_0^+; \kappa)) \rho \lambda^+ \Delta_A^+ g_1
\]

+ \rho h_-(\delta_0^-; \kappa) \lambda^- \Delta_q^- g_1 + h_-(\delta_0^+; \kappa) (1 - \rho) \lambda^- \Delta_q^- g_1 + (1 - h_-(\delta_0^-; \kappa)) \rho \lambda^- \Delta_A^- g_1 \right\} + o(\gamma),
\]

(D.2)

subject to the boundary conditions \( g_0(T, \cdot) = g_1(T, \cdot) = 0 \) and where \( f_0 \) is a function of the \( g_0 \) solution and acts as a source:

\[
f_0 = -\frac{1}{2} L(g_0^2) - \lambda^+ \left( \frac{1}{2} h_+(\delta_0^+; \kappa) (\delta_0^+)^2 + \rho \delta_0^+ h_+(\delta_0^+; \kappa) \Delta_A^+ g_0 + \frac{1}{2} \rho \Delta_A^+ g_0^2 \right)
\]

- \lambda^- \left( \frac{1}{2} h_-(\delta_0^-; \kappa) (\delta_0^-)^2 + \rho \delta_0^- h_-(\delta_0^-; \kappa) \Delta_A^- g_0 + \frac{1}{2} \rho \Delta_A^- g_0^2 \right) .
\]

(D.3)

Focusing on the \( o(1) \) term in (D.2), which is given by its first line, we have

\[
\left\{ \begin{array}{c}
(\partial_t + L) g_0 + \rho \lambda^+ \Delta_A^+ g_0 + \rho \lambda^- \Delta_A^- g_0 + \lambda^+ \delta_0^+ h_+(\delta_0^+; \kappa) + \lambda^- \delta_0^- h_-(\delta_0^-; \kappa) = 0, \\
g_0(T, \cdot) = 0.
\end{array} \right.
\]

(D.4)

Clearly, this PIDE admits the Feynman-Kac representation provided in (14).

The \( o(\gamma) \) term in (D.2), given by setting the expression in the braces to zero, is more difficult.
Nonetheless, since \( q \) only appears explicitly in the term \(-\frac{1}{2}\sigma^2 q^2\), and appears implicitly only in the shift operators, setting the first order term to zero implies that the function \( g_1 \) admits a solution that is polynomial and at most quadratic in \( q \) as in (15) subject to the boundary conditions \( a(T, \cdot) = b(T, \cdot) = c(T, \cdot) = 0 \). Using the ansatz in (15), one finds

\[
\begin{align*}
\Delta^\pm_q g_1 &= (c \mp b) + q (\mp 2c), \quad (D.5a) \\
\Delta^\pm_\lambda g_1 &= \Delta^\pm_\lambda a + q \Delta^\pm_\lambda b + q^2 \Delta^\pm_\lambda c, \quad (D.5b) \\
\Delta^\pm_{q, \lambda} g_1 &= (\Delta^\pm_\lambda a + \mathcal{S}^\pm_\lambda c \mp \mathcal{S}^\pm_\lambda b) + q \left( \Delta^\pm_\lambda b \mp 2\mathcal{S}^\pm_\lambda c \right) + q^2 \Delta^\pm_\lambda c. \quad (D.5c)
\end{align*}
\]

Furthermore, since the source term proportional to \( q^2 \) is a constant, and the boundary condition for \( c \) is \( c(T, \cdot) = 0 \), \( c \) is independent of all state variables and is a function only of time. This leads to a number of simplifications and the \( o(\gamma) \) equation becomes

\[
0 = q^2 \left\{ \partial_t c - \frac{1}{2} \sigma^2 \right\} + q \left\{ (\partial_t + \mathcal{L}) b + \rho \lambda^+ \Delta^+_\lambda b + \rho \lambda^- \Delta^-_\lambda b + 2 h_-(\delta_0^-; \kappa) \lambda^- c - 2 h_+(\delta_0^+; \kappa) \lambda^+ c \right\} + \left\{ (\partial_t + \mathcal{L}) a + \rho \lambda^+ \Delta^+_\lambda a + \rho \lambda^- \Delta^-_\lambda a + f \right\},
\]

which is provided in (16d). It is easy to see that setting the coefficient of \( q^2 \) to zero implies that \( c = -\frac{1}{2}\sigma^2 (T - t) \) as stated. Furthermore, inserting this solution into the linear term in \( q \) shows that \( b \) is independent of \( \kappa \) and is a function only of time and \( \lambda \). Setting this linear term to zero and then appealing to a Feynman-Kac argument leads to the representation (16b). Finally, inserting the solutions for \( b, c \) and \( g_0 \) into the constant in \( q \) coefficient, setting that term to zero and appealing to a Feynman-Kac argument leads to the representation (16a).  

\[\Box\]

Appendix D.1. Optimal Trading Strategy With Exponential \( h(\cdot) \), Feedback Control Form.

Although an exact optimal control is not analytically tractable in general, we can extend the result of Avellaneda and Stoikov (2008) to the bivariate Hawkes framework.

**Proposition 5.** If \( h^\pm(\delta; \kappa) = e^{-\kappa^\pm \delta} \) for \( \delta > 0 \) and \( \mathbb{P} \left[ \inf_{t \in [0, T]} \kappa_t^\pm > 0 \right] = 1 \), then the feedback control of the optimal trading strategy for the HJB equation (6) is given by

\[
\begin{align*}
\dot{\delta}_t^* &= \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{\kappa^-} \right) - \frac{1}{\gamma} \ln \frac{\rho e^{-\gamma \mathcal{S}^-g}}{\rho e^{-\gamma \mathcal{S}^-g} + (1 - \rho) e^{-\gamma g}}, \\
\dot{\delta}_t^+ &= \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{\kappa^+} \right) - \frac{1}{\gamma} \ln \frac{\rho e^{-\gamma \mathcal{S}^+g}}{\rho e^{-\gamma \mathcal{S}^+g} + (1 - \rho) e^{-\gamma g}}, \quad (D.7)
\end{align*}
\]

where, the ansatz \( \Phi = -\exp \{-\gamma(x + q s + g(t, q, \lambda, \kappa))\} \) with boundary condition \( g(T, \cdot) = 0 \) has been applied. Furthermore, the solution in (D.7) is unique.
Proof. Applying the first order conditions to the supremum terms and using the specified ansatz leads, after some simplifications, to stated result. Uniqueness is trivial. □

Appendix D.2. Special Case: \( h_{\pm}(\delta; \kappa) \) are Constants

If we want a closed form expression for the function \( b \), then we need to make further assumptions on \( h_{\pm}(\delta; \kappa) \). Both exponential and power law fill rates lead to zeroth order optimal controls which have \( h_{\pm}(\delta; \kappa) \) being a constant irrespective of the dynamics on the shape parameter \( \kappa_{\pm} \). This leads us to investigate the class of models for which \( h_{\pm}(\delta; \kappa) \) is a constant. The following proposition provides an explicit form for the function \( b \) which is key in computing the optimal limit orders.

Proposition 6. Explicit Solution for \( b(t, \lambda, \kappa) \). If \( h_{\pm}(\delta; \kappa) = h_{\pm} \) are constants \( \mathbb{P}\text{-a.s.} \), then the function \( b(t, \lambda, \kappa) \) is independent of \( \kappa \), and is explicitly

\[
b(t, \lambda) = \sigma^2 \xi' \left( \left( A^{-1}(T-t) - A^{-2} \left( I - e^{-A(T-t)} \right) \right) [\lambda - A^{-1}\zeta] + \frac{1}{2}(T-t)^2 A^{-1}\zeta \right),
\]

where \( I \) is the \( 2 \times 2 \) identity matrix and \( \xi = \begin{pmatrix} -h_- \\ h_+ \end{pmatrix} \).

Proof. Note that \( \mathbb{E} \left[ \int_{t}^{T} \lambda_u^\pm (T-u) \, du \bigg| \mathcal{F}_t \right] = \int_{t}^{T} \mathbb{E} \left[ \lambda_u^\pm \big| \mathcal{F}_t \right] \, (T-u) \, du = \int_{t}^{T} m_t^\pm(u) \, (T-u) \, du \). Using the form of \( m_t^\pm(u) \) provided in (B.2) and integrating over \( u \) implies that

\[
\int_{t}^{T} \begin{pmatrix} m_t^-(u) \\ m_t^+(u) \end{pmatrix} (T-u) \, du = \left( A^{-1}(T-t) - A^{-2} \left( I - e^{-A(T-t)} \right) \right) \begin{pmatrix} \lambda_t^- \\ \lambda_t^+ \end{pmatrix} - A^{-1}\zeta + A^{-1}\zeta \frac{1}{2}(T-t)^2.
\]

This result is valid under the restriction that \( A \) is invertible, which is implied by the arrival rate of market orders (2) and Lemma 1. Moreover, since \( b(t, \lambda) = \sigma^2 \int_{t}^{T} \{ h_+ \cdot m_t^+(u) - h_- \cdot m_t^-(u) \} (T-u) \, du \) we arrive at (D.8). □

Example 7. Exponential Fill Rate. Take \( \kappa_{\pm} = f_{\pm}(\kappa) \), where \( f_{\pm} : \mathbb{R}^k \mapsto \mathbb{R}^+ \) are continuous functions. If \( h_{\pm}(\delta; \kappa) = e^{-\kappa_{\pm}\delta} \) for \( \delta > 0 \) and \( \mathbb{P} \left[ \inf_{t \in [0,T]} \kappa_t^\pm > 0 \right] = 1 \), then \( h_{\pm}(\delta_{0}^\pm; \kappa) = e^{-1} \) is constant and Proposition 6 applies.
Example 8. Power Fill Rate. Take $\kappa^\pm = f^\pm(\kappa)$, where $f^\pm : \mathbb{R}^\pm \to \mathbb{R}^+$ are continuous functions, and $\alpha^\pm > 1$ as fixed constants. If $h^\pm(\delta; \kappa) = \left[1 + (\kappa^\pm \delta)^{\alpha^\pm}\right]^{-1}$ for $\delta > 0$ and $\mathbb{P}\left[\inf_{t \in [0, T]} \kappa^\pm_t > 0 \right] = 1$, then $\delta_0^\pm = (\alpha^\pm - 1)^{-1/\alpha^\pm} (\kappa^\pm)^{-1}$ and $h^\pm(\delta_0^\pm; \kappa) = \frac{\alpha^\pm - 1}{\alpha^\pm}$ is constant and Proposition 6 applies.

Notice that the Poisson model of trade arrivals can be recovered by setting $\rho = 0$. Furthermore, if the initial states $\lambda^\pm_0$ are equal and $\kappa^\pm$ are equal then $b \equiv 0$.

Appendix E. Simulation Procedure

Here we describe in more detail the approach to simulating the PnL distribution of the HF strategy. Note that this produces an exact simulation – specifically, there are no discretization errors.

1. Generate the duration until the next market order (whether it is a buy or sell) given the current level of activity $\lambda^\pm_n$:
   - In between orders, the total rate of order arrival is
     $$\lambda_t = \lambda_t^+ + \lambda_t^- = 2\theta + (\lambda^+_n + \lambda^-_n - 2\theta) e^{-\beta(t - t_n)}.$$  

   To obtain a random draw of the time of the next trade, note that
   $$\mathbb{P}(t_{n+1} - t_n > \Delta T | \mathcal{F}_{t_n}) = \exp\left\{-2\theta \Delta T - (\lambda^+_n + \lambda^-_n - 2\theta) \frac{1 - e^{-\beta \Delta T}}{\beta}\right\}.$$  

   Therefore, draw a uniform $u \sim U(0, 1)$ and find the root\(^{15}\) of the equation
   $$\tau e^\tau = \frac{\lambda^+_n - 2\theta}{2\theta} e^\varsigma$$  

   where $\varsigma = \frac{\lambda^+_n - 2\theta}{2\theta} + \frac{\beta}{2\theta} \ln u$. Then, $T_{n+1} = \frac{1}{\beta}(\tau - \varsigma)$ is a sample for the next duration and $t_{n+1} = t_n + T_{n+1}$.

2. Decide if the trade is a buy or sell market order
   - The probability that the market order is a buy order is
     $$p_{\text{buy}} = \frac{\theta + (\lambda^+_n - \theta) e^{-\beta T_{n+1}}}{2\theta + (\lambda^+_n + \lambda^-_n - 2\theta) e^{-\beta T_{n+1}}}.$$  

   Therefore, draw a uniform $u \sim U(0, 1)$ and if $u < p_{\text{buy}}$ the order is a buy order, otherwise it is a sell order.

\(^{15}\)This is efficiently computed using the Lambert-W function since $A_0$ is typically small.
3. Decide whether the market order filled the agent’s posted limit order.
   • Compute the posted limit order at the time of the market order
     \[ \lambda_{t_{n+1}}^- = \theta + (\lambda_{t_n}^- - \theta) e^{-\beta T_{n+1}}, \quad \lambda_{t_{n+1}}^+ = \theta + (\lambda_{t_n}^+ - \theta) e^{-\beta T_{n+1}}. \]
   • Draw a uniform \( u \sim U(0, 1) \).
   • If the market order was a sell order, then if \( u < e^{-\kappa^- T_{n+1}} \) the agent’s buy limit order was lifted.
   • If the market order was a buy order, then if \( u < e^{-\kappa^+ T_{n+1}} \) the agent’s sell limit order was hit.

4. Update the mid-price of the asset.
   • \( S_{t_{n+1}} = S_{t_n} + \sigma \sqrt{T_{n+1}} Z \), where \( Z \sim N(0, 1) \).

5. Update the inventory and agent’s cash on hand.
   • If the agent’s sell limit order was hit, then \( X_{t_{n+1}} = X_{t_n} + (S_{t_{n+1}} + \delta_{t_{n+1}}^+) \) and \( q_{t_{n+1}} = q_{t_n} - 1 \).
   • If the agent’s buy limit order was hit, then \( X_{t_{n+1}} = X_{t_n} - (S_{t_{n+1}} - \delta_{t_{n+1}}^-) \) and \( q_{t_{n+1}} = q_{t_n} + 1 \).

6. Decide if the trade is influential and update activities.
   • Draw a uniform \( u \sim (0, 1) \), if \( u < \rho \) the trade is influential, set \( H_{n+1} = 1 \), otherwise set \( H_{n+1} = 0 \).
   • If the market order was a sell order then set
     \[ \lambda_{t_{n+1}}^- = \theta + (\lambda_{t_n}^- - \theta) e^{-\beta T_{n+1}} + \eta H_{n+1}, \quad \lambda_{t_{n+1}}^+ = \theta + (\lambda_{t_n}^+ - \theta) e^{-\beta T_{n+1}} + \nu H_{n+1}, \]
     \[ \kappa_{t_{n+1}}^- = \theta_{\kappa} + (\kappa_{t_n}^- - \theta_{\kappa}) e^{-\beta_{\kappa} T_{n+1}} + \eta_{\kappa} H_{n+1}, \quad \kappa_{t_{n+1}}^+ = \theta_{\kappa} + (\kappa_{t_n}^+ - \theta_{\kappa}) e^{-\beta_{\kappa} T_{n+1}} + \nu_{\kappa} H_{n+1}, \]
   • otherwise set,
     \[ \lambda_{t_{n+1}}^- = \theta + (\lambda_{t_n}^- - \theta) e^{-\beta T_{n+1}} + \nu H_{n+1}, \quad \lambda_{t_{n+1}}^+ = \theta + (\lambda_{t_n}^+ - \theta) e^{-\beta T_{n+1}} + \eta H_{n+1}, \]
     \[ \kappa_{t_{n+1}}^- = \theta_{\kappa} + (\kappa_{t_n}^- - \theta_{\kappa}) e^{-\beta_{\kappa} T_{n+1}} + \nu_{\kappa} H_{n+1}, \quad \kappa_{t_{n+1}}^+ = \theta_{\kappa} + (\kappa_{t_n}^+ - \theta_{\kappa}) e^{-\beta_{\kappa} T_{n+1}} + \eta_{\kappa} H_{n+1}. \]

7. Repeat from step 1 until \( t_{n+1} \geq T \).

The PnL for the Poisson benchmark case is obtained by setting \( \rho = 0 \) when calculating the optimal posting and cancelation of the limit orders.
Lemma 9. Conditional Mean of $\kappa_t$. Under the dynamics given in (3), the conditional mean $\tilde{m}_t^\pm(u) := \mathbb{E}[\kappa_u^\pm|\mathcal{F}_t]$ is

$$
\tilde{m}_t^\pm(u) = \theta_\kappa + \frac{\rho}{\beta_\kappa} \left[ \eta_\kappa m_t^\pm(u) + \nu_\kappa m_t^\mp(u) \right] + \left[ \kappa_t^\pm - \theta_\kappa - \frac{\rho}{\beta_\kappa} (\eta_\kappa \lambda_t^\pm + \nu_\kappa \lambda_t^\mp) \right] e^{-\beta_\kappa (u-t)} \quad (F.1)
$$

where $m_t^\pm(u)$ are given in Appendix B, and $\mathbf{A}, \zeta$ are given in Lemma 1.

Proof. Proceeding as in the proof of Lemma 1 in Appendix B, $\tilde{m}_t^\pm(u)$ satisfies the (uncoupled) system of ODEs

$$
\frac{d\tilde{m}_t^\pm(u)}{du} + \beta_\kappa \tilde{m}_t^\pm(u) = \beta_\kappa \theta_\kappa + \rho \left[ \eta_\kappa m_t^\pm(u) + \nu_\kappa m_t^\mp(u) \right] \quad (F.2)
$$

where $m_t^\pm(u)$ is given by (B.2). Solving (F.2) with the boundary condition $\tilde{m}_t^\pm(t) = \kappa_t^\pm$ gives the stated result. \qed