INTRODUCTION

Recently a number of researchers have pointed out the remarkable Shannon capacity gains available using multi-element antennas at both receiver and transmitter of a wireless system, often called *multiple-in, multiple-out* (MIMO) systems. For example, Foschini and Ganz [1] show that a capacity of about 19 bits/s/Hz is possible (with 99% confidence) with 4 transmit and receive antennas at a signal-to-noise ratio of 21 dB. Telatar [2] has independently shown similar results, and Tarokh et al. [3] have devised "space time codes" to achieve some part of these capacities. This work has all been based on the assumption of independent Rayleigh fading between each receive and transmit antenna. This implicitly assumes a high degree of scattering of signals between receive and transmit arrays, and in particular that there is a substantial angular spread in the signals at both sites.

Recently also the spatial or Geometrically-Based Stochastic model has been proposed for the mobile radio channel [4]. This is able to predict both angular and time delay spreads in the channel response, in the form of the Angle-Delay Power Spectrum (ADPS). The location of scatterers and reflectors relative to transmitter and receiver are assigned using some stochastic distribution, as shown in Fig. 1, although not all the clusters of scatterers may be present in all cases. From this the angle-of-arrival and excess delay of the multipath components due to each are calculated, assuming that signals are only scattered once between transmitter and receiver (this is known as the *single bounce* assumption). In this paper we will be interested primarily in the angle of arrival.

![Diagram of scatterers](image-url)

**Fig. 1. Distribution of scatterers in the spatial model**

This paper applies the spatial model to the calculation of capacities for a multi-element antenna system, thus explicitly taking account of the scattering environment. Thus, for example, we determine limits to the capacity gains available when only a limited number of scatterers are present. We also determine the required size and the optimum antenna spacing when the angular scattering of the channel is comparatively small (as it may be at the base station of cellular mobile system). Accordingly in the next section we give the formula for the Shannon capacity of MIMO systems, showing how the spatial model may be used to obtain the capacity, and in the third section we give some capacity results.
DERIVATION OF CAPACITY

The MIMO channel with $n_R$ receive and $n_T$ transmit antennas can readily be represented by an $(n_R \times n_T)$ matrix $H$, whose $i^{th}$ element gives the propagation from the $j^{th}$ transmit antenna to the $i^{th}$ receive antenna. Applying the singular value decomposition to this matrix, we may obtain a pair of unitary matrices $U$ and $V$ such that:

$$UHV = D$$  \[1\]

where $D$ is a diagonal matrix with diagonal elements made up of the eigenvalues of $HH^H$ ($X^H$ denotes the Hermitian transpose of $X$, i.e. the complex conjugate transpose). This transformation effectively forms $n$ separate channels, where $n \leq \min(n_R, n_T)$ is the rank of $HH^H$, with the gain of each given by the eigenvalue. We then apply Shannon’s capacity formula to each of these (remembering that the total transmitted power must be distributed between the channels), obtaining:

$$C = W \sum_{i=1}^{n} \log_2 \left( 1 + \frac{\lambda_{ij} S}{N} \right) = W \log_2 \left| 1 + HH^H S \right|$$  \[2\]

where $|X|$ denotes the determinant of $X$. Note that this capacity is attainable even if the channel is unknown at the transmitter. We now assume that the transmitter and receiver antennas are linear arrays with spacing $l$ between elements, and that $n_S$ scatterers are distributed in the space between the antennas. The response at the receiving antenna to a signal of complex amplitude $A$ and direction of arrival (DoA) $\phi$ can be written:

$$r = Aw_R(\phi) \text{where } w_{Rk}(\phi) = \exp\left(2\pi j \sin(\phi) \frac{kl}{\lambda}\right), \quad k = 0 \ldots n_R - 1$$  \[3\]

The phase reference position here is the 0th antenna element. Similarly the transmitted signal in the direction $\phi$ from the transmitting antenna with input signal vector $s$ is $A = w_T(\phi)^T s$. Thus the channel matrix:

$$H = \sum_{p=1}^{n_S} A_p w_R(\phi_{Rp}) w_T(\phi_{Tp})$$  \[4\]

where $A_p$, $\phi_{Rp}$ and $\phi_{Tp}$ are respectively the complex scattering coefficient, and the angle of arrival at the receive and at the transmit antenna. Note that in each term of this summation the rows (and columns) are related by a constant factor, and hence we may see that the rank of $H$ is limited to $n_S$. Thus the capacity increase is also limited to that given by $n_S$ parallel channels. We may use the spatial model to assign random locations to the scatterers and thus calculate $\phi_{Rp}$ and $\phi_{Tp}$ for each scatterer. Hence we determine $H$, from which the capacity may be obtained.

CAPACITY ESTIMATES USING SPATIAL MODEL

Using these principles we investigate the capacity of a fading multipath channel including a finite number of scatterers. Strictly the Shannon capacity of a fading channel is zero, so following [1] we use the concept of outage capacity. This is defined as the capacity achieved in a given proportion of randomly-chosen instances of the channel. Therefore we determine the probability distribution function of the capacity using Monte Carlo simulation of a random distribution of scatterers. We have considered two simple scenarios which illustrate the effects of more realistic scatterer distributions. In both we assume that the two antenna arrays, with $n_T = n_R = n$ and element spacing $l = \lambda/2$, are broadside on to one another a distance $d$ apart. In scenario (a) $n_S$ scatterers are uniformly randomly distributed within a square of side $d$ symmetrically placed between the antennas, as shown in Fig. 2(a). In (b) the scatterers are uniformly distributed within a circle around the receive antenna (Fig. 2(b)), as in Lee’s local scatterers model [5]. The scattering coefficient of each is Rayleigh distributed. The mean square amplitude of each is $\frac{n}{n_S}$, so that the channel matrix is normalised, in the sense that:

$$\sum_{i,j=1}^{n} |H_{ij}|^2 = n$$  \[5\]

(This criterion differs slightly from that used in [1], and results in general in a lower capacity, because it does not include the increased antenna gain due to the use of multiple antennas).
Fig. 2. Simulation scenarios: (a) distribution of scatterers in a square between antennas; in a circle around receive antenna

Fig. 3 shows the complementary cumulative distribution function (c.d.f.) of capacity for eight scatterers in scenario (a) with various numbers of receive/transmit antenna elements. The abscissa of this plot gives the proportion of cases in which the given capacity is available (in %). The capacities are compared with the capacity of eight uncoupled independently-fading Rayleigh channels, which is the limit for a channel matrix with rank 8 and Rayleigh fading elements. We note that the capacity increases with $n$ for small $n$, but tends to a limit as $n$ increases. This contrasts with the independently-fading Rayleigh channel examined in [1], in which capacity increases without limit with $n$. However we find that if the number of scatterers is much larger than the number of antennas, the independent Rayleigh channel gives a good approximation to the capacity. Antenna spacing in all cases is $\lambda/2$, which gives the maximum capacity. We observe that capacity decreases with smaller spacing for a given number of elements, but remains approximately constant at larger spacing.

Fig. 3. Complementary cumulative distribution function for 8 scatterers in scenario (a) and various numbers of transmit/receive antenna elements ($n_T = n_R = n$), compared with capacity of uncoupled Rayleigh channel, $n = 8$.

Fig. 4. shows the capacity of the channel with the scatterer distribution of scenario (b), with eight scatterers and eight transmit/receive antenna elements. The ratio $d/r = 5$. Capacity is plotted for transmit antenna spacings between $\lambda/2$ and $3\lambda$. It is compared with the capacity for scenario (a) with the same number of scatterers and antenna elements. We observe that in this case capacity reaches a maximum at a spacing of $2\lambda$. Note that the cluster of scatterers subtends an angle of 0.4 radians at the transmitter, and that the $2\lambda$-spaced array has an aliased beam at 0.52 radians ($\pi/6$). Thus any further increase in antenna spacing would simply mean that two aliased beams fell within the cluster of scatterers.
Note also that the linear array used at the receiver has a forward-backward ambiguity: that is, signals from the two sides of the array subtending the same angle from broadside cannot be distinguished. However, the use of a double array that eliminates this ambiguity does not increase capacity. In fact Fig. 4 shows that with sufficient transmit antenna spacing scenario (b) has the same capacity as scenario (a), in which the forward-backward ambiguity does not apply.

Fig. 4. Complementary cumulative distribution function for 8 scatterers in scenario (b) and 8 transmit/receive antenna elements \( n_T = n_R = n \) with different transmit antenna spacings \( l_T \lambda \), compared with capacity for scenario (a), \( n = 8 \).

CONCLUSIONS

We conclude firstly that in a practical situation where a limited number of significant scatterers are present the capacity of MIMO systems cannot be increased indefinitely by increasing the number of transmit/receive antennas. A limit is reached which is given by the capacity of \( n_S \) uncoupled Rayleigh fading channels. This applies regardless of the element spacing of the antennas: we cannot always provide uncorrelated fading at the elements by increasing the spacing. However, if the number of antenna elements is much less than the number of scatterers, the Rayleigh channel capacity is a good approximation.

However, where the scatterers lie in a cluster around the receive antenna, which is a common situation in mobile systems, capacity is improved by increasing the transmit element spacing, until the spacing between aliassed beams from the array is close to the angle subtended by the scatterer cluster.

REFERENCES


